Decentralized Optimal Control and Stabilization of Interconnected Systems With Asymmetric Information

Na Wang, Xiao Liang, Hongdan Li, and Xiao Lu

***Abstract*—The paper addresses the decentralized optimal control and stabilization problems for interconnected systems subject to asymmetric information. Compared with previous work, a closed-loop optimal solution to the control problem and sufficient and necessary conditions for the stabilization problem of the interconnected systems are given for the first time. The main challenge lies in three aspects: Firstly, the asymmetric information results in coupling between control and estimation and failure of the separation principle. Secondly, two extra unknown variables are generated by asymmetric information (different information filtration) when solving forward-backward stochastic difference equations. Thirdly, the existence of additive noise makes the study of mean-square boundedness an obstacle. The adopted technique is proving and assuming the linear form of controllers and establishing the equivalence between the two systems with and without additive noise. A dual-motor parallel drive system is presented to demonstrate the validity of the proposed algorithm.**

***Index Terms*—Asymmetric information, decentralized control, forward-backward stochastic difference equations, interconnected system, stalibization.**

I. Introduction

INTERCONNECTED systems have been found widely in a considerable quantity of fields and practical application scenarios, such as smart grids, formation flight, sensor network and cyber-physical systems [1]–[4], which consist of numerical subsystems. To realize a common goal, subsystems exchange and share partial information through the network to make an individual decision where communication delay occurs inevitably due to the limited bandwidth and cache capacity of nodes. Owing to the information available to each subsystem being incomplete and different, decentralized control with asymmetric information has proven to be an effective control scheme to achieve a desirable performance of the system.

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The decentralized control with asymmetric information is a kind of control scheme where the decision-maker of each subsystem or station accesses different information to make a decision, which could be traced back to team decision problem [5] and was further studied by Radner [6]. The decentralized optimal control problem with asymmetric information is challenging since the optimal control law may be nonlinear [7], [8]. Paramount attention has been paid to decentralized control problem over the past decades [9]–[16]. Just to name a few, [12] provided a dynamic program for the decentralized control problem with local and remote controllers employing the approach of common information. In [13], decentralized optimal control for a networked control system with asymmetric observations was studied with the assumption of linear controllers. The stabilizing solution was derived under decentralized controllers for the multiplicative-noise stochastic systems in [14]. Nevertheless, the results derived in the aforementioned literature mainly pertain to the decentralized control subject to special asymmetric information structure, e.g., $\mathcal{F}_1 \subseteq \mathcal{F}_2$, where $\mathcal{F}_1, \mathcal{F}_2$ denote the information sets available to two controllers of the system respectively, while the more general asymmetric information structure, i.e., $\mathcal{F}_1 \neq \mathcal{F}_2$ and $\mathcal{F}_1 \nsubseteq \mathcal{F}_2$ has not been investigated.

Information structure may determine the complexity and tractability of the decentralized optimal control problem in essence, hence playing a decisive role, as mentioned in [7], [8]. The decentralized control with a more general information structure where the relationship between the two information sets available to two controllers does not belong to inclusion has attracted lots of research and topics [17]–[22]. In particular, with a periodic sharing information pattern, [19] addressed the optimal control problem and proved that the periodic sharing pattern has a non-classical separation property. In [20], one step delayed control sharing was studied applying the dynamic programming method. In [21], the distributed LQG problem with the varying communication delay case was dealt with based on the information hierarchy graph.

In [22], optimal control law was derived for a class of lower block triangular systems. Great progress has been made, however, most works mentioned above are involved in the finite-horizon optimal control problem and few efforts have been devoted to the infinite-horizon stabilization problem even though it is a fundamental problem.

Given the discussion above, the decentralized control problem subject to asymmetric information has not been fully explored, which makes further study necessary. We are inspired by the previous work [23] where the optimal control policy was derived using a decomposition method of system state and control input based on noise history with the control weighting matrix positive definite. It is noted that the optimal controllers proposed in [23] are dependent on the noise history, which is essentially an open-loop control and hard to realize from the perspective of implementation due to the unmeasurable additive noises. Furthermore, the infinite-horizon stabilization problem was not taken into consideration either.

To get a closed-loop optimal solution as well as a complete stabilization solution, an interconnected system subject to asymmetric information is considered in this paper, where the state information of a subsystem is transmitted to another subsystem with one step communication delay to make the control policy, whereas the control input does not share. This kind of model derives from production cases, the economic dispatch of the power system for example. The whole power demand is distributed among the generating units, where each unit (subsystem) has some local information on its own environment and shares information through the network with time delay to minimize the total operating cost. Compared with [23], for the finite-horizon case, the solution to the forward-backward stochastic difference equations (FBSDEs) is presented by applying the stochastic maximum principle. Based on the solution, we obtain closed-loop optimal controllers and performance in terms of the solution to the Riccati equation under the assumption that the control weighting matrix is positive semidefinite. In addition, for the infinite-horizon case, we show the system is mean-square bounded if and only if the algebraic Riccati equation admits a unique positive definite solution.

In this paper, the problems of decentralized optimal control and stabilization for interconnected systems involving asymmetric information are investigated. The main challenge lies in threefold: The first is the coupling between control and estimation resulting from asymmetric information structure, which makes the classical separation principle no longer applicable. The second refers to finding the solution to the FBSDEs. Specifically, the information available to two controllers is partial because of one step communication delay, and the different information filtration (asymmetric information) leads to two extra unknown variables when solving FBSDEs. The above two obstacles could be overcome by proving that the asymmetric information structure is featured by the partially nested information structure and thus a linear form of the optimal controllers could be deduced. Thirdly, the study of the stabilization problem for the system involving additive noise is challenging, and merely a sufficient condition could be derived [24]. It shall be dealt with by constructing an equivalent relationship between the systems involving and without additive noise.

The contribution of the work is summarized below. A complete solution to the problem of decentralized optimal control and stabilization for the interconnected system subject to asymmetric information is provided for the first time. For the finite-horizon case, the equivalent conditions for the unique solvability of the optimal control problem with asymmetric information are obtained by adopting the stochastic maximum principle. Based on the solution of the FBSDEs, the necessary and sufficient conditions for the decentralized optimal control problem are given, as well as the analytical expression for the closed-loop optimal controllers in terms of the solution to the proposed Riccati equation. For the infinite-horizon case, the necessary condition of the mean-square stabilization is presented for the system without additive noises. On the basis of the results and relationship between the two systems, the necessary and sufficient conditions for the mean-square boundedness of the system with additive noise are attained.

The remainder of the paper is structured as follows: In Section II, the finite-horizon decentralized optimal control problem is studied. The stabilization and optimal control problem for the infinite-horizon are investigated in Section III. Numerical examples are provided in Section IV. Concluding remarks are presented in Section V. Relevant proofs are detailed in the appendices.

The following notations and definitions will be used. *Notations:* $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. I means the unit matrix with the appropriate dimension. $\Delta > 0$ (or ≥ 0) denotes that Δ is a positive definite (or positive semidefinite) matrix. $\mathbb{E}$ stands for the mathematical expectation operator. $\mathcal{Z}(k)$ denotes the σ-algebra generated by the random variable z_k. $Tr(\cdot)$ means the trace of a matrix, and $A^\dagger$ denotes the Moore-Penrose inverse of matrix A and A' represents the transpose of the matrix A. $0_{i\times j}$ represents a matrix of dimensions $i \times j$ with all zero elements and $\mathbf{0}$ means a zero matrix with compatible dimensions. $F \doteq blkdiag\{A, B, C, D\}$ stands for a block diagonal matrix created by aligning the input matrices A, B, C, D along the diagonal of F.

II. Finite Horizon Case

A. Problem Formulation

The system to be studied is given by

$$\begin{bmatrix} x_{k+1}^1 \\ x_{k+1}^2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_k^1 \\ x_k^2 \end{bmatrix} + \begin{bmatrix} B_1 & \mathbf{0} \\ \mathbf{0} & B_2 \end{bmatrix} \begin{bmatrix} u_k^1 \\ u_k^2 \end{bmatrix} + \begin{bmatrix} \omega_k^1 \\ \omega_k^2 \end{bmatrix} \tag{1}$$

where $x_k^i \in \mathbb{R}^{n_i}$, $u_k^i \in \mathbb{R}^{m_i}$ and $\omega_k^i \in \mathbb{R}^{n_i}$ respectively are state, control input and process noise of the ith subsystem. A_{ij}, B_i are constant matrices with compatible dimensions. Initial state x_0^i is a Gaussian variable with $x_0^i \sim \mathcal{N}(\bar{x}_0^i, \Sigma_0^i), \Sigma_0^i > 0$. The noise ω_k^i is Gaussian process with $\omega_k^i \sim \mathcal{N}(0, W^i)$. Moreover, ω_k^i is related with ω_k^j for $i \neq j$; x_0^i and ω_k^i are independent of each other.

