A Mean-Field Game for a Forward-Backward Stochastic System With Partial Observation and Common Noise

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***Abstract*—This paper considers a linear-quadratic (LQ) mean-field game governed by a forward-backward stochastic system with partial observation and common noise, where a coupling structure enters state equations, cost functionals and observation equations. Firstly, to reduce the complexity of solving the mean-field game, a limiting control problem is introduced. By virtue of the decomposition approach, an admissible control set is proposed. Applying a filter technique and dimensional-expansion technique, a decentralized control strategy and a consistency condition system are derived, and the related solvability is also addressed. Secondly, we discuss an approximate Nash equilibrium property of the decentralized control strategy. Finally, we work out a financial problem with some numerical simulations.**

***Index Terms*—Decentralized control strategy, ϵ-Nash equilibrium, forward-backward stochastic system, mean-field game, partial observation.**

I. Introduction

THE stochastic differential game problem within large-population system has attracted increasing attentions from various areas. A large-population system is distinguished with numerous agents, where the states or the cost functionals are coupled via a coupling structure. In view of the highly complicated coupling term, it is not feasible or effective to study the exact Nash equilibrium relying on all agents' exact states. Alternatively, an available and effective idea is to design an approximate Nash equilibrium only based on each individual's information. The mean-field method independently proposed by [1] and [2] provides an effective technique to solve the large-population game problem. With the mean-field method, a complex mean-field game problem can be converted into a series of classical control problems; as a result, the curse of dimensionality is overcome and computational complexity is reduced. Reference [2] studied a mean-field game, where the dynamic systems are asymmetric, and the analysis for the ϵ-Nash equilibrium was given. Reference [3] established some results showing the unique solvability of stochastic mean-field games. Some recent works can be found in: [4], [5] for mean-field games with the Stackelberg structure, [6]–[8] for game models with the linear-quadratic (LQ) framework, [9]−[11] for game models with jumps, [12]–[14] for game models with state or control constraints, [15], [16] for game problems with social optimality, [17], [18] for backward stochastic mean-field games, [19] for Nash equilibriums of game problems, and [20]–[24] for stochastic mean-field control problems.

We point out that in the mean-field game, there are numerous agents with complicated interactions, and the state-average is approximated by a frozen term; thus, the optimal strategy can be computed off-line. However, in the mean-field control problem, the mathematical expectation of state is a part of the state, which is influenced by the control process. As a result, the strategies derived from a mean-field game and mean-field control are called ϵ-Nash equilibrium and optimal control, respectively.

Note that the mentioned works above focus on the mean-field game problem governed by a stochastic differential equation (SDE) or backward SDE (BSDE). However, we often encounter such a scenario in reality. For example, the wealth level and education investment level satisfy an SDE and a BSDE (see Section V), respectively. It is well known that forward-backward SDE (FBSDE) is a well-defined dynamic system, which provides a tool to characterize and analyse the problem above. A coupled (fully or partially coupled) FBSDE involves the feature of both SDE and BSDE, and it is a combination of them in structure, which may degenerate to either one if the other vanishes. Furthermore, FBSDE is applied to illustrate many behaviors of economics, finance and other fields, such as large scale investors, recursive utility, etc.

In some existing mean-field game literature, the authors assume that all agents can access the full information. However, in reality it is unrealistic for agents to do so. Due to the

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dynamic system, the agents need to make decisions based on real-time information. For example, in an integrated energy system [25] affected by weather, temperature and humidity, it is difficult to guarantee the accuracy of measured data. Thus, the study of the control and game problem with incomplete information has important practical value; one can refer to [26]–[30] for more information.

Inspired by the content above, we study a mean-field game governed by FBSDE with partial observation and common noise, which plays a vital role in both theoretical research and practical application. Although there are some existing works on a mean-field game with an incomplete information scheme, this work presents many advancements. In order to avoid confusion, we list the differences and contributions of this work item by item.

1) The large-population system is more general in the paper. In this work, the dynamic system is more general than that of [31], [32], where the diffusion term $\mathcal{Z}_i(\cdot)$ (see (1) below) enters the drift term of BSDE. It is well known that the solution of BSDE is a pair, which has two parts, the backward state and the diffusion term. The analysis and processing of the diffusion term is challenging, and thus it is usually absent from the drift term in many research works. As a consequence, the resulting Hamiltonian system involves fully coupled conditional mean-field FBSDEs, and it is extremely challenging to solve. In order to overcome this difficulty, employing convex analysis theory, we prove the unique solvability of Problem II and the optimality system. Moreover, [33] studied a mean-field game driven by BSDE with partial information, and derived an ϵ-Nash equilibrium via the stochastic maximum principle and optimal filtering.

2) Compared with [33], the state of this paper is governed by an FBSDE with partial observation instead of a BSDE with partial information. The BSDE studied in [33] is usually used to describe some financial problems with prescribed terminal conditions, which can not characterize the recursive utility optimization problems, principal-agent problems in continuous time, etc. FBSDE provides an effective tool to investigate the above problems. Employing the optimal filter technique, decomposition technique and dimensional-expansion technique, we obtain a feedback form of the decentralized control strategy relying on the optimal filter of the forward state instead of backward state given in [33].

3) Different from [31], employing a dimensional-expansion technique and introducing two ordinary differential equations (ODEs), we obtain the solvability of the consistency condition. Since the initial and terminal conditions of consistency condition (20)–(25), (27), (28) and (34) below are mixed, their solvability is extremely difficult to derive. By virtue of the dimensional-expansion technique and two ODEs, the solvability of the consistency condition is derived. However, the solvability of the consistency condition in [31] is discussed by a contraction mapping technique with a strong assumption, and it holds in some special cases. Thus, our results obtained are more universal.

4) Unlike [27], by virtue of Riccati equation approach, we obtain a feedback form of the decentralized control strategy. Introducing eight ODEs, we decouple the complicated Hamiltonian system, and propose a feedback form of the decentralized control strategy. However, [27] gave an open-loop form of the optimal control via the stochastic maximum principle.

5) Last but not least, this work significantly improves the description and resolution of the mean-field game with partial observation. In addition, this work compensates for the deficiencies and flaws, and the results obtained are more elaborate and rigorous than some existing works. See [32] for more results regarding a mean-field game with partial observation.

The rest of this paper is structured as follows. We formulate a mean-field game problem in Section II. We investigate a limiting control problem associated with an individual agent, providing a decentralized control strategy via the consistency condition and optimal filter in Section III. Section IV is dedicated to the ϵ-Nash equilibrium property of a decentralized control strategy. We give a financial example and provide some remarks in Sections V and VI, respectively.

II. Problem Formulation and Preliminary

Let $\mathcal{N}=\{1,\ldots,N\}$. Let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{0\le t\le T},\mathbb{P})$ denote a complete, filtered probability space, where $\{W(\cdot),W_i(\cdot):i\in\mathcal{N}\}$ is a $(N+1)$-dimensional Brownian motion on it. Let $\mathbb{E}$ be the expectation with respect to $\mathbb{P}$, and $\mathcal{F}_t^W=\sigma\{W(r):0\le r\le t\}$. Let $\mathbb{S}$ be an Euclidean space with norm $|\cdot|$. Let A^τ be the transpose of matrix A. For any stochastic process $x(\cdot)$, we call $x(\cdot)$ is $\mathcal{L}^2$-bounded, if $\mathbb{E}\int_0^T|x(t)|^2dt$ is bounded. For convenience, we introduce two spaces as follows.

$\mathcal{L}^2_{\mathcal{G}}(0,T;\mathbb{S})=\{\xi:[0,T]\times\Omega\to\mathbb{S}|\xi(\cdot)$ is the $\mathcal{G}_t$-adapted stochastic process satisfying $\mathbb{E}\int_0^T|\xi(t)|^2dt<+\infty\}$;

$\mathcal{L}^\infty(0,T;\mathbb{S})=\{\xi:[0,T]\to\mathbb{S}|\xi(\cdot)$ is the uniformly bounded stochastic process$\}$.