By expanding the dimension of state and noise respectively, system (1) could be rewritten as

$$x_{k+1} = Ax_k + \bar{B}_1 u_k^1 + \bar{B}_2 u_k^2 + \omega_k \tag{2}$$

where $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $\bar{B}_1 = \begin{bmatrix} B_1 \\ 0_{n_2\times m_1} \end{bmatrix}$, $\bar{B}_2 = \begin{bmatrix} 0_{n_1\times m_2} \\ B_2 \end{bmatrix}$, $x_k = \begin{bmatrix} x_k^1 \\ x_k^2 \end{bmatrix}$, $\omega_k = \begin{bmatrix} \omega_k^1 \\ \omega_k^2 \end{bmatrix}$. The noise and initial state of expanded dimension respectively satisfy $\omega_k \sim \mathcal{N}(0, W)$, $x_0 \sim \mathcal{N}(\bar{x}_0, \Sigma_0), \Sigma_0 > 0$.

The associated cost functional is of the form

$$J = \mathbb{E}\Big\{ \sum_{k=0}^{N} \Big[x_k' Q x_k + (u_k^1)' R_1 u_k^1 + (u_k^2)' R_2 u_k^2 \Big] + x_{N+1}' H x_{N+1} \Big\} \tag{3}$$

where $Q \geq 0, R_1 \geq 0, R_2 \geq 0$ and $H \geq 0$ are the weighting matrices with compatible dimensions.

Remark 1: It should be emphasized that the control weighting matrix R_i in the cost functional (3) is positive semidefinite, which is a weaker condition than that of the work in [23] where the assumption $R > 0$ was made.

In the interconnected system (2), the state of a subsystem is transmitted to another subsystem through the network connection with one step time required. Thus the information available to two controllers (subsystems) is as follows:

$$u_k^1 : \left\{ x_0^1, \ldots, x_{k-1}^1, x_k^1, x_0^2, \ldots, x_{k-1}^2 \right\}$$
$$u_k^2 : \left\{ x_0^2, \ldots, x_{k-1}^2, x_k^2, x_0^1, \ldots, x_{k-1}^1 \right\}.$$

We introduce the following information sets:

$$\mathcal{F}_k^c = \{x_0, \ldots, x_{k-1}\} \tag{4}$$

$$\mathcal{F}_k^1 = \left\{x_0, \ldots, x_{k-1}, x_k^1\right\} = \left\{\mathcal{F}_k^c, x_k^1\right\} \tag{5}$$

$$\mathcal{F}_k^2 = \left\{x_0, \ldots, x_{k-1}, x_k^2\right\} = \left\{\mathcal{F}_k^c, x_k^2\right\}. \tag{6}$$

Remark 2: In this paper, we assume there exists one step communication delay to guarantee the information propagation is no slower than the dynamics propagation through the plant. Actually, such an assumption of one step communication delay is commonly found in previous research [20], [22].

Problem 1: Find $\mathcal{F}_k^1$-measurable controller u_k^1 and $\mathcal{F}_k^2$-measurable controller u_k^2 to minimize the cost functional (3) subject to system (2).

Remark 3: Note that the proposed information structure is characterized by $\mathcal{F}_{k-1}^i \subset \mathcal{F}_k^j$. It conforms to a partially nested information pattern, which indicates that the optimal control policy is linear (see Definition 3 and Theorem 2 in [25]).

Remark 4: The problem studied in this paper is different from the previous works [23], [26]. In [26], the system with uncorrelated subsystem noises was dealt with, however, the results could not be extended to the case with correlated subsystem noises. By contrast, we remove the assumption that subsystem process noises are uncorrelated. Compared with [23], we aim to find the optimal closed-loop solution with the relaxer condition rather than an open-loop solution.

B. Equivalent Conditions of the Solvability of Problem 1

In this section, stochastic maximum principle is applied to (2) and (3) to transform the solvability of the Problem 1 into that of the FBSDEs.

Lemma 1: Assume Problem 1 could be solved uniquely, the optimal controllers meet the following equations:

$$0_{m_1\times 1} = \mathbb{E}[\bar{B}_1' \lambda_k | \mathcal{F}_k^1] + R_1 u_k^1 \tag{7}$$

$$0_{m_2\times 1} = \mathbb{E}[\bar{B}_2' \lambda_k | \mathcal{F}_k^2] + R_2 u_k^2 \tag{8}$$

where λ_k satisfies

$$\lambda_{k-1} = \mathbb{E}[A' \lambda_k | \mathcal{F}_k^1, \mathcal{F}_k^2] + Q x_k \tag{9}$$

$$\lambda_N = H x_{N+1}. \tag{10}$$

Conversely, if FBSDEs (2) and (7)−(10) admit a unique solution, then Problem 1 could be solved uniquely.

Proof: The details of the proof are omitted here, which is similar to [27], [28]. ■

It is evident that the information sets available to u_k^1 and u_k^2 have an interaction, that is, $\mathcal{F}_k^c = \mathcal{F}_k^1 \cap \mathcal{F}_k^2$, which indicates that u_k^1 and u_k^2 are related to each other. To make the solvability of the FBSDEs easier and obtain the optimal controllers, the equivalent FBSDEs are obtained via the following definitions:

$$\hat{u}_k^1 = \mathbb{E}[u_k^1 | \mathcal{F}_k^c], \quad \tilde{u}_k^1 = u_k^1 - \hat{u}_k^1 \tag{11}$$

$$\hat{u}_k^2 = \mathbb{E}[u_k^2 | \mathcal{F}_k^c], \quad \tilde{u}_k^2 = u_k^2 - \hat{u}_k^2. \tag{12}$$

Obviously, we have that

$$\mathbb{E}[\tilde{u}_k^1 | \mathcal{F}_k^1] = \tilde{u}_k^1, \qquad \mathbb{E}[\tilde{u}_k^2 | \mathcal{F}_k^2] = \tilde{u}_k^2. \tag{13}$$

And system (2) can be further transformed as

$$x_{k+1} = Ax_k + B\hat{u}_k + \bar{B}_1 \tilde{u}_k^1 + \bar{B}_2 \tilde{u}_k^2 + \omega_k \tag{14}$$

with $B = \begin{bmatrix} \bar{B}_1 & \bar{B}_2 \end{bmatrix}$, $\hat{u}_k = \begin{bmatrix} \hat{u}_k^1 \\ \hat{u}_k^2 \end{bmatrix}$.

Lemma 2: Problem 1 is uniquely solvable if and only if the FBSDEs below are uniquely solvable:

$$0_{m\times 1} = \mathbb{E}[B' \lambda_k | \mathcal{F}_k^c] + R\hat{u}_k \tag{15}$$

$$0_{m1\times 1} = \mathbb{E}[\bar{B}_1' \lambda_k | \mathcal{F}_k^1] - \mathbb{E}[\bar{B}_1' \lambda_k | \mathcal{F}_k^c] + R_1 \tilde{u}_k^1 \tag{16}$$

$$0_{m2\times 1} = \mathbb{E}[\bar{B}_2' \lambda_k | \mathcal{F}_k^2] - \mathbb{E}[\bar{B}_2' \lambda_k | \mathcal{F}_k^c] + R_2 \tilde{u}_k^2 \tag{17}$$

where $R = blkdiag\{R_1, R_2\}$, λ_k is as (9) and (10).

Proof: In view of Lemma 1, the key to verify Lemma 2 is establishing the equivalence of (7), (8) and (15)−(17). Taking conditional expectation on both sides of (7) and (8) with regard to $\mathcal{F}_k^c$, it follows:

$$0_{m1\times 1} = \mathbb{E}[\bar{B}_1' \lambda_k | \mathcal{F}_k^c] + R_1 \hat{u}_k^1 \tag{18}$$

$$0_{m2\times 1} = \mathbb{E}[\bar{B}_2' \lambda_k | \mathcal{F}_k^c] + R_2 \hat{u}_k^2. \tag{19}$$

Equation (15) could be obtained by augmentating $\hat{u}_k^1$ and $\hat{u}_k^2$. Then subtracting (18) from (7), we have (16). Similarly, subtracting (19) from (8), (17) follows. Therefore, (7) and (8) could be equivalently represented as (15)–(17) and Lemma 2 holds. ■

C. Main Results

The major results shall be presented in three steps in this section: deduce the feedback forms of the controllers, give the optimal estimation and demonstrate the solution to Problem 1. The first part is shown now.

As mentioned in Remark 3, the optimal controllers we aim to find are in linear forms. To this end, the linear feedback forms of $\hat{u}_k^c$, $\tilde{u}_k^1$ and $\tilde{u}_k^2$ will be deduced based on Lemma 2 and the projection principle, respectively. Firstly, from (15), the $\hat{u}_k^c$ is $\mathcal{F}_k^c$-measurable. Coupled with the relationship between x and λ presented in (10), it is inferred that $\hat{u}_k^c$ is in the feedback form of $\mathbb{E}\left[x_k|\mathcal{F}_k^c\right]$, i.e.,

$$\hat{u}_k = K_k\mathbb{E}\left[x_k|\mathcal{F}_k^c\right]. \tag{20}$$

Secondly, by virtue of the projection principle, we have

$$\mathbb{E}[\bar{B}_1'\lambda_k|\mathcal{F}_k^1] = \mathbb{E}[\bar{B}_1'\lambda_k|\mathcal{F}_k^c] + \Xi_k^1\left\{x_k^1 - \mathbb{E}\left[x_k^1|\mathcal{F}_k^c\right]\right\}$$
$$\mathbb{E}[\bar{B}_2'\lambda_k|\mathcal{F}_k^2] = \mathbb{E}[\bar{B}_2'\lambda_k|\mathcal{F}_k^c] + \Xi_k^2\left\{x_k^2 - \mathbb{E}\left[x_k^2|\mathcal{F}_k^c\right]\right\}$$

where $\Xi_k^i = \mathbb{E}\left[\bar{B}_i'\lambda_k\{x_k^i - \mathbb{E}\left[x_k^i|\mathcal{F}_k^c\right]\}'\right]\mathbb{E}\left[\{x_k^i - \mathbb{E}\left[x_k^i|\mathcal{F}_k^c\right]\}\{x_k^i - \mathbb{E}\left[x_k^i|\mathcal{F}_k^c\right]\}'\right]^{-1}, i = 1,2$. Along with (16) and (17), it implies $\tilde{u}_k^i$ could be expressed as the feedback form of the innovation process $x_k^i - \mathbb{E}\left[x_k^i|\mathcal{F}_k^c\right]$, i.e.,

$$\tilde{u}_k^1 = K_k^1\left\{x_k^1 - \mathbb{E}\left[x_k^1|\mathcal{F}_k^c\right]\right\} \tag{21}$$

$$\tilde{u}_k^2 = K_k^2\left\{x_k^2 - \mathbb{E}\left[x_k^2|\mathcal{F}_k^c\right]\right\}. \tag{22}$$

Consequently, we can conclude from (20)–(22) the key to obtain optimal controllers is to get feedback gain K_k and K_k^i and related iterative process of states estimated by the partial information (4)–(6).

Now we are in a position to present iterative process of $\mathbb{E}\left[x_k|\mathcal{F}_k^c\right]$, $\mathbb{E}\left[x_k^i|\mathcal{F}_k^c\right]$, $\mathbb{E}\left[x_k|\mathcal{F}_k^i\right]$ and $\mathbb{E}\left[x_k^i|\mathcal{F}_k^i\right]$. Since $x_k^i = I_i x_k, (I_1 = [\ I_{n_1} \quad 0_{n_1\times n_2}\ ], I_2 = [\ 0_{n_2\times n_1} \quad I_{n_2}\ ])$, we give the estimation of augmented state x_k instead of x_k^i for simplicity.

$$\begin{aligned}\hat{x}_{k+1/k+1}^c &= \mathbb{E}[x_{k+1}|\mathcal{F}_{k+1}^c]\\ &= Ax_k + B\hat{u}_k + \bar{B}_1\tilde{u}_k^1 + \bar{B}_2\tilde{u}_k^2\end{aligned} \tag{23}$$

$$\tilde{x}_{k+1/k+1}^c = x_{k+1} - \hat{x}_{k+1/k+1}^c \tag{24}$$

with initial value $\hat{x}_{0/0}^c = \bar{x}_0$.

$$\begin{aligned}\hat{x}_{k+1/k} &= \mathbb{E}[x_{k+1}|\mathcal{F}_k^1]\\ &= A\hat{x}_{k/k} + B\hat{u}_k + \bar{B}_1\tilde{u}_k^1 + \bar{B}_2\mathbb{E}\left[\tilde{u}_k^2|\mathcal{F}_k^1\right]\\ &= (A + \bar{B}_1K_k^1I_1 + \bar{B}_2K_k^2I_2)\hat{x}_{k/k}\\ &\quad + (BK_k - \bar{B}_1K_k^1I_1 - \bar{B}_2K_k^2I_2)\hat{x}_{k/k}^c\end{aligned} \tag{25}$$

$$\begin{aligned}\hat{\hat{x}}_{k+1/k} &= \mathbb{E}[x_{k+1}|\mathcal{F}_k^2]\\ &= A\hat{\hat{x}}_{k/k} + B\hat{u}_k + \bar{B}_1\mathbb{E}\left[\tilde{u}_k^1|\mathcal{F}_k^2\right] + \bar{B}_2\tilde{u}_k^2\\ &= (A + \bar{B}_1K_k^1I_1 + \bar{B}_2K_k^2I_2)\hat{\hat{x}}_{k/k}\\ &\quad + (BK_k - \bar{B}_1K_k^1I_1 - \bar{B}_2K_k^2I_2)\hat{x}_{k/k}^c.\end{aligned} \tag{26}$$

The estimation error covariance is as follows:

$$\begin{aligned}&\mathbb{E}[(x_{k+1} - \hat{x}_{k+1/k+1}^c)(x_{k+1} - \hat{x}_{k+1/k+1}^c)']\\ &= \mathbb{E}\Big\{[(Ax_k + B\hat{u}_k + \bar{B}_1\tilde{u}_k^1 + \bar{B}_2\tilde{u}_k^2 + \omega_k)\\ &\quad - (Ax_k + B\hat{u}_k + \bar{B}_1\tilde{u}_k^1 + \bar{B}_2\tilde{u}_k^2)]\\ &\quad \times [(Ax_k + B\hat{u}_k + \bar{B}_1\tilde{u}_k^1 + \bar{B}_2\tilde{u}_k^2 + \omega_k)\\ &\quad - (Ax_k + B\hat{u}_k + \bar{B}_1\tilde{u}_k^1 + \bar{B}_2\tilde{u}_k^2)]'\Big\} = W.\end{aligned} \tag{27}$$

As shown in Lemma 2, to solve Problem 1 is converted into solving FBSDEs (9), (10), (14)–(17). To get the solution to FBSDEs and derive optimal closed-loop feedback gain explicitly, we introduce the following two backward recursion equations one of which is a Riccati equation:

$$P_k = A'P_{k+1}A + K_k'\Upsilon_kK_k + Q \tag{28}$$

$$\tilde{P}_k = A'P_{k+1}A + K_k'\Upsilon_k\tilde{K}_k + Q \tag{29}$$

where

$$\begin{cases}K_k = -\Upsilon_k^{-1}M_k\\ \tilde{K}_k = blkdiag\{K_k^1, K_k^2\}\\ K_k^i = -(\Upsilon_k^i)^{-1}M_k^i, \quad (i,j = 1,2\&i \neq j)\\ \Upsilon_k = B'P_{k+1}B + R\\ \Upsilon_k^i = \bar{B}_i'P_{k+1}\bar{B}_i + R_i\\ M_k = B'P_{k+1}A\\ M_k^i = \bar{B}_i'P_{k+1}(A + \bar{B}_jK_k^jI_j)I_i^\dagger\end{cases} \tag{30}$$

with terminal values $P_{N+1} = \tilde{P}_{N+1} = H$.