In this work, we investigate a mean-field game involving N agents, where the dynamics system of agent $\mathcal{A}_i$ satisfies an FBSDE

$$\begin{cases}d\mathcal{X}_i(t)=\Big[A(t)\mathcal{X}_i(t)+B(t)u_i(t)+G(t)\mathcal{X}^{(N)}(t)\\ \qquad\qquad+\bar{G}(t)\Big]dt+\sigma(t)dW_i(t)+\bar{\sigma}(t)dW(t)\\ -d\mathcal{Y}_i(t)=\Big[C_1(t)\mathcal{Y}_i(t)+C_2(t)\mathcal{Z}_i(t)+C_3(t)\mathcal{X}_i(t)\\ \qquad\qquad+D(t)u_i(t)+F(t)\mathcal{X}^{(N)}(t)+\bar{F}(t)\Big]dt\\ \qquad\qquad-\sum_{j=1}^N\mathcal{Z}_{ij}(t)dW_j(t)-\mathcal{Z}_i(t)dW(t)\\ \mathcal{X}_i(0)=a_{i0},\quad \mathcal{Y}_i(T)=H\mathcal{X}_i(T),\quad i\in\mathcal{N}\end{cases}\tag{1}$$

where the coefficients $A(\cdot)$, $B(\cdot)$, $G(\cdot)$, $\bar{G}(\cdot)$, $\sigma(\cdot)$, $\bar{\sigma}(\cdot)$, $C_1(\cdot)$, $C_2(\cdot)$, $C_3(\cdot)$, $D(\cdot)$, $F(\cdot)$, $\bar{F}(\cdot)$ are deterministic functions on $[0,T]$, a_{i0} is a random variable, H is a constant; $u_i(\cdot)$, $\mathcal{X}_i(\cdot)$ and $(\mathcal{Y}_i(\cdot),\mathcal{Z}_{i1}(\cdot),\ldots,\mathcal{Z}_{iN}(\cdot),\mathcal{Z}_i(\cdot))$ represent the control strategy, the forward and backward components of state of agent $\mathcal{A}_i$, respectively; $\mathcal{X}^{(N)}(\cdot)=\frac{1}{N}\sum_{i=1}^N\mathcal{X}_i(\cdot)$; $W_i(\cdot)$ and $W(\cdot)$ stand for the individual and common random noises, respectively. Here, the common noise $W(\cdot)$ can be interpreted as some global uncertainties, such as the macro-economic scenario, tax policy and interest rate, which influences all agents' states in a large-population system. Different from the common noise, the individual noise $W_i(\cdot)$ can be regarded as some local uncertainties, which only influences agent $\mathcal{A}_i$. The full information is denoted by $\mathcal{F}_t = \sigma\{W(r), W_i(r), a_{i0} : 0 \le r \le t, i \in \mathcal{N}\}$. Let $\mathcal{U}_i = \{u_i(\cdot)|u_i(\cdot) \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R})\}$, $i \in \mathcal{N}$.

The observation process and cost functional are

$$\begin{cases} dY_i(t) = \Big[f(t)\mathcal{X}_i(t) + g(t)\mathcal{X}^{(N)}(t) + h(t)\Big]dt + dW_i(t) \\ Y_i(0) = 0, \quad i \in \mathcal{N} \end{cases} \tag{2}$$

and

$$\begin{aligned} \mathcal{J}_i(u_i(\cdot), u_{-i}(\cdot)) = \frac{1}{2}\mathbb{E}\Big\{ \int_0^T \Big[Q_1(t)\Big(\mathcal{X}_i(t) - \mathcal{X}^{(N)}(t)\Big)^2 \\ + 2\bar{Q}_1(t)\Big(\mathcal{X}_i(t) - \mathcal{X}^{(N)}(t)\Big) + R(t)u_i^2(t) \\ + 2r(t)u_i(t)\Big]dt + K_1\mathcal{X}_i^2(T) + 2\bar{K}_1\mathcal{X}_i(T) \\ + K_2\mathcal{Y}_i^2(0) + 2\bar{K}_2\mathcal{Y}_i(0)\Big\} \end{aligned} \tag{3}$$

where $u_{-i}(\cdot) = (u_1(\cdot), \ldots, u_{i-1}(\cdot), u_{i+1}(\cdot), \ldots, u_N(\cdot))$, the coefficients $f(\cdot), g(\cdot), h(\cdot), Q_1(\cdot), \bar{Q}_1(\cdot), R(\cdot), r(\cdot)$ are deterministic functions on $[0,T]$, and $K_1, \bar{K}_1, K_2, \bar{K}_2$ are constants.

Assumption 1: i) The coefficients of (1)–(3) satisfy

$$\begin{aligned} &A(\cdot), B(\cdot), G(\cdot), \bar{G}(\cdot), \sigma(\cdot), \bar{\sigma}(\cdot) \in \mathcal{L}^\infty(0,T;\mathbb{R}) \\ &C_1(\cdot), C_2(\cdot), C_3(\cdot), D(\cdot), F(\cdot), \bar{F}(\cdot) \in \mathcal{L}^\infty(0,T;\mathbb{R}) \\ &f(\cdot), g(\cdot), h(\cdot) \in \mathcal{L}^\infty(0,T;\mathbb{R}), \quad H \in \mathbb{R} \\ &Q_1(\cdot), \bar{Q}_1(\cdot), R(\cdot), r(\cdot) \in \mathcal{L}^\infty(0,T;\mathbb{R}) \\ &Q_1(\cdot), R(\cdot) > 0, \quad K_1, K_2 > 0, \quad \bar{K}_1, \bar{K}_2 \in \mathbb{R}. \end{aligned}$$

ii) $\{a_{i0}\}_{i=1}^N$ are mutually independent and have the same distribution with $\mathbb{E}|a_{i0}|^2 < +\infty$, mean a_0 and variance $\sigma_0 > 0$, independent of $\{W(\cdot), W_i(\cdot), i \in \mathcal{N}\}$.

Remark 1: Note that the partially-coupled forward-backward stochastic system (1) and observation process (2) rely on control via $\mathcal{X}_i(\cdot)$ and $\mathcal{X}^{(N)}(\cdot)$, which makes the large-population game problem more challenging and has more important theoretical significance, compared with [5], [8], [13], [15], [18], [32]. Moreover, due to the fact that $\mathcal{X}^{(N)}(\cdot)$ is $\mathcal{F}_t$-adapted, the terms $\sum_{j=1}^N \mathcal{Z}_{ij}(\cdot)dW_j(\cdot)$ and $\mathcal{Z}_i(\cdot)dW(\cdot)$ are introduced in the second equation of (1) to ensure the adaptiveness of $\mathcal{Y}_i(\cdot)$. However, $\mathcal{Z}_{ij}(\cdot)$ is not introduced in the drift term of the second equation of (1). Otherwise, it will be extremely difficult to design a decentralized control strategy. Hereafter, we will drop the time variable t for simplicity.

Lemma 1: Under Assumption 1, for any $u_i \in \mathcal{U}_i$ $(i \in \mathcal{N})$, (1) and (2) admit unique solutions $(\mathcal{X}_i, \mathcal{Y}_i, \mathcal{Z}_{i1}, \ldots, \mathcal{Z}_{iN}, \mathcal{Z}_i) \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^{N+3})$ and $Y_i \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R})$, respectively.

Proof: See Appendix A. ■

Now we state an FBSDE mean-field game problem.

Problem I: Seek $\tilde{u}(\cdot) = (\tilde{u}_1(\cdot), \ldots, \tilde{u}_N(\cdot))$ such that

$$\mathcal{J}_i(\tilde{u}_i(\cdot), \tilde{u}_{-i}(\cdot)) = \inf_{u_i(\cdot) \in \bar{\mathcal{U}}_i} \mathcal{J}_i(u_i(\cdot), \tilde{u}_{-i}(\cdot))$$

where $\tilde{u}_{-i} = (\tilde{u}_1, \ldots, \tilde{u}_{i-1}, \tilde{u}_{i+1}, \ldots, \tilde{u}_N)$, and $\bar{\mathcal{U}}_i$ $(i \in \mathcal{N})$ is given in Definition 1 below.

In this paper, our aim is to seek an ϵ-Nash equilibrium of game Problem I, whose main process is addressed as follows. Employing the mean-field method, we convert the game Problem I into a limiting control Problem II. Decoupling the optimality system, we propose a decentralized control strategy via the consistency condition, whose approximate Nash equilibrium property is also verified with FBSDE theory.

III. A Limiting Control Problem

This section aims to investigate a limiting control problem associated with Problem I. Due to the common noise W, we employ an $\mathcal{F}_t^W$-adapted and $\mathcal{L}^2$-bounded stochastic process x_0 to approximate $\mathcal{X}^{(N)}$ as $N \to +\infty$.