The solution to Problem 1 is shown as follows.

Theorem 1: Problem 1 has a unique solution if and only if the difference equations (28)–(30) are well defined, i.e., Υ_k and Υ_k^i are invertible for $k = N,\ldots,0$. Under the situation, the associated optimal controllers are given by

$$\tilde{u}_k^1 = -[I\ \ 0]\,\Upsilon_k^{-1}M_k\hat{x}_{k/k}^c - (\Upsilon_k^1)^{-1}M_k^1(x_k^1 - \hat{x}_{k/k}^{1c}) \tag{31}$$

$$\tilde{u}_k^2 = -[0\ \ I]\,\Upsilon_k^{-1}M_k\hat{x}_{k/k}^c - (\Upsilon_k^2)^{-1}M_k^2(x_k^2 - \hat{x}_{k/k}^{2c}) \tag{32}$$

where $\Upsilon_k, \Upsilon_k^i, M_k$, and M_k^i are as in (30). And the solution to FBSDEs satisfies the following relationship:

$$\lambda_{k-1} = P_k\hat{x}_{k/k}^c + \tilde{P}_k(x_k - \hat{x}_{k/k}^c). \tag{33}$$

Moreover, the associated performance is as

$$J_N^* = \mathbb{E}[x_0'P_0\hat{x}_{0/0}^c + x_0'\tilde{P}_0(x_0 - \hat{x}_{0/0}^c)] + \sum_{k=0}^{N}Tr[W\tilde{P}_{k+1}]. \tag{34}$$

Proof: Please see Appendix A. ■

Remark 5: Differently from [23] which derived an open-loop optimal solution using a decomposition method for system state and control input based on noise history, Theorem 1 presents a closed-loop optimal solution to finite-horizon decentralized control problem subject to asymmetric information by applying the stochastic maximum principle. It is noted that the stochastic maximum principle, an effective and powerful tool in optimal control theory, is widely employed to address optimization problems, see [13], [27], [28]. Moreover, the infinite-horizon stabilization problem will be dealt with in the next section, while it was not considered in [23].

III. Stabilization Problem and Infinite Case

A. Problem Formulation

The infinite-horizon stabilization problem will be investigated in this section. To begin with, the system (2) without additive noise is considered. Thus, the system studied can be expressed as

$$x_{k+1} = Ax_k + \bar{B}_1 u_k^1 + \bar{B}_2 u_k^2 \tag{35}$$

where the initial state x_0 is a Gaussian random vector with mean and covariance $\bar{x}_0$ and Σ_0, respectively. The cost functional is given as

$$J = \mathbb{E}\sum_{k=0}^{\infty}\Big[x_k' Q x_k + (u_k^1)' R_1 u_k^1 + (u_k^2)' R_2 u_k^2\Big]. \tag{36}$$

Remark 6: It should be emphasized that for the case without additive noise, the $\hat{x}_{k/k}^c$ and $(x_k^i - \hat{x}_{k/k}^{ic})$ to be stated below are reduced respectively to:

$$\begin{cases} \hat{x}_{k/k}^c = x_k \\ x_k^i - \hat{x}_{k/k}^{ic} = 0. \end{cases} \tag{37}$$

Firstly, we introduce two definitions.

Definition 1: Without u_k^i, the system (35) is said to be mean-square stable if it holds that $\lim_{k\to\infty} E(x_k' x_k) = 0$ for any initial value x_0.

Definition 2: We call system (35) mean-square stabilizable if there exists $\mathcal{F}_{k-1}^1$-measurable $u_k^1 = \Gamma_1 \hat{x}_{k/k}^c - L_1(x_k^1 - \hat{x}_{k/k}^{1c})$ and $\mathcal{F}_k^2$-measurable $u_k^2 = \Gamma_2 \hat{x}_{k/k}^c - L_2(x_k^2 - \hat{x}_{k/k}^{2c}), k \ge 0$ with constant matrices Γ_1, Γ_2, L_1 and L_2, the closed-loop system of (35) could be asymptotically mean-square stable for any initial values x_0.

Definition 3: The following system: $x_{k+1} = Ax_k + \bar{B}_1 u_k^1 + \bar{B}_2 u_k^2, y_k = Dx_k$ or (A, D) is called exactly observable if for any $N \ge n$, $y_k = 0, a.s.\ \forall 0 \le k \le N \Rightarrow x_0 = 0$

Then two standard assumptions are made below. The first guarantees the existence of unique pair of controllers, and the second is a standard assumption for the mean-square stabilization problem [29]:

Assumption 1: $R_1 > 0, R_2 > 0$ and $Q = D'D \ge 0$.

Assumption 2: $(A, Q^{1/2})$ is exactly observable.

Before the main problem and results are given, the Lemma 3 is shown.

For convenience, P_k, $\tilde{P}_k$, $\tilde{K}_k$, K_k^i, Υ_k, Υ_k^i, M_k, M_k^i in (28)−(30) are rewritten as $P_k(N)$, $\tilde{P}_k(N)$, $\tilde{K}_k(N)$, $K_k^i(N)$, $\Upsilon_k(N)$, $\Upsilon_k^i(N)$, $M_k(N)$, $M_k^i(N)$ to make the time horizon N explicit for the finite-horizon case.

Lemma 3: With Assumptions 1 and 2, if there exist stabilizing controllers to make the system (35) mean-square stabilizable, the algebraic Riccati equation (38) admits a unique positive definite solution P

$$P = A'PA + K'\Upsilon K + Q \tag{38}$$

$$\tilde{P} = A'PA + K'\Upsilon\tilde{K} + Q \tag{39}$$

where

$$\begin{cases} K = -\Upsilon^{-1}M \\ \tilde{K} = blkdiag\{K^1, K^2\} \\ K^i = -(\Upsilon^i)^{-1}M^i, \quad (i, j = 1, 2\ \&\ i \ne j) \\ \Upsilon = B'PB + R \\ \Upsilon^i = \bar{B}_i' P\bar{B}_i + R_i \\ M = B'PA \\ M^i = \bar{B}_i' P(A + \bar{B}_j K^j I_j) I_i^{\dagger}. \end{cases} \tag{40}$$

Proof: Please see Appendix B. ■

Remark 7: Noting that since the additive noise is not considered for the system here, the algebraic equations (38)−(40) in Lemma 3 are reduced to (38) and (41).

$$\begin{cases} K = -\Upsilon^{-1}M \\ \Upsilon = B'PB + R \\ M = B'PA. \end{cases} \tag{41}$$

Next we will investigate the boundedness problem of the system (2). The associated performance is given by

$$\tilde{J} = \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N}\mathbb{E}[x_k' Q x_k + (\hat{u}_k)' R\hat{u}_k + (\tilde{u}_k^1)' R_1 \tilde{u}_k^1 + (\tilde{u}_k^2)' R_2 \tilde{u}_k^2]. \tag{42}$$

Problem 2 to be investigated is shown below.

Problem 2: Find a pair of controllers $\mathcal{F}_k^1$-measurable u_k^1 and $\mathcal{F}_k^2$-measurable u_k^2 to make system (2) mean-square bounded and to minimize cost functional (42).

B. Solution to Problem 2

The main results are presented in the following theorem.

Theorem 2: Under Assumptions 1 and 2, the system (2) is mean-square bounded if and only if there exists a unique positive definite solution P to the algebraic equation (38). In this case, the stabilizing controllers satisfy

$$u_k^1 = -[I\ \ 0]\Upsilon^{-1}M\hat{x}_{k/k}^c - (\Upsilon^1)^{-1}M^1(x_k^1 - \hat{x}_{k/k}^{1c}) \tag{43}$$

$$u_k^2 = -[0\ \ I]\Upsilon^{-1}M\hat{x}_{k/k}^c - (\Upsilon^2)^{-1}M^2(x_k^2 - \hat{x}_{k/k}^{2c}). \tag{44}$$

Moreover, the stabilizing controllers could minimize the cost functional (42) which is calculated as

$$J^* = \mathbb{E}[x_0' P \hat{x}_{0/0}^c + x_0' \tilde{P}(x_0 - \hat{x}_{0/0}^c)]. \tag{45}$$

Proof: Please see Appendix C. ■

IV. Numerical Examples

Consider a dual-motor parallel drive system as depicted in Fig. 1. The dynamics of a single motor system can be described as

$$J\ddot{\theta} = -((k_v + \Delta k_v)\dot{\theta} + (k_c + \Delta k_c)\mathrm{sign}(\dot{\theta})) + bu - Td \tag{46}$$

where J, θ, k_v and k_c denote inertia moment, position output, viscous friction coefficient and Coulomb friction coefficient, respectively. sign() is signum function, b is control gain, T_d is external disturbance, and Δk_c and Δk_v are parametric perturbations.