Introduce a limiting state equation

$$\begin{cases} dx_i = \Big(Ax_i + Bu_i + Gx_0 + \bar{G}\Big)dt + \sigma dW_i + \bar{\sigma}dW \\ -dy_i = \Big(C_1y_i + C_2z_i + C_3x_i + Du_i + Fx_0 + \bar{F}\Big)dt \\ \qquad - z_{ii}dW_i - z_idW \\ x_i(0) = a_{i0}, \quad y_i(T) = Hx_i(T), \quad i \in \mathcal{N} \end{cases} \tag{4}$$

a limiting observation process and a limiting cost functional

$$\begin{cases} d\bar{Y}_i = (fx_i + gx_0 + h)dt + dW_i \\ \bar{Y}_i(0) = 0 \end{cases} \tag{5}$$

$$\begin{aligned} J_i(u_i(\cdot)) = \frac{1}{2}\mathbb{E}\Big\{ \int_0^T \Big[Q_1(x_i - x_0)^2 + 2\bar{Q}_1(x_i - x_0) \\ + Ru_i^2 + 2ru_i\Big]dt + K_1x_i^2(T) + 2\bar{K}_1x_i(T) \\ + K_2y_i^2(0) + 2\bar{K}_2y_i(0)\Big\}. \end{aligned}$$

As for the stochastic control problem with observation process, it is natural to select the strategy u_i based on observation process $\bar{Y}_i$, where $\bar{Y}_i$ relies on u_i. Then, the classical variational approach is unavailable due to the circular dependence between u_i and $\bar{Y}_i$. In order to overcome this obstacle, employing the decomposition technique, we split the state equation and observation process into

$$(x_i, y_i, z_{ii}, z_i) = \Big(x_i^0, y_i^0, z_{ii}^0, z_i^0\Big) + \Big(x_i^1, y_i^1, z_{ii}^1, z_i^1\Big)$$

and

$$\bar{Y}_i = \bar{Y}_i^0 + \bar{Y}_i^1$$

where $(x_i^0, y_i^0, z_{ii}^0, z_i^0)$ and $\bar{Y}_i^0$ are independent of u_i $(i \in \mathcal{N})$.

Define the processes $(x_i^0, y_i^0, z_{ii}^0, z_i^0)$ and $\bar{Y}_i^0$ by

$$\begin{cases} dx_i^0 = Ax_i^0dt + \sigma dW_i \\ -dy_i^0 = \Big(C_1y_i^0 + C_2z_i^0 + C_3x_i^0\Big)dt - z_{ii}^0dW_i - z_i^0dW \\ x_i^0(0) = a_{i0}, \quad y_i^0(T) = Hx_i^0(T), \quad i \in \mathcal{N} \end{cases} \tag{6}$$

and

$$\begin{cases} d\bar{Y}_i^0 = fx_i^0dt + dW_i \\ \bar{Y}_i^0(0) = 0. \end{cases} \tag{7}$$

Let $u_i \in \mathcal{L}^2_{\mathcal{F}^{W_i,W}}(0,T;\mathbb{R})$ $(i \in \mathcal{N})$ be a control process. Further, we define $(x_i^1, y_i^1, z_{ii}^1, z_i^1)$ and $\bar{Y}_i^1$ by

$$\begin{cases} dx_i^1 = \left(Ax_i^1 + Bu_i + Gx_0 + \bar{G}\right)dt + \bar{\sigma}dW \\ -dy_i^1 = \left(C_1y_i^1 + C_2z_i^1 + C_3x_i^1 + Du_i + Fx_0 + \bar{F}\right)dt \\ \qquad -z_{ii}^1 dW_i - z_i^1 dW \\ x_i^1(0) = 0, \quad y_i^1(T) = Hx_i^1(T), \quad i \in \mathcal{N} \end{cases} \tag{8}$$

and

$$\begin{cases} d\bar{Y}_i^1 = (fx_i^1 + gx_0 + h)dt \\ \bar{Y}_i^1(0) = 0. \end{cases} \tag{9}$$

It is easy to determine that (6)–(9) are uniquely solvable. Introduce

$$\begin{aligned} x_i &= x_i^0 + x_i^1,\ y_i = y_i^0 + y_i^1,\ z_{ii} = z_{ii}^0 + z_{ii}^1 \\ z_i &= z_i^0 + z_i^1,\ \bar{Y}_i = \bar{Y}_i^0 + \bar{Y}_i^1. \end{aligned} \tag{10}$$

Itô's formula and (6)–(10) imply that (x_i, y_i, z_{ii}, z_i) and $\bar{Y}_i$ are the unique solutions of (4) and (5).

Let

$$\begin{aligned} \mathcal{F}_t^{\bar{Y}_i^0, W} &= \sigma\left\{\bar{Y}_i^0(r), W(r) : 0 \le r \le t\right\} \\ \mathcal{F}_t^{\bar{Y}_i, W} &= \sigma\left\{\bar{Y}_i(r), W(r) : 0 \le r \le t\right\}. \end{aligned}$$

$$\bar{\mathcal{U}}_i^0 = \Big\{u_i(\cdot) |\ u_i(\cdot) \text{ is } \mathcal{F}_t^{\bar{Y}_i^0, W}\text{-adapted, and } \mathbb{E}\int_0^T |u_i(t)|^2 dt < +\infty\Big\}, \quad i \in \mathcal{N}.$$

Then, we define $u_i \in \mathcal{F}_t^{\bar{Y}_i, W}$ as the admissible control. Note that the limiting process x_0 is $\mathcal{F}_t^W$-adapted, which results in the presence of W in the filtration $\mathcal{F}_t^{\bar{Y}_i, W}$.

Definition 1: A control u_i ($i \in \mathcal{N}$) is called admissible for agent $\mathcal{A}_i$, if $u_i \in \bar{\mathcal{U}}_i^0$ is $\mathcal{F}_t^{\bar{Y}_i, W}$-adapted. We denote by $\bar{\mathcal{U}}_i$ the admissible control set.

Lemma 2: For any $u_i \in \bar{\mathcal{U}}_i$ ($i \in \mathcal{N}$), $\mathcal{F}_t^{\bar{Y}_i, W} = \mathcal{F}_t^{\bar{Y}_i^0, W}$.

Proof: For any $u_i \in \bar{\mathcal{U}}_i$, since u_i is $\mathcal{F}_t^{\bar{Y}_i^0, W}$-adapted, then it follows from (8) that x_i^1 is $\mathcal{F}_t^{\bar{Y}_i^0, W}$-adapted, so is $\bar{Y}_i^1$. Then, $\bar{Y}_i = \bar{Y}_i^0 + \bar{Y}_i^1$ is $\mathcal{F}_t^{\bar{Y}_i^0, W}$-adapted, i.e., $\mathcal{F}_t^{\bar{Y}_i, W} \subseteq \mathcal{F}_t^{\bar{Y}_i^0, W}$. In a similar way, we obtain $\mathcal{F}_t^{\bar{Y}_i^0, W} \subseteq \mathcal{F}_t^{\bar{Y}_i, W}$. ■

We address a limiting control problem of Problem I.

Problem II: For agent $\mathcal{A}_i$ ($i \in \mathcal{N}$), seek $u_i^* \in \bar{\mathcal{U}}_i$ such that

$$J_i(u_i^*) = \inf_{u_i \in \bar{\mathcal{U}}_i} J_i(u_i). \tag{11}$$

Then u_i^* is an optimal (decentralized) control strategy of Problem II, $(x_i^*, y_i^*, z_{ii}^*, z_i^*)$ and $\bar{Y}_i^*$ denote the state and observation associated with u_i^*.

Note that since $\bar{\mathcal{U}}_i$ ($i \in \mathcal{N}$) depends on u_i via $\bar{Y}_i$, the classical variational approach is not proper for investigating Problem II. It follows from Definition 1 that $\bar{\mathcal{U}}_i \subseteq \bar{\mathcal{U}}_i^0$, then $\inf_{u_i' \in \bar{\mathcal{U}}_i} J_i(u_i') \ge \inf_{u_i \in \bar{\mathcal{U}}_i^0} J_i(u_i)$. On the other hand, similar to Lemma 2.3 in [34], it holds $\inf_{u_i' \in \bar{\mathcal{U}}_i} J_i(u_i') \le \inf_{u_i \in \bar{\mathcal{U}}_i^0} J_i(u_i)$. Then,

$$\inf_{u_i' \in \bar{\mathcal{U}}_i} J_i(u_i') = \inf_{u_i \in \bar{\mathcal{U}}_i^0} J_i(u_i).$$

Based on Lemma 1, we can investigate the optimality of $J_i(u_i)$ on $\bar{\mathcal{U}}_i^0$. Moreover, since $\bar{\mathcal{U}}_i^0$ is independent of control, employing the classical variational method, we establish stationarity condition (16) and Hamiltonian systems (17) and (18) in Lemma 3 below.

Lemma 1 implies

$$\hat{h}_i(t) = \mathbb{E}\Big[h_i(t)\Big|\mathcal{F}_t^{\bar{Y}_i, W}\Big] = \mathbb{E}\Big[h_i(t)\Big|\mathcal{F}_t^{\bar{Y}_i^0, W}\Big].$$

In what follows, we give the filtering equation of the first equation of (4) with respect to $\mathcal{F}_t^{\bar{Y}_i, W}$. Set

$$\begin{aligned} &\tilde{f} = (f, 0)^\tau,\ \tilde{g} = (g, 0)^\tau,\ \tilde{h} = (h, 0)^\tau \\ &\widetilde{Y}_i = (\bar{Y}_i, W)^\tau,\ \overline{W}_i = (W_i, W)^\tau,\ \hat{\sigma} = (\sigma, \bar{\sigma}). \end{aligned}$$

Then the first equation of (4) and (5) are written as

$$\begin{cases} dx_i = \left(Ax_i + Bu_i + Gx_0 + \bar{G}\right)dt + \hat{\sigma}d\overline{W}_i \\ x_i(0) = a_{i0}, \quad i \in \mathcal{N} \end{cases} \tag{12}$$

$$\begin{cases} d\widetilde{Y}_i = (\tilde{f}x_i + \tilde{g}x_0 + \tilde{h})dt + d\overline{W}_i \\ \widetilde{Y}_i(0) = (0, 0)^\tau. \end{cases} \tag{13}$$

Applying Theorem 2.1 in [35], the optimal filtering $\hat{x}_i$ of (12) with respect to $\widetilde{Y}_i$ yields

$$\begin{cases} d\hat{x}_i = \left(A\hat{x}_i + Bu_i + Gx_0 + \bar{G}\right)dt + (\sigma + fP) \\ \qquad \times [d\bar{Y}_i - (f\hat{x}_i + gx_0 + h)dt] + \bar{\sigma}dW \\ \hat{x}_i(0) = a_0 \end{cases} \tag{14}$$

where P is given by Bernoulli equation

$$\begin{cases} \dot{P} + 2(\sigma f - A)P + f^2P^2 = 0 \\ P(0) = \sigma_0 \end{cases} \tag{15}$$

which admits a unique solution.

To investigate Problem II, we present the following lemma first, which tells us that Problem II is uniquely solvable with Assumption 1.

Lemma 2: Let Assumption 1 hold. Then, Problem II has a unique decentralized control strategy.

Proof: See Appendix B. ■

Employing the classical variational method, we get

Lemma 3: Under Assumption 1, we have

$$u_i^* = R^{-1}\left(D\hat{p}_i^* - B\hat{q}_i^* - r\right),\ i \in \mathcal{N} \tag{16}$$

where $(x_i^*, y_i^*, z_{ii}^*, z_i^*)$ and $(p_i^*, q_i^*, k_{ii}^*, k_i^*)$ satisfy

$$\begin{cases} dx_i^* = \left(Ax_i^* + Bu_i^* + Gx_0 + \bar{G}\right)dt + \sigma dW_i + \bar{\sigma}dW \\ -dy_i^* = \left(C_1y_i^* + C_2z_i^* + C_3x_i^* + Du_i^* + Fx_0 + \bar{F}\right)dt \\ \qquad -z_{ii}^* dW_i - z_i^* dW \\ x_i^*(0) = a_{i0}, \quad y_i^*(T) = Hx_i^*(T) \end{cases} \tag{17}$$

$$\begin{cases} dp_i^* = C_1p_i^* dt + C_2p_i^* dW \\ -dq_i^* = \left[Aq_i^* - C_3p_i^* + Q_1(x_i^* - x_0) + \bar{Q}_1\right]dt \\ \qquad -k_{ii}^* dW_i - k_i^* dW \\ p_i^*(0) = -K_2y_i^*(0) - \bar{K}_2 \\ q_i^*(T) = -Hp_i^*(T) + K_1x_i^*(T) + \bar{K}_1. \end{cases} \tag{18}$$

Equations (16)–(18) are called the optimality system of Problem II. By Lemma 2, Problem II is uniquely solvable, which signifies the unique solvability of (16)–(18). In what follows, we aim to decouple (17) and (18).

Assumption 2: $1+\pi_1(t)\pi_4(t)\neq 0$, where π_1,π_4 are given by (24) and (27) below, respectively.

Theorem 1: Under Assumption 1, (17) and (18) admit unique solutions with (16). Moreover, we have the relations as follows:

i)

$$\begin{cases} q_i^* = \alpha p_i^* + \beta x_i^* + \gamma, \quad i \in \mathcal{N} \\ k_{ii}^* = \beta\sigma, k_i^* = \alpha C_2 p_i^* + \beta\bar{\sigma} \end{cases} \tag{19}$$

where

$$\begin{cases} \dot{\beta} + 2A\beta - R^{-1}B^2\beta^2 + Q_1 = 0 \\ \beta(T) = K_1 \end{cases} \tag{20}$$

$$\begin{cases} \dot{\alpha} + (C_1 + A - R^{-1}B^2\beta)\alpha + R^{-1}BD\beta - C_3 = 0 \\ \alpha(T) = -H \end{cases} \tag{21}$$

$$\begin{cases} \dot{\gamma} + (A - R^{-1}B^2\beta)\gamma + (\beta G - Q_1)\mathbb{E}x_0 + \beta\bar{G} + \bar{Q}_1 \\ \quad - R^{-1}B\beta r = 0 \\ \gamma(T) = \bar{K}_1. \end{cases} \tag{22}$$

ii)

$$\begin{cases} y_i^* = \pi_1 p_i^* + \pi_2 x_i^* + \pi_3, \quad i \in \mathcal{N} \\ z_{ii}^* = \pi_2\sigma, \ z_i^* = \pi_1 C_2 p_i^* + \pi_2\bar{\sigma} \end{cases}$$

where

$$\begin{cases} \dot{\pi}_2 + (A + C_1 - R^{-1}B^2\beta)\pi_2 + C_3 - R^{-1}BD\beta = 0 \\ \pi_2(T) = H \end{cases} \tag{23}$$

$$\begin{cases} \dot{\pi}_1 + \left(2C_1 + C_2^2\right)\pi_1 + R^{-1}(D - B\alpha)(B\pi_2 + D) = 0 \\ \pi_1(T) = 0 \end{cases} \tag{24}$$

$$\begin{cases} \dot{\pi}_3 + C_1\pi_3 + \left[-R^{-1}B(r + B\gamma) + \bar{G} + C_2\bar{\sigma}\right]\pi_2 \\ \quad + (G\pi_2 + F)\mathbb{E}x_0 + \bar{F} - R^{-1}D(r + B\gamma) = 0 \\ \pi_3(T) = 0. \end{cases} \tag{25}$$

iii)

$$p_i^* = -\pi_4 y_i^* + \pi_5, \quad i \in \mathcal{N} \tag{26}$$

where

$$\begin{cases} \dot{\pi}_4 - 2C_1\pi_4 + \left[C_2^2\pi_1 + R^{-1}D(D - B\alpha)\right]\pi_4^2 = 0 \\ \pi_4(0) = K_2 \end{cases} \tag{27}$$

$$\begin{cases} \dot{\pi}_5 + \left[C_2^2\pi_1\pi_4 + R^{-1}D\pi_4(D - B\alpha) - C_1\right]\pi_5 \\ \quad + [C_2\bar{\sigma}\pi_2 - R^{-1}D(r + B\gamma) + \bar{F} \\ \quad + (F + C_3 - R^{-1}DB\beta)\mathbb{E}x_0]\pi_4 = 0 \\ \pi_5(0) = -\bar{K}_2. \end{cases} \tag{28}$$

iv) With Assumption 2, we have

$$y_i^* = (1 + \pi_1\pi_4)^{-1}(\pi_2 x_i^* + \pi_3 + \pi_1\pi_5), \quad i \in \mathcal{N}. \tag{29}$$

Proof: See Appendix C. ■

Remark 2: Note that (20)–(24) are independent of $\mathbb{E}x_0$, in the light of Proposition 4.2 in [36], Riccati equation (20) is uniquely solvable. Then (21), (23) and (24) are uniquely solvable. Moreover, Bernoulli equation (27) results in a unique solution. However, (22), (25) and (28) depend on $\mathbb{E}x_0$, whose solvability will be given in Lemma 4 below.

Theorem 2: Under Assumptions 1 and 2, we get

$$u_i^* = \mathbb{A}_2\hat{x}_i^* + \mathbb{A}_1\pi_3 + \mathbb{A}_3\pi_5 - R^{-1}(B\gamma + r), \ i \in \mathcal{N} \tag{30}$$

where

$$\begin{aligned} d\hat{x}_i^* &= \left(A\hat{x}_i^* + Bu_i^* + Gx_0 + \bar{G}\right)dt + (\sigma + fP) \\ &\quad \times\left[d\bar{Y}_i - \left(f\hat{x}_i^* + gx_0 + h\right)dt\right] + \bar{\sigma}dW \\ &= \Big\{(A + B\mathbb{A}_2)\hat{x}_i^* + B\left[\mathbb{A}_1\pi_3 + \mathbb{A}_3\pi_5 - R^{-1}(B\gamma + r)\right] \\ &\quad + Gx_0 + \bar{G} + (\sigma + fP)f\left(x_i^* - \hat{x}_i^*\right)\Big\}dt \\ &\quad + (\sigma + fP)dW_i + \bar{\sigma}dW \end{aligned} \tag{31}$$

with $\hat{x}_i^*(0) = a_0$, and

$$\begin{cases} \mathbb{A}_1 = R^{-1}\pi_4(B\alpha - D)(1 + \pi_1\pi_4)^{-1} \\ \mathbb{A}_2 = \mathbb{A}_1\pi_2 - R^{-1}B\beta, \mathbb{A}_3 = \mathbb{A}_1\pi_1 + R^{-1}(D - B\alpha). \end{cases} \tag{32}$$

Proof: Inserting the first equality of (19), (26) and (29) into (16), we obtain feedback form (30) with (32). Moreover, it follows from (5) and (14), (31) holds. ■

Remark 3: We point out that Problem II is distinguished from [27] mainly in two aspects. i) The admissible control set contains the common noise W. Due to the presence of W, we construct the admissible control set $\bar{\mathcal{U}}_i$ depending on W in Definition 1. Otherwise, once W is absent from $\bar{\mathcal{U}}_i$, Lemma 1 will not hold. Without such equivalence, it turns out to be really difficult and challenging to study Problem II. ii) Introducing eight ODEs shown in Theorem 1, we get the decentralized control strategy in a feedback form, instead of an open-loop form given by [27].