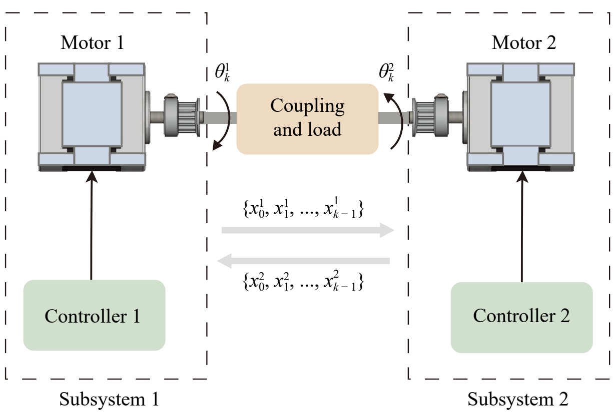

Fig. 1. Overview of dual-motor drive system.

In this system, the dual-motor should achieve the goal of tracking a specified rotational angle while ensuring that the velocities of both motors tend to 0. Thus, a linearized model gives a sufficient description of the system behavior with these conditions. The discrete-time model for a dual-motor is given as (2) through linearing and one step forward discretization to (46). The matrices $A, \bar{B}_1$ and $\bar{B}_2$ are detailed as

$$A = \begin{bmatrix} 1 & T_s & 0 & 0 \\ \gamma_1 & \delta_1 & \gamma_1 & 0 \\ 0 & 0 & 1 & T_s \\ \gamma_2 & 0 & \gamma_2 & \delta_2 \end{bmatrix}, \quad \bar{B}_1 = \begin{bmatrix} 0 \\ k_{u1} \\ 0 \\ 0 \end{bmatrix}, \quad \bar{B}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ k_{u2} \end{bmatrix}$$

where $\gamma_i = -T_s k_r / J_i, \delta_i = -T_s k_v / J_i + 1, (i = 1, 2)$, T_s is the sampling time. The state $x_k = [\ (\Delta\theta_k^1)' \quad (v_k^1)' \quad (\Delta\theta_k^2)' \quad (v_k^2)'\ ]'$, where $\Delta\theta_k^i = \theta_k^i - \theta_d$ is tracking deviation, θ_k^i and θ_d are position output and desired rotational angle. As shown in Fig. 1, the two motors are installed facing each other, thus the two motors rotate in opposite directions. v_k^i is the velocity of the ith motor, and the corresponding states for both subsystems are $x_k^i = [\ (\Delta\theta_k^i)' \quad (v_k^i)'\ ]'$.

The objective of this paper is to design a pair of optimal controllers to achieve position tracking and make velocities of both motors tend to 0. The associated parameters referred to [30] are as follows: $\theta_d = 0$ rad, $T_s = 0.01$ s, $\gamma_1 = -0.0526$, $\gamma_2 = -0.0577$, $\delta_1 = 0.993$, $\delta_2 = 0.9923$, $k_{u1} = 0.4509$, $k_{u2} = 0.4308$, $N = 1000$, $\bar{x}_0^i = \begin{bmatrix} 0 & 0 \end{bmatrix}'$, $\Sigma_0^i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $W = \begin{bmatrix} 1 & 0 & 1.2 & 0 \\ 0 & 1 & 0 & 1.2 \\ 1.2 & 0 & 1.44 & 0 \\ 0 & 1.2 & 0 & 1.44 \end{bmatrix} \times 10^{-3}$. Set the weighting matrices $Q = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$, $R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

The velocities and tracking deviations of the two motors are shown in Fig. 2 and 3, respectively. It can be seen from Fig. 2 that the regulated velocities of the two motors tend to 0 as expected by applying the optimal controllers in Theorem 2. As seen from Fig. 3, $\Delta\theta_k^{1*}$ and $\Delta\theta_k^{2*}$ are regulated tracking deviations, while $\Delta\theta_k^1$ and $\Delta\theta_k^2$ are the tracking deviations without control. It shows that the regulated position keeps gradually around the desired position and outperforms the case without control, which indicates that the proposed control strategy is effective.

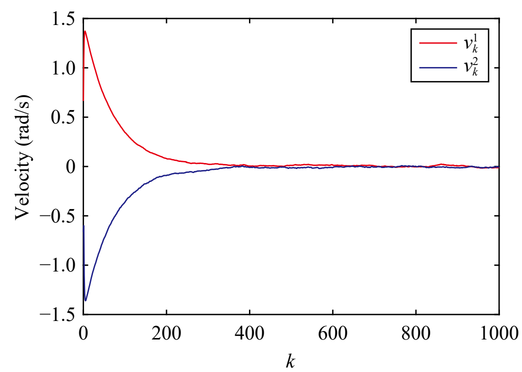

Fig. 2. Velocities of the two motors.

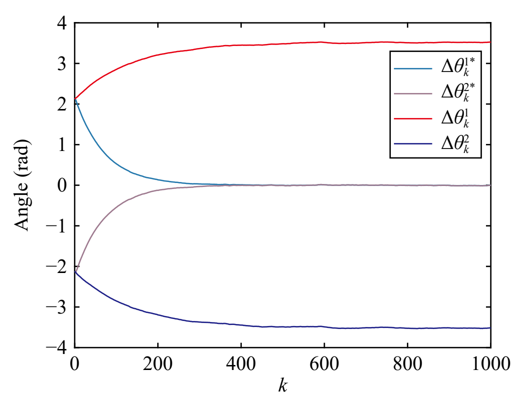

Fig. 3. Comparison of tracking deviations with and without control.

V. Conclusion

The decentralized optimal control and stabilization problems for interconnected systems subject to asymmetric information have been investigated. Firstly, employing the stochastic maximum principle, an equivalent solvability condition for the problem of optimization has been presented in terms of FBSDEs. Based on the equivalence, a complete solution, the necessary and sufficient conditions and an closed-loop explicit form of controllers, has been obtained for the finite-horizon case. Furthermore, under standard assumptions, it has been shown that the system is mean-square bounded for the infinite-horizon case if and only if the algebraic Riccati equation admits a unique positive definite solution. In the future, our work will be extended to the multiplicative noises model, interconnected systenms with d-step delayed information sharing model and a large-scale system consisting of multi-subsystem with d-step delay.

Appendix A
Proof of Theorem 1

Using Lemma 2, we will show that (15)−(17) are uniquely solvable if and only if Υ_k, Υ_k^1 and Υ_k^2 are invertible for $k=0,\ldots,N$ by induction.

For $k=N$, from (10), (14), and (15), we have

$$\begin{aligned}0_{m\times 1} &= \mathbb{E}[B'\lambda_N|\mathcal{F}_N^c]+R\hat{u}_N\\ &= \mathbb{E}[B'H(Ax_N+B\hat{u}_N+\bar{B}_1\tilde{u}_N^1+\bar{B}_2\tilde{u}_N^2+\omega_N)|\mathcal{F}_N^c]\\ &\quad +R\hat{u}_N\\ &=(B'HB+R)\hat{u}_N+B'HA\hat{x}_{N/N}^c\\ &=\Upsilon_N\hat{u}_N+M_N\hat{x}_{N/N}^c. \end{aligned}\tag{47}$$

From Lemma 2, we could obtain that (15) for $k=N$ is uniquely solvable if and only if Υ_N is invertible. Along with (20), K_N is calculated as $K_N=-\Upsilon_N^{-1}M_N$. Thus, $\hat{u}_N$ could be derived as $\hat{u}_N=-\Upsilon_N^{-1}M_N\hat{x}_{N/N}^c$.