In what follows, we analyse the limiting process x_0 and (22), (25) and (28). Introduce the decentralized control strategy:

$$u_i^* = \mathbb{A}_2\hat{x}_i^* + \mathbb{A}_1\pi_3 + \mathbb{A}_3\pi_5 - R^{-1}(B\gamma + r), \ i \in \mathcal{N}. \tag{33}$$

Inserting (33) into the first equation of (1), we have

$$\begin{cases} dX_i^* = \Big\{AX_i^* + B\left[\mathbb{A}_2\hat{x}_i^* + \mathbb{A}_1\pi_3 + \mathbb{A}_3\pi_5 - R^{-1}(B\gamma + r)\right] \\ \qquad + GX^{*(N)} + \bar{G}\Big\}dt + \sigma dW_i + \bar{\sigma}dW \\ X_i^*(0) = a_{i0}, \quad i \in \mathcal{N} \end{cases}$$

which implies that

$$\begin{aligned} dX^{*(N)} &= \Big\{(A + G)X^{*(N)} + B\mathbb{A}_2\hat{x}^{*(N)} + B\Big[\mathbb{A}_1\pi_3 \\ &\quad + \mathbb{A}_3\pi_5 - R^{-1}(B\gamma + r)\Big] + \bar{G}\Big\}dt \\ &\quad + \sigma\frac{1}{N}\sum_{i=1}^{N} dW_i + \bar{\sigma}dW \end{aligned}$$

with $\mathcal{X}^{*(N)}(0)=\frac{1}{N}\sum_{i=1}^N a_{i0}$, and $\rho^{(N)}=\frac{1}{N}\sum_{i=1}^N \rho_i$ with $\rho=\mathcal{X}^*,\hat{x}^*$. Taking $N\to+\infty$, we arrive at

$$\begin{cases} dx_0=\Big[(A+G+B\mathbb{A}_2)x_0+B(\mathbb{A}_1\pi_3+\mathbb{A}_3\pi_5) \\ \qquad -R^{-1}B(B\gamma+r)+\bar{G}\Big]dt+\bar{\sigma}dW \\ x_0(0)=a_0 \end{cases} \tag{34}$$

where we approximate $\hat{x}^{*(N)}$ by x_0, and it will be proven in Lemma 6 below. Equations (20)–(25), (27) and (28) together with (34) are called the consistency condition.

Taking $\mathbb{E}[\cdot]$ on both sides of (34), it yields

$$\begin{cases} d\mathbb{E}x_0=\Big[(A+G+B\mathbb{A}_2)\mathbb{E}x_0+B(\mathbb{A}_1\pi_3 \\ \qquad +\mathbb{A}_3\pi_5)-R^{-1}B(B\gamma+r)+\bar{G}\Big]dt \\ \mathbb{E}x_0(0)=a_0. \end{cases} \tag{35}$$

Assumption 3: We assume that $U(\cdot)$ is invertible, where $U(\cdot)$ is given in Appendix D.

Lemma 4: Under Assumptions 1–3, (22), (25), (28) and (35) are solvable.

Proof: See Appendix D. ■

According to the analysis above, limiting equation (34) is solvable, and its solution x_0 is $\mathcal{F}_t^W$-adapted and $\mathcal{L}^2$-bounded.

IV. ϵ-NASH EQUILIBRIUM OF PROBLEM I

Now we focus on verifying the ϵ-Nash equilibrium property of $(u_1^*,\dots,u_N^*)$ obtained in Section III.

Definition 2: $(u_1^*,\dots,u_N^*)$ is called an ϵ-Nash equilibrium of Problem I, if there exists $\epsilon=\epsilon(N)\geq 0$ with $\lim_{N\to+\infty}\epsilon(N)=0$ such that

$$\mathcal{J}_i(u_i^*,u_{-i}^*)\leq \mathcal{J}_i(u_i,u_{-i}^*)+\epsilon,\quad i\in\mathcal{N}$$

when an admissible alternative strategy $u_i\in\bar{\mathcal{U}}_i$ is taken by agent $\mathcal{A}_i$.

Theorem 3: Under Assumptions 1–3, $u_i^*=\mathbb{A}_2\hat{x}_i^*+\mathbb{A}_1\pi_3+\mathbb{A}_3\pi_5-R^{-1}(B\gamma+r)$ $(i\in\mathcal{N})$ is an ϵ-Nash equilibrium of Problem I with $\epsilon=O(\frac{1}{\sqrt{N}})$, where $\hat{x}_i^*,\mathbb{A}_j$ $(j=1,2,3),\pi_3,\pi_5$ are given by (31), (32), (25) and (28), respectively.

The proof of Theorem 3 will be addressed later.

Under the ϵ-Nash equilibrium, the system of Problem I is

$$\begin{cases} d\mathcal{X}_i^*=\Big\{A\mathcal{X}_i^*+B\Big[\mathbb{A}_2\hat{x}_i^*+\mathbb{A}_1\pi_3+\mathbb{A}_3\pi_5-R^{-1}(B\gamma \\ \qquad +r)\Big]+G\mathcal{X}^{*(N)}+\bar{G}\Big\}dt+\sigma dW_i+\bar{\sigma}dW \\ -d\mathcal{Y}_i^*=\Big\{C_1\mathcal{Y}_i^*+C_2\mathcal{Z}_i^*+C_3\mathcal{X}_i^*+D\Big[\mathbb{A}_2\hat{x}_i^*+\mathbb{A}_1\pi_3 \\ \qquad +\mathbb{A}_3\pi_5-R^{-1}(B\gamma+r)\Big]+F\mathcal{X}^{*(N)}+\bar{F}\Big\}dt \\ \qquad -\sum_{j=1}^N\mathcal{Z}_{ij}^*dW_j-\mathcal{Z}_i^*dW \\ \mathcal{X}_i^*(0)=a_{i0},\quad \mathcal{Y}_i^*(T)=H\mathcal{X}_i^*(T),\quad i\in\mathcal{N} \end{cases} \tag{36}$$

and the corresponding system of Problem II is

$$\begin{cases} dx_i^*=\Big\{Ax_i^*+B\Big[\mathbb{A}_2\hat{x}_i^*+\mathbb{A}_1\pi_3+\mathbb{A}_3\pi_5-R^{-1}(B\gamma+r)\Big] \\ \qquad +Gx_0+\bar{G}\Big\}dt+\sigma dW_i+\bar{\sigma}dW \\ -dy_i^*=\Big\{C_1y_i^*+C_2z_i^*+C_3x_i^*+D\Big[\mathbb{A}_2\hat{x}_i^*+\mathbb{A}_1\pi_3 \\ \qquad +\mathbb{A}_3\pi_5-R^{-1}(B\gamma+r)\Big]+Fx_0+\bar{F}\Big\}dt \\ \qquad -z_{ii}^*dW_i-z_i^*dW \\ x_i^*(0)=a_{i0},\quad y_i^*(T)=Hx_i^*(T) \end{cases} \tag{37}$$

where

$$\begin{aligned} d\hat{x}_i^* &= \Big(A\hat{x}_i^*+Bu_i^*+Gx_0+\bar{G}\Big)dt+(\sigma+fP) \\ &\quad\times\Big[d\bar{Y}_i-\Big(f\hat{x}_i^*+gx_0+h\Big)dt\Big]+\bar{\sigma}dW \\ &=\Big\{(A+B\mathbb{A}_2)\hat{x}_i^*+B\Big[\mathbb{A}_1\pi_3+\mathbb{A}_3\pi_5-R^{-1}(B\gamma \\ &\quad +r)\Big]+Gx_0+\bar{G}+(\sigma+fP)f\Big(x_i^*-\hat{x}_i^*\Big)\Big\}dt \\ &\quad +(\sigma+fP)dW_i+\bar{\sigma}dW \end{aligned} \tag{38}$$

with $\hat{x}_i^*(0)=a_0$.