In virtue of (16) and (21), we have

$$\begin{aligned}0_{m_1\times 1} &= \mathbb{E}[\bar{B}_1'\lambda_N|\mathcal{F}_N^1]-E[\bar{B}_1'\lambda_N|\mathcal{F}_N^c]+R_1\tilde{u}_N^1\\ &= \mathbb{E}[B'H(Ax_N+B\hat{u}_N+\bar{B}_1\tilde{u}_N^1+\bar{B}_2\tilde{u}_N^2+\omega_N)|\mathcal{F}_N^1]\\ &\quad -\mathbb{E}[B'H(Ax_N+B\hat{u}_N+\bar{B}_1\tilde{u}_N^1+\bar{B}_2\tilde{u}_N^2\\ &\quad +\omega_N)|\mathcal{F}_N^c]+R_1\tilde{u}_N^1\\ &=(\bar{B}_1'H\bar{B}_1'+R_1)\tilde{u}_k^1\\ &\quad +(\bar{B}_1'HA+\bar{B}_1'H\bar{B}_2'K_N^2I_2)(\hat{x}_{N/N}-\hat{x}_{N/N}^c) \end{aligned}\tag{48}$$

$$=\Upsilon_N^1K_N^1I_1(\hat{x}_{N/N}-\hat{x}_{N/N}^c)+M_k^1(\hat{x}_{N/N}-\hat{x}_{N/N}^c).\tag{49}$$

From (48), we derive that (16) is uniquely solvable for $k=N$ if and only if Υ_N^1 is invertible. (49) holds for any $(\hat{x}_{N/N}-\hat{x}_{N/N}^c)$, hence we get $K_N^1=-(\Upsilon_k^1)^{-1}M_k^1$ and $\tilde{u}_N^1=-(\Upsilon_N^1)^{-1}\times M_N^1(x_N^1-\hat{x}_{N/N}^{1c})$.

Similarly, using (17) and (22), it follows:

$$\begin{aligned}0_{m_2\times 1} &= \mathbb{E}[\bar{B}_2'\lambda_N|\mathcal{F}_N^2]-\mathbb{E}[\bar{B}_2'\lambda_N|\mathcal{F}_N^c]+R_2\tilde{u}_N^2,\\ &= \mathbb{E}[B'H(Ax_N+B\hat{u}_N+\bar{B}_1\tilde{u}_N^1+\bar{B}_2\tilde{u}_N^2+\omega_N)|\mathcal{F}_N^2]\\ &\quad -\mathbb{E}[B'H(Ax_N+B\hat{u}_N+\bar{B}_1\tilde{u}_N^1+\bar{B}_2\tilde{u}_N^2\\ &\quad +\omega_N)|\mathcal{F}_N^c]+R_1\tilde{u}_N^1\\ &=(\bar{B}_2'H\bar{B}_2'+R_2)\tilde{u}_k^2\\ &\quad +(\bar{B}_2'HA+\bar{B}_2'H\bar{B}_1'K_N^1I_1)(\hat{\hat{x}}_{N/N}-\hat{x}_{N/N}^c) \end{aligned}\tag{50}$$

$$=\Upsilon_N^2K_N^2I_2(\hat{\hat{x}}_{N/N}-\hat{x}_{N/N}^c)+M_k^2(\hat{\hat{x}}_{N/N}-\hat{x}_{N/N}^c).\tag{51}$$

From (50), we derive that (17) is uniquely solvable for $k=N$ if and only if Υ_N^2 is invertible. (50) holds for any $(\hat{\hat{x}}_{N/N}-\hat{x}_{N/N}^c)$, hence we get $K_N^2=-(\Upsilon_k^2)^{-1}M_k^2$ and $\tilde{u}_N^2=-(\Upsilon_N^2)^{-1}\times M_N^2(x_N^2-\hat{x}_{N/N}^{2c})$.

Subsequently, the λ_{N-1} will be calculated. Using (9), we obtain that

$$\begin{aligned}\lambda_{N-1} &= \mathbb{E}[A'\lambda_N|\mathcal{F}_N^1,\mathcal{F}_N^2]+Qx_N,\\ &= \mathbb{E}[A'H(Ax_N+B\hat{u}_N+\bar{B}_1\tilde{u}_N^1+\bar{B}_2\tilde{u}_N^2\\ &\quad +\omega_N)|\mathcal{F}_N^1,\mathcal{F}_N^2]+Qx_N\\ &=(A'HA+Q+A'HBK_N)\hat{x}_{N/N}^c\\ &\quad +(A'HA+Q+A'HB\tilde{K}_N)(x_N-\hat{x}_{N/N}^c)\\ &=P_N\hat{x}_{N/N}^c+\tilde{P}_N(x_N-\hat{x}_{N/N}^c). \end{aligned}\tag{52}$$

To complete the induction, we assume the following assertions hold for $k=n+1,\ldots,N$. Namely, we assume

1) Equations (15)−(17) are uniquely solvable if and only if Υ_k and Υ_k^i are invertible;

2) λ_{k-1} is of the form (33).

Then we shall prove that the above assertions still hold for $k=n$.

Substituting (9) and (14) into (15), we have

$$\begin{aligned}0_{m\times 1} &= \mathbb{E}[B'\lambda_n|\mathcal{F}_n^c]+R\hat{u}_n,\\ &= \mathbb{E}[B'P_{n+1}\hat{x}_{n+1/n+1}^c\\ &\quad B'\tilde{P}_{n+1}(x_{n+1}-\hat{x}_{n+1/n+1}^c)|\mathcal{F}_n^c]+R\hat{u}_n,\\ &=(B'P_{n+1}B+R)\hat{u}_n+B'P_{n+1}A\hat{x}_{n/n}^c,\\ &=\Upsilon_n\hat{u}_n+M_n\hat{x}_{n/n}^c. \end{aligned}\tag{53}$$

From (53), the solvability of (15) for $k=n$ is that Υ_n is invertible, thus $\hat{u}_n$ in (31) could be verified.

Next we will calculate u_k^i. Using an argument similar to (48)−(51), it follows:

$$\begin{aligned}0_{m_1\times 1} &= \mathbb{E}[\bar{B}_1'\lambda_N|\mathcal{F}_N^1]-\mathbb{E}[\bar{B}_1'\lambda_N|\mathcal{F}_N^c]+R_1\tilde{u}_N^1\\ &=(\bar{B}_1'H\bar{B}_1'+R_1)\tilde{u}_k^1+(\bar{B}_1'HA\\ &\quad +\bar{B}_1'H\bar{B}_2'K_N^2I_2)(\hat{x}_{N/N}-\hat{x}_{N/N}^c) \end{aligned}\tag{54}$$

$$=\Upsilon_N^1K_N^1I_1(\hat{x}_{N/N}-\hat{x}_{N/N}^c)+M_k^1(\hat{x}_{N/N}-\hat{x}_{N/N}^c)\tag{55}$$

$$\begin{aligned}0_{m_2\times 1} &= \mathbb{E}[\bar{B}_2'\lambda_N|\mathcal{F}_N^2]-E[\bar{B}_2'\lambda_N|\mathcal{F}_N^c]+R_2\tilde{u}_N^2\\ &=(\bar{B}_2'H\bar{B}_2'+R_2)\tilde{u}_k^2+(\bar{B}_2'HA\\ &\quad +\bar{B}_2'H\bar{B}_1'K_N^1I_1)(\hat{\hat{x}}_{N/N}-\hat{x}_{N/N}^c) \end{aligned}\tag{56}$$

$$=\Upsilon_N^2K_N^2I_2(\hat{\hat{x}}_{N/N}-\hat{x}_{N/N}^c)+M_k^2(\hat{\hat{x}}_{N/N}-\hat{x}_{N/N}^c).\tag{57}$$

From (54) and (56), the solvability of (16) and (17) for $k=n$ is that Υ_N^i is invertible. $\tilde{u}_k^1$ and $\tilde{u}_k^2$ are obtained as the form of (55)−(57).

Further, λ_{n-1} will be given from (9),

$$\begin{aligned}\lambda_{n-1} &= \mathbb{E}[A'\lambda_n|\mathcal{F}_n^1,\mathcal{F}_n^2]+Qx_n\\ &= \mathbb{E}[A'P_{n+1}\hat{x}^c_{n+1/n+1}+A'\tilde{P}_{n+1}(x_{n+1}\\ &\quad -\hat{x}^c_{n+1/n+1})|\mathcal{F}_n^1,\mathcal{F}_n^2]+Qx_n\\ &= (A'P_{n+1}A+Q+K_n'\Upsilon_nK_n)\hat{x}^c_{n/n}\\ &\quad +(A'P_{n+1}A+Q+K_n'\Upsilon_n\tilde{K}_n)(x_n-\hat{x}^c_{n/n})\\ &= P_n\hat{x}^c_{n/n}+\tilde{P}_n(x_n-\hat{x}^c_{n/n})\end{aligned}$$

which is the same as (33). Here ends the induction.