Lemma 5: $\sup_{t\in[0,T]}\mathbb{E}|\varrho(t)|^2$ and $\mathbb{E}\int_0^T|\tilde{\varrho}(t)|^2dt$ are bounded, where $\varrho=x_0,x_i^*,\hat{x}_i^*,y_i^*$ and $\tilde{\varrho}=u_i^*,z_{ii}^*,z_i^*$, $i\in\mathcal{N}$.

Proof: See Appendix E. ■

Lemma 6:

$$\sup_{t\in[0,T]}\mathbb{E}\Big|\mathcal{X}^{*(N)}(t)-x_0(t)\Big|^2=O\Big(\frac{1}{N}\Big) \tag{39}$$

$$\sup_{t\in[0,T]}\mathbb{E}\Big|x^{*(N)}(t)-x_0(t)\Big|^2=O\Big(\frac{1}{N}\Big) \tag{40}$$

$$\sup_{t\in[0,T]}\mathbb{E}\Big|\hat{x}^{*(N)}(t)-x_0(t)\Big|^2=O\Big(\frac{1}{N}\Big) \tag{41}$$

where $\phi^{(N)}=\frac{1}{N}\sum_{i=1}^N\phi_i$ with $\phi=\mathcal{X}^*,x^*,\hat{x}^*$.

Proof: See Appendix F. ■

Lemma 7:

$$\sup_{i\in\mathcal{N}}\sup_{t\in[0,T]}\mathbb{E}\Big|\mathcal{X}_i^*(t)-x_i^*(t)\Big|^2=O\Big(\frac{1}{N}\Big) \tag{42}$$

$$\sup_{i\in\mathcal{N}}\sup_{t\in[0,T]}\mathbb{E}\Big|\mathcal{Y}_i^*(t)-y_i^*(t)\Big|^2=O\Big(\frac{1}{N}\Big). \tag{43}$$

Proof: See Appendix G. ■

Lemma 8:

$$|\mathcal{J}_i(u_i^*,u_{-i}^*)-J_i(u_i^*)|=O\Big(\frac{1}{\sqrt{N}}\Big),\quad i\in\mathcal{N}. \tag{44}$$

Proof: See Appendix H. ■

In what follows, we proceed to give the asymptotic analysis of $(u_1^*,\dots,u_N^*)$. For any fixed i, suppose that agent $\mathcal{A}_i$ takes a perturbation strategy $u_i\in\bar{\mathcal{U}}_i$ and the corresponding state is

$$\begin{cases} dl_i=\Big(Al_i+Bu_i+Gl^{(N)}+\bar{G}\Big)dt+\sigma dW_i+\bar{\sigma}dW \\ -dm_i=\Big(C_1m_i+C_2n_i+C_3l_i+Du_i+Fl^{(N)}+\bar{F}\Big)dt \\ \qquad -n_{ii}dW_i-n_idW \\ l_i(0)=a_{i0},\quad m_i(T)=Hl_i(T),\quad i\in\mathcal{N} \end{cases} \tag{45}$$

whereas agent $\mathcal{A}_k$ $(k \neq i)$ keeps optimal strategy u_k^* with the state

$$\begin{cases} dl_k = \Big\{Al_k + B\Big[\mathbb{A}_2\hat{x}_k^* + \mathbb{A}_1\pi_3 + \mathbb{A}_3\pi_5 - R^{-1}(B\gamma \\ \qquad + r)\Big] + Gl^{(N)} + \bar{G}\Big\}dt + \sigma dW_k + \bar{\sigma}dW \\ -dm_k = \Big\{C_1 m_k + C_2 n_k + C_3 l_k + D\Big[\mathbb{A}_2\hat{x}_k^* + \mathbb{A}_1\pi_3 \\ \qquad + \mathbb{A}_3\pi_5 - R^{-1}(B\gamma + r)\Big] + Fl^{(N)} + \bar{F}\Big\}dt \\ \qquad - n_{kk}dW_k - n_k dW \\ l_k(0) = a_{k0}, \quad m_k(T) = Hl_k(T), \quad k \in \mathcal{N}. \end{cases} \tag{46}$$

If $(u_1^*, \ldots, u_N^*)$ is an ϵ-Nash equilibrium of Problem I, consider the perturbation $u_i \in \bar{\mathcal{U}}_i$ satisfying

$$\mathcal{J}_i(u_i, u_{-i}^*) \leq \mathcal{J}_i(u_i^*, u_{-i}^*). \tag{47}$$

Recall

$$\begin{aligned} \mathcal{J}_i(u_i, u_{-i}^*) = \frac{1}{2}\mathbb{E}\bigg\{ \int_0^T \bigg[Q_1\bigg(\mathcal{X}_i - \mathcal{X}^{(N)} + \frac{\bar{Q}_1}{Q_1}\bigg)^2 \\ - \frac{\bar{Q}_1^2}{Q_1} + R\Big(u_i + \frac{r}{R}\Big)^2 - \frac{r^2}{R}\bigg]dt + K_1\bigg(\mathcal{X}_i(T) + \frac{\bar{K}_1}{K_1}\bigg)^2 \\ - \frac{\bar{K}_1^2}{K_1} + K_2\bigg(\mathcal{Y}_i(0) + \frac{\bar{K}_2}{K_2}\bigg)^2 - \frac{\bar{K}_2^2}{K_2}\bigg\}. \end{aligned} \tag{48}$$

Applying (47) and (48) with Lemma 8, it holds

$$\begin{aligned} \mathbb{E}\int_0^T R\Big(u_i + \frac{r}{R}\Big)^2 dt &\leq 2\mathcal{J}_i(u_i, u_{-i}^*) + \frac{\bar{K}_1^2}{K_1} + \frac{\bar{K}_2^2}{K_2} \\ &+ \mathbb{E}\int_0^T \bigg(\frac{\bar{Q}_1^2}{Q_1} + \frac{r^2}{R}\bigg)dt = 2J_i(u_i^*) + \frac{\bar{K}_1^2}{K_1} + \frac{\bar{K}_2^2}{K_2} \\ &+ \mathbb{E}\int_0^T \bigg(\frac{\bar{Q}_1^2}{Q_1} + \frac{r^2}{R}\bigg)dt + O\bigg(\frac{1}{\sqrt{N}}\bigg) \end{aligned}$$

where

$$\begin{aligned} J_i(u_i^*) = \frac{1}{2}\mathbb{E}\bigg\{ \int_0^T \Big[Q_1(x_i^* - x_0)^2 + 2\bar{Q}_1\left(x_i^* - x_0\right) \\ + R(u_i^*)^2 + 2ru_i^*\Big]dt + K_1(x_i^*(T))^2 + 2\bar{K}_1 x_i^*(T) \\ + K_2(y_i^*(0))^2 + 2\bar{K}_2 y_i^*(0)\bigg\} \leq c. \end{aligned}$$

Hence,

$$\mathbb{E}\int_0^T u_i^2(t)dt \leq c.$$

Correspondingly, in Problem II, assume that agent $\mathcal{A}_i$ takes $u_i \in \bar{\mathcal{U}}_i$ and the corresponding state is

$$\begin{cases} dl_i^0 = \left(Al_i^0 + Bu_i + Gx_0 + \bar{G}\right)dt + \sigma dW_i + \bar{\sigma}dW \\ -dm_i^0 = \left(C_1 m_i^0 + C_2 n_i^0 + C_3 l_i^0 + Du_i + Fx_0 + \bar{F}\right)dt \\ \qquad - n_{ii}^0 dW_i - n_i^0 dW \\ l_i^0(0) = a_{i0}, \quad m_i^0(T) = Hl_i^0(T), \quad i \in \mathcal{N} \end{cases} \tag{49}$$

whereas agent $\mathcal{A}_k$ $(k \neq i)$ keeps optimal strategy u_k^* with the state

$$\begin{cases} dl_k^0 = \Big\{Al_k^0 + B\Big[\mathbb{A}_2\hat{x}_k^* + \mathbb{A}_1\pi_3 + \mathbb{A}_3\pi_5 - R^{-1}(B\gamma \\ \qquad + r)\Big] + Gx_0 + \bar{G}\Big\}dt + \sigma dW_k + \bar{\sigma}dW \\ -dm_k^0 = \Big\{C_1 m_k^0 + C_2 n_k^0 + C_3 l_k^0 + D\Big[\mathbb{A}_2\hat{x}_k^* + \mathbb{A}_1\pi_3 \\ \qquad + \mathbb{A}_3\pi_5 - R^{-1}(B\gamma + r)\Big] + Fx_0 + \bar{F}\Big\}dt \\ \qquad - n_{kk}^0 dW_k - n_k^0 dW \\ l_k^0(0) = a_{k0}, \quad m_k^0(T) = Hl_k^0(T), \quad k \in \mathcal{N}. \end{cases} \tag{50}$$

Lemma 9: $\sup_{t\in[0,T]}\mathbb{E}|l_i^0(t)|^2, \sup_{t\in[0,T]}\mathbb{E}|m_i^0(t)|^2, \mathbb{E}\int_0^T |n_{ii}^0(t)|^2 dt$ and $\mathbb{E}\int_0^T |n_i^0(t)|^2 dt$ are bounded, $i \in \mathcal{N}$.