In brief, the uniquely solvability of (15)–(17) is transformed into the invertibility of Υ_k and Υ_k^i. Then employing Lemma 2, we conclude that Problem 1 has a unique solution if and only if Υ_k and Υ_k^i are invertible for $k=N,\dots,0$. Moreover, the optimal controllers are of the form of (31) and (32), and the solution to FBSDEs (9), (10), (14)–(17), are also verified as (33).

Finally, the optimal cost functional will be calculated using (31) and (33).

From (9), it yields

$$\begin{aligned}&\mathbb{E}[x_k'\lambda_{k-1}-x_{k+1}'\lambda_k]\\ &\quad=\mathbb{E}[x_k'Qx_k+(u_k^1)'R_1u_k^1+(u_k^2)'R_2u_k^2]-\mathbb{E}[\omega_k'\lambda_k]. \qquad (58)\end{aligned}$$

Adding from $k=0$ to $k=N$ on both sides of (58), one has

$$\begin{aligned}&\mathbb{E}[x_0'\lambda_{-1}-x_{N+1}'\lambda_N]\\ &\quad=\mathbb{E}[x_0'\lambda_{-1}-x_{N+1}'Hx_{N+1}']\\ &\quad=\sum_{k=0}^N\mathbb{E}[x_k'Qx_k+(u_k^1)'R_1u_k^1+(u_k^2)'R_2u_k^2]-\sum_{k=0}^N\mathbb{E}[\omega_k'\lambda_k]. \qquad (59)\end{aligned}$$

From (59) with optimal controllers in (31) and (32), it follows:

$$\begin{aligned}J_N^* &= \mathbb{E}[x_0'\lambda_{-1}-x_{N+1}'\lambda_N]\\ &= \mathbb{E}[x_0'\lambda_{-1}]+\sum_{k=0}^N E[\omega_k'\lambda_k]\\ &= \sum_{k=0}^N\mathbb{E}[x_k'Qx_k+(u_k^1)'R_1u_k^1+(u_k^2)'R_2u_k^2]+\sum_{k=0}^N\mathbb{E}[\omega_k'\lambda_k]\\ &= \mathbb{E}[x_0'P_0\hat{x}^c_{0/0}+x_0'\tilde{P}_0(x_0-\hat{x}^c_{0/0})]\\ &\quad \sum_{k=0}^N\mathbb{E}\Big\{\omega_k'[P_{k+1}\hat{x}^c_{k+1/k+1}+\tilde{P}_{k+1}(x_{k+1}-\hat{x}^c_{k+1/k+1})]\Big\}\\ &= \mathbb{E}[x_0'P_0\hat{x}^c_{0/0}+x_0'\tilde{P}_0(x_0-\hat{x}^c_{0/0})]+\sum_{k=0}^N Tr[W_k\tilde{P}_{k+1}]. \qquad (60)\end{aligned}$$

Thus (34) is obtained. ■

APPENDIX B
PROOF OF LEMMA 3

Under Assumptions 1 and 2, suppose there exist controllers $u_k^1=\Gamma_1x_k$ and $u_k^2=\Gamma_2x_k$ with constant matrices Γ_1 and Γ_2 to make closed-loop system (35) mean-square stabilizable, the assertion that there exists a unique positive definite solution P to (38) shall be proved.

Firstly, we will prove that $P_k(N)$ is convergent.

Without additive noise, the optimal cost (34) is turned to be

$$J_N^*=\mathbb{E}[x_0'P_0(N)x_0]. \qquad (61)$$

Evidently, we have $\mathbb{E}[x_0'P_0(N)x_0]=J_N^*\le J_{N+1}^*=\mathbb{E}[x_0'P_0(N+1)x_0]$. As x_0 is arbitrary, it could be obtained that $P_0(N)$ decreases monotonically with respect to N. Then we shall show the boundedness of $P_0(N)$. Since controllers $u_k^1=\Gamma_1x_k$ and $u_k^2=\Gamma_2x_k$ could mean-square stabilize system (35), we have $\lim_{k\to\infty}\mathbb{E}(x_k'x_k)=0$. Following from [31], we have that there exists constant $f_1>0$ satisfying

$$\sum_{k=0}^\infty\mathbb{E}(x_k'x_k)\le f_1\mathbb{E}(x_0'x_0).$$

Consequently, noting from (36) that $Q,\Gamma_1'R_1\Gamma_1$ and $\Gamma_2'R_2\Gamma_2$ are all bounded, there exists constant f_2 such that

$$\begin{aligned}J &= \mathbb{E}\sum_{k=0}^\infty\Big[x_k'Qx_k+(u_k^1)'R_1u_k^1+(u_k^2)'R_2u_k^2\Big]\\ &= \mathbb{E}\sum_{k=0}^\infty\Big[x_k'Qx_k+x_k'\Gamma_1'R_1\Gamma_1x_k'+x_k'\Gamma_2'R_2\Gamma_2x_k'\Big]\\ &\le f_2[f_1\mathbb{E}(x_0'x_0)].\end{aligned}$$

Thus for any $N>0$, we could obtain from (61)

$$\mathbb{E}[x_0'P_0(N)x_0]=J_N^*\le J\le f_2[f_1\mathbb{E}(x_0'x_0)]$$

which indicates that $P_0(N)$ is bounded. Along with the monotonic decreasing of $P_0(N)$, we have $P_0(N)$ is convergent, i.e., $\lim_{N\to\infty}P_0(N)=P$.

Letting $P_{N+1}=\tilde{P}_{N+1}=0$, the variables in (28)–(30) are time invariant for N, i.e.,

$$\begin{aligned}&P_k(N)=P_{k-s}(N-s),\tilde{P}_k(N)=\tilde{P}_{k-s}(N-s)\\ &K_k(N)=K_{k-s}(N-s),\tilde{K}_k(N)=\tilde{K}_{k-s}(N-s)\\ &K_k^i(N)=K_{k-s}^i(N-s),\Upsilon_k(N)=\Upsilon_{k-s}(N-s)\\ &\Upsilon_k^i(N)=\Upsilon_{k-s}^i(N-s),M_k(N)=M_{k-s}(N-s)\\ &M_k^i(N)=M_{k-s}^i(N-s),s\le k\le N,0\le s\le N.\end{aligned}$$

Hence, it follows $\lim_{N\to\infty}P_k(N)=\lim_{N\to\infty}P_0(N-k)=P$. Thus, $P_k(N)$ is convergent.

Secondly, we will show that there is $N_0>0$ satisfying $P_0(N)>0$ for any $N>N_0$. If not, for any $N\ge0$, there exists $x\ne0$ satisfying $\mathbb{E}[x_0'P_0(N)x_0]=0$. Assume $x_0=x$, the optimal cost functional (34) satisfies

$$\begin{aligned}J_N^* &= \mathbb{E}\sum_{k=0}^\infty\Big[x_k^{*\prime}Qx_k^*+(u_k^{1*})'R_1u_k^{1*}+(u_k^{2*})'R_2u_k^{2*}\Big]\\ &= \mathbb{E}[x'\tilde{P}_0(N)x]=0\end{aligned}$$

where x_k^* is the optimal state trajectory, u_k^{1*} and u_k^{2*} represent the optimal controllers. As stated in Assumption 1, $R_1>0$, $R_2>0$ and $Q\ge0$, we have

$$u_k^{1*}=0,\quad u_k^{2*}=0,\quad Q^{1/2}x_k^*=0,\quad 0\le k\le N,N\ge0.$$

Assumption 2 implies $x_0 = x = 0$, which contradicts $x = 0$. Namely, there exists N_0 satisfying $P_0(N) > 0$ for $N \geq N_0$. Thus, $P = \lim_{N\to\infty} P_0(N) > 0$ has been shown.

Finally, the uniqueness of the solution to (28) is shown as follows. Assume that there exists another solution P^s to (28) satisfying $P^s > 0$. Recall (61), the optimal cost functional is as $J^* = \mathbb{E}[x_0' P^s x_0] = \mathbb{E}[x_0' P x_0]$. Since x_0 is arbitrary, it follows that $P^s = P$. ■

Appendix C
Proof of Theorem 2

Sufficient: With Assumptions 1 and 2, if there exists a positive definite solution P to (38) we will demonstrate (2) is mean-square bounded. In virtue of (2), (43) and (44), one has

$$\begin{aligned}\mathbb{E}[x_{k+1}' x_{k+1}] &= \mathbb{E}\Big\{[Ax_k + BKx_{k/k}^c + \bar{B}_1 K^1 I_1 (x_k - \hat{x}_{k/k}^c) \\ &\quad + \bar{B}_2 K^2 I_2 (x_k - \hat{x}_{k/k}^c) + \omega_k]' \\ &\quad \times [Ax_k + BK\hat{x}_{k/k}^c + \bar{B}_1 K^1 I_1 (x_k - \hat{x}_{k/k}^c) \\ &\quad + \bar{B}_2 K^2 I_2 (x_k - \hat{x}_{k/k}^c) + \omega_k]\Big\} \\ &= \mathbb{E}[x_k'(A+BK)'(A+BK)x_k] + \Big\{Tr[W(A \\ &\quad + \bar{B}_1 K^1 I_1 + \bar{B}_2 K^2 I_2)'(A + \bar{B}_1 K^1 I_1 + \bar{B}_2 K^2 I_2)] \\ &\quad - Tr[W(A+BK)'(A+BK)] + W\Big\}. \end{aligned} \tag{62}$$

It is evident that the second term is constant, hence $\lim_{k\to\infty}\mathbb{E}(x_k' x_k)$ is mean-square bounded iff system (63) is mean-square stable.

$$\theta_{k+1} = (A+BK)\theta_k \tag{63}$$

with initial value $\theta_0 = x_0$.

Now we will demonstrate the mean-square stability of system (63). To this end, the Lyapunov function candidate F_k is defined

$$F_k = \mathbb{E}[\theta_k' P \theta_k].$$

In virtue of (38)−(40), one has

$$\begin{aligned}F_{k+1} - F_k &= \mathbb{E}\Big\{\theta_k'[(A+BK)'P(A+BK)]\theta_k - \theta_k' P\theta_k\Big\} \\ &= -\mathbb{E}[\theta_k'(K'RK+Q)\theta_k] \leq 0\end{aligned}$$

which indicates that F_k monotonically decreases with respect to k. As $P > 0$, $F_k \geq 0$ is bounded below, i.e, F_k is convergent. Selecting an integer l satisfying $l > 0$, by adding from $k = l$ to $k = l+N$ on both sides of above equation and letting $l \to \infty$, one obtains

$$\lim_{l\to\infty} \sum_{k=l}^{k=l+N} \mathbb{E}[\theta_k'(K'RK+Q)\theta_k] = \lim_{l\to\infty} F_l - F_{N+l+1} = 0.$$

Along with Assumption 1, we have $\lim_{l\to\infty}\mathbb{E}[\theta_l'\theta_l] = 0$. Hence, we obtain that $\lim_{k\to\infty}\mathbb{E}[x_k' x_k]$ is bounded, that is, (43) and (44) make system (2) mean-square bounded. Finally, we shall prove that (43) and (44) minimize the performance (42). Define

$$\tilde{V}(k) = \mathbb{E}[x_k' P \hat{x}_{k/k}^c + x_k' \tilde{P}(x_k - \hat{x}_{k/k}^c)]. \tag{64}$$

Noting that when $k \to \infty$, $\tilde{V}(k)$ is bounded owing to the boundedness of $\lim_{k\to\infty}\mathbb{E}[x_k' x_k]$ and $\lim_{k\to\infty}\mathbb{E}[\tilde{x}_{k/k}^{c'} x_{k/k}^c]$.

It follows:

$$\begin{aligned}&\tilde{V}(k+1) - \tilde{V}(k) \\ &\quad = \mathbb{E}\Big\{x_k' Q x_k + \hat{u}_k' R \hat{u}_k + (\tilde{u}_k^1)' R_1 \tilde{u}_k^1 + (\tilde{u}_k^2)' R_2 \tilde{u}_k^2 \\ &\qquad - (\hat{u}_k + K\hat{x}_{k/k}^c)'\Upsilon(\hat{u}_k + K\hat{x}_{k/k}^c) \\ &\qquad - [\tilde{u}_k^1 + K^1(x_k^1 - \hat{x}_{k/k}^{1c})]'\Upsilon^1[\tilde{u}_k^1 + K^1(x_k^1 - \hat{x}_{k/k}^{1c})] \\ &\qquad - [\tilde{u}_k^2 + K^2(x_k^2 - \hat{x}_{k/k}^{2c})]'\Upsilon^2[\tilde{u}_k^2 + K^2(x_k^2 - \hat{x}_{k/k}^{2c})]\Big\} \\ &\qquad - tr\tilde{P}W. \end{aligned} \tag{65}$$

Via taking summation from $k = 0$ to $k = N$ on both sides of (65) and letting $N \to \infty$, we have

$$\begin{aligned}\tilde{J} &= \lim_{N\to\infty}\frac{1}{N}\Big\{\tilde{V}(0) - \tilde{V}(N+1) + \sum_{k=0}^{N} tr\tilde{P}W \\ &\quad + (\hat{u}_k + K\hat{x}_{k/k}^c)'\Upsilon(\hat{u}_k + K\hat{x}_{k/k}^c) \\ &\quad + [\tilde{u}_k^1 + K^1(x_k^1 - \hat{x}_{k/k}^{1c})]'\Upsilon^1[\tilde{u}_k^1 + K^1(x_k^1 - \hat{x}_{k/k}^{1c})] \\ &\quad + [\tilde{u}_k^2 + K^2(x_k^2 - \hat{x}_{k/k}^{2c})]'\Upsilon^2[\tilde{u}_k^2 + K^2(x_k^2 - \hat{x}_{k/k}^{2c})]\Big\} \\ &= tr\tilde{P}W + (\hat{u}_k + K\hat{x}_{k/k}^c)'\Upsilon(\hat{u}_k + K\hat{x}_{k/k}^c) \\ &\quad + [\tilde{u}_k^1 + K^1(x_k^1 - \hat{x}_{k/k}^{1c})]'\Upsilon^1[\tilde{u}_k^1 + K^1(x_k^1 - \hat{x}_{k/k}^{1c})] \\ &\quad + [\tilde{u}_k^2 + K^2(x_k^2 - \hat{x}_{k/k}^{2c})]'\Upsilon^2[\tilde{u}_k^2 + K^2(x_k^2 - \hat{x}_{k/k}^{2c})].\end{aligned}$$

Since the Υ and Υ^i are positive definite, the optimal controllers to make above equation minimize are as (43) and (44), and the associated cost functional is the same as (42).

Necessity: Suppose that (2) is mean-square bounded, we will prove there exists a unique positive definite solution P to (38).

From the perspective of the complete response of the linear time invariant system, we could split system (2) into two parts, i.e., $x_k = s_k + y_k$, where s_k and y_k are as

$$s_{k+1} = As_k + BKs_k = (A+BK)s_k \tag{66}$$

$$\begin{aligned}y_{k+1} &= Ay_k + BKy_{k/k}^c + \bar{B}_1 K^1 I_1(y_k - \hat{y}_{k/k}^c) \\ &\quad + \bar{B}_2 K^2 I_2 (y_k - \hat{y}_{k/k}^c) + \omega_k \\ &= (A + \bar{B}_1 K^1 I_1 + \bar{B}_2 K^2 I_2)(y_k - \hat{y}_{k/k}^c) \\ &\quad + (A+BK)\hat{y}_{k/k}^c + \omega_k\end{aligned} \tag{67}$$

with initial values $s_0 = x_0$ and $y_0 = 0$, respectively. It is easy to get that s_k is orthogonal to y_k. Therefore, the mean-square boundedness of the system (2) is equivalent to the mean-square stabilization of the system (35). Together with Lemma 3, we derive that the algebraic equation (38) admits a unique solution $P > 0$ from the mean-square boundedness of the system (2). ■

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