Proof: See Appendix I. ■

Similar to Lemmas 6–8, we draw two lemmas as follows. Adopting the similar arguments addressed in Lemmas 6–8, we can prove Lemmas 10 and 11 below, where the detailed procedures are omitted to save space.

Lemma 10:

$$\sup_{t\in[0,T]} \mathbb{E}\left|l^{(N)}(t) - x_0(t)\right|^2 = O\left(\frac{1}{N}\right) \tag{51}$$

$$\sup_{i\in\mathcal{N}}\sup_{t\in[0,T]} \mathbb{E}\left|l_i(t) - l_i^0(t)\right|^2 = O\left(\frac{1}{N}\right) \tag{52}$$

$$\sup_{i\in\mathcal{N}}\sup_{t\in[0,T]} \mathbb{E}\left|m_i(t) - m_i^0(t)\right|^2 = O\left(\frac{1}{N}\right) \tag{53}$$

where $l^{(N)} = \frac{1}{N}\sum_{j=1}^N l_j$.

Lemma 11:

$$|\mathcal{J}_i(u_i, u_{-i}^*) - J_i(u_i)| = O\left(\frac{1}{\sqrt{N}}\right), \quad i \in \mathcal{N}. \tag{54}$$

Proof of Theorem 2: According to (44) and (54), it holds

$$\mathcal{J}_i(u_i^*, u_{-i}^*) \leq \mathcal{J}_i(u_i, u_{-i}^*) + O\left(\frac{1}{\sqrt{N}}\right).$$

Taking $\epsilon = O(\frac{1}{\sqrt{N}})$, we complete the proof. ■

Now we summarize the process of seeking an ϵ-Nash equilibrium of Problem I, which also shows the process of searching for the optimal (decentralized) control strategy (see Fig. 1 below for convenience): i) Firstly, employing mean-field method, we obtain an auxiliary Problem II. ii) Secondly, by virtue of optimal filter technique, decomposition technique and dimensional-expansion technique, we obtain an optimal (decentralized) control strategy. iii) Finally, applying the FBSDE theory, we verify the decentralized control strategy obtained is an ϵ-Nash equilibrium of Problem I. Moreover, the process of verifying asymptotic optimality can be illustrated by Fig. 2 below.

V. A Financial Example

In this section, we discuss a financial problem, which facilitates the study of mean-field game Problem I.

Suppose that there are N counties in province $\mathcal{P}$, in general

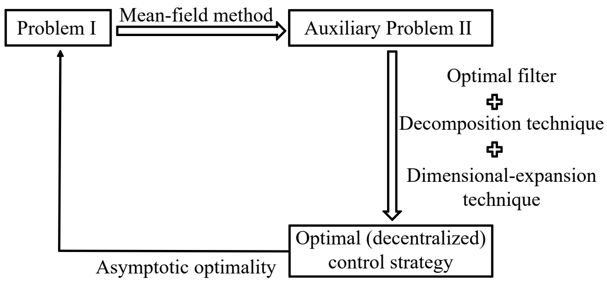


Fig. 1. The research route.

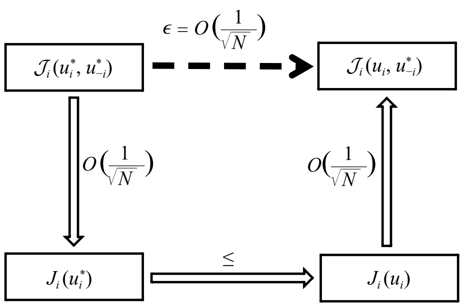


Fig. 2. The process of verifying asymptotic optimality.

they are closely related with each other. Suppose that the initial wealth level of county $\mathcal{A}_i$ $(i \in \mathcal{N})$ is denoted by a_{i0}, which is positive. Let $\mathcal{X}_i(\cdot)$ $(i \in \mathcal{N})$ denote the wealth level of county $\mathcal{A}_i$, let $u_i(\cdot)$ denote the control strategy of county $\mathcal{A}_i$, such as attracting investment and talent policy, and let $\mathcal{X}^{(N)}(\cdot) = \frac{1}{N}\sum_{i=1}^N \mathcal{X}_i(\cdot)$ stand for the influence of the average wealth of province $\mathcal{P}$. Moreover, each county plans to spend $H\mathcal{X}_i(T)$ $(i \in \mathcal{N})$ on educational infrastructure at time T, where $H \in (0,1)$ is a positive constant and $T > 0$. We denote by $\mathcal{Y}_i(\cdot)$ $(i \in \mathcal{N})$ the expense of the educational infrastructure of county $\mathcal{A}_i$, which is affected by the wealth level of province $\mathcal{P}$ and the control strategy of county $\mathcal{A}_i$. Meanwhile, each county hopes to minimize its own cost functional $\mathcal{J}_i$, $i \in \mathcal{N}$. Under the notations and interpretations above, $\mathcal{X}_i(\cdot)$ and $\mathcal{Y}_i(\cdot)$ are modeled by

$$\begin{cases} d\mathcal{X}_i(t) = \left[A\mathcal{X}_i(t) + Bu_i(t) + G\mathcal{X}^{(N)}(t)\right]dt + \sigma dW_i(t) \\ \qquad\qquad + \bar{\sigma} dW(t) \\ -d\mathcal{Y}_i(t) = \left[C_1\mathcal{Y}_i(t) + C_3\mathcal{X}_i(t) + Du_i(t) + F\mathcal{X}^{(N)}(t)\right]dt \\ \qquad\qquad - \sum_{j=1}^N \mathcal{Z}_{ij}(t)dW_j(t) - \mathcal{Z}_i(t)dW(t) \\ \mathcal{X}_i(0) = a_{i0}, \quad \mathcal{Y}_i(T) = H\mathcal{X}_i(T), \quad i \in \mathcal{N} \end{cases} \tag{55}$$

where $\mathcal{X}^{(N)} = \frac{1}{N}\sum_{i=1}^N \mathcal{X}_i$, $\mathcal{Z}_{ij}$ and $\mathcal{Z}_i$ are some adjustment factors, which guarantees the well-posedness of the second equation of (55); W_i stands for some local uncertainties, like population mobility, household consumption and resource consumption; W denotes some global uncertainties such as natural disaster; the coefficients in (55) are constants.

Assume that each county can only observe a part of the total wealth level of province $\mathcal{P}$, such as produced capital. In fact, the total wealth level of province $\mathcal{P}$ is difficult to access. On the one hand, accurate calculation of wealth costs significant time and energy; on the other hand, some wealth can not be measured, like intangible capital. For county $\mathcal{A}_i$ $(i \in \mathcal{N})$, let $\check{Y}_i(\cdot)$ represent the observable part of the wealth of province $\mathcal{P}$, where $f_0, g_0 \in (0,1)$ represent the proportions of the observable parts of the wealth of county $\mathcal{A}_i$ and the average wealth of other counties, respectively. Then

$$\check{Y}_i(t) = f_0\mathcal{X}_i(t) + g_0\mathcal{X}^{(N-1)}(t) + N_i(t), \quad i \in \mathcal{N} \tag{56}$$

where $\check{Y}_i$ is also affected by the white noise process N_i. Integrating on (56), we get

$$Y_i(t) = \int_0^t \left[f_0\mathcal{X}_i(s) + g_0\mathcal{X}^{(N-1)}(s)\right]ds + \int_0^t dN_i(s). \tag{57}$$

Let $W_i(t) = \int_0^t dN_i(s)$. Then, W_i is a Brownian motion. Employing Itô's formula to (57), it yields

$$\begin{cases} dY_i(t) = \left[f_0\mathcal{X}_i(t) + g_0\mathcal{X}^{(N-1)}(t)\right]dt + dW_i(t) \\ Y_i(0) = 0, \quad i \in \mathcal{N}. \end{cases} \tag{58}$$

Let

$$\begin{cases} f = f_0 - g_0\dfrac{1}{N-1} \\ g = \dfrac{g_0 N}{N-1}. \end{cases}$$

Then, (58) is written as

$$\begin{cases} dY_i(t) = \left[f\mathcal{X}_i(t) + g\mathcal{X}^{(N)}(t)\right]dt + dW_i(t) \\ Y_i(0) = 0, \quad i \in \mathcal{N}. \end{cases}$$

Problem III: Each county hopes to select a suitable control strategy $\tilde{u}_i \in \bar{\mathcal{U}}_i$ to minimize

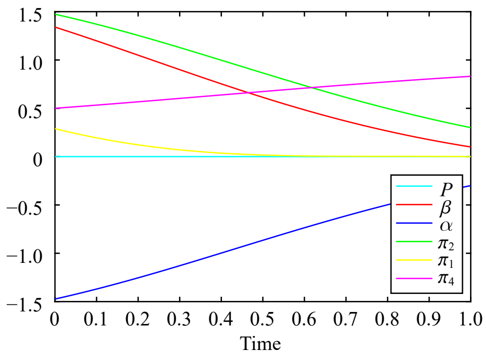
$$\mathcal{J}_i(u_i(\cdot), \tilde{u}_{-i}(\cdot)) = \frac{1}{2}\mathbb{E}\left\{\int_0^T \left[Q_1\left(\mathcal{X}_i(t) - \mathcal{X}^{(N)}(t)\right)^2 + Ru_i^2(t)\right]dt + K_1\left(\mathcal{X}_i(T) - M\right)^2 + K_2\mathcal{Y}_i^2(0)\right\} \tag{59}$$

where $\tilde{u}_{-i} = (\tilde{u}_1, \ldots, \tilde{u}_{i-1}, \tilde{u}_{i+1}, \ldots, \tilde{u}_N)$, $i \in \mathcal{N}$, Q_1, R, K_1, M, K_2 are positive constants; $\bar{\mathcal{U}}_i$ is given in Definition 1 above.

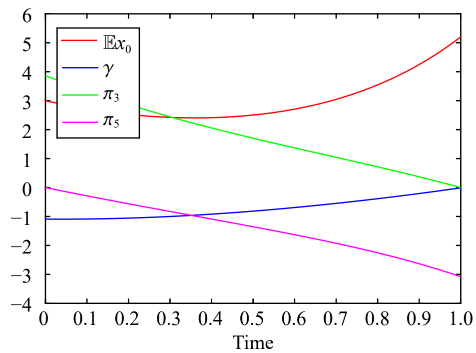
In (59), the running cost implies that each county wants to minimize the departure of the wealth level $\mathcal{X}_i$ from the "average", in order to achieve coordinated development; the terminal cost $(\mathcal{X}_i(T) - M)^2$ measures the deviation between the terminal wealth level of county $\mathcal{A}_i$ at time T and the terminal target M; the initial cost measures that each county wants to minimize its initial expense of educational infrastructure.

To be more intuitive, we consider Problem III with 100 counties. Take $T = 1$, $A = 1$, $B = -1$, $G = 0.3$, $\sigma = 0.5$, $\bar{\sigma} = 0.8$, $C_1 = 0.2$, $C_3 = 0.5$, $D = 0.5$, $F = 0.8$, $H = 0.3$, $f = 0.5$, $g = 0.5$, $Q_1 = 0.5$, $R = 1$, $K_1 = 0.1$, $M = 0.1$, $K_2 = 0.5$, $a_{i0} \sim U(1,5)$ $(1 \leq i \leq 100)$.

Employing the Euler method, we plot the curves of $(P, \beta, \alpha, \pi_2, \pi_1, \pi_4)$ shown in Fig. 3(a). Note that $\mathbb{E}x_0, \gamma, \pi_3, \pi_5$ given by (22), (25), (28) and (35) are coupled. Employing the decoupling method used in Appendix B, we obtain the curves of $(\mathbb{E}x_0, \gamma, \pi_3, \pi_5)$ shown in Fig. 3(b). Based on Fig. 3 and applying Monte Carlo method, we obtain the curve of ϵ-Nash equilibrium strategy $(u_1^*, \ldots, u_{100}^*)$ shown in Fig. 4, which also



(a) The curves of $(P, \beta, \alpha, \pi_2, \pi_1, \pi_4)$



(b) The curves of $(\mathbb{E}x_0, \gamma, \pi_3, \pi_5)$

Fig. 3. Numerical solutions of $(P,\beta,\alpha,\pi_2,\pi_1,\pi_4)$ and $(\mathbb{E}x_0,\gamma,\pi_3,\pi_5)$.

illustrates the efficiency of Theorem 3 obtained. According to Theorem 3, u_i^* is represented by $\hat{x}_i^*$, where $\hat{x}_i^*$ is the solution of SDE (31). Therefore, the trajectory of u_i^* is related to the trend of SDE. Besides, from Fig. 4, it can be seen that the control strategy u_i^* gradually tends to 0, which means that government intervention is decreasing. It is worth pointing out that if $u_i^* < 0$, it implies that county $\mathcal{A}_i$ takes some negative actions, like reducing staff. Besides, the wealth level $\mathcal{X}_i^*$ and the expense of the educational infrastructure $\mathcal{Y}_i^*$ corresponding to $(u_1^*,\ldots,u_{100}^*)$ are shown in Fig. 5. As shown in Fig. 5(a), wealth levels are on the rise overall, which is consistent with reality. Inversely, Fig. 5(b) shows that the expenses of educational infrastructure are on the decline, which means that counties spend more money on educational infrastructure at the initial time 0, and less expense is needed as the educational facilities become better.

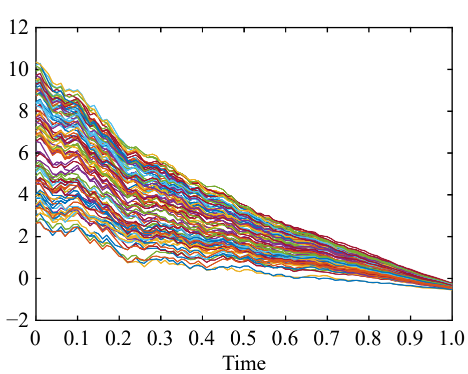
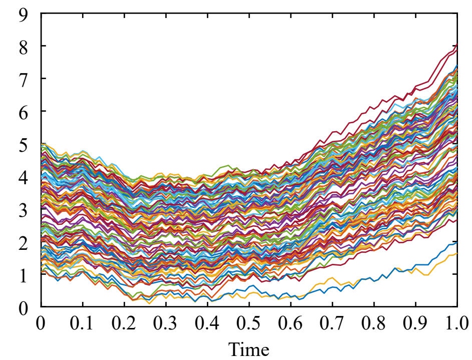
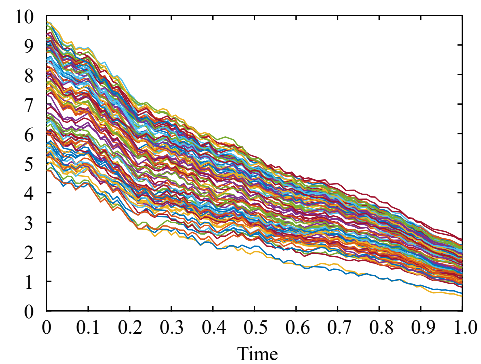


Fig. 4. Numerical solution of u_i^* $(1 \le i \le 100)$.



(a) $\mathcal{X}_i^*$



(b) $\mathcal{Y}_i^*$

Fig. 5. Numerical solutions of $\mathcal{X}_i^*$ and $\mathcal{Y}_i^*$ $(1 \le i \le 100)$.

Furthermore, it follows from (39) in Lemma 6 that the difference between $\mathcal{X}^{*(N)}$ and x_0 is small enough in the sense of expectation when N is big enough. Based on the curves of $(\mathbb{E}x_0,\gamma,\pi_3,\pi_5)$ shown in Fig. 3(b), the curve of x_0 can be obtained. As a result, one sample trajectory of each of the stochastic processes x_0 and $\mathcal{X}^{*(100)} = \frac{1}{100}\sum_{i=1}^{100}\mathcal{X}_i^*$ is shown in Fig. 6. As we see in Fig. 6, it is possible that the two trajectories coincide well at some times and have some slight deviations at the other times.

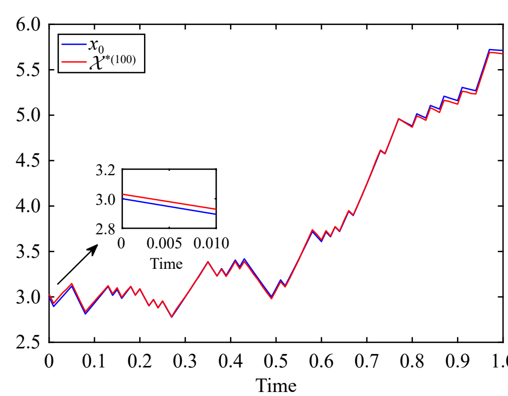


Fig. 6. Numerical solutions of x_0 and $\mathcal{X}^{*(100)}$.

VI. Conclusion and Outlook

This paper discusses a mean-field game of FBSDE in the framework of partial observation. Employing the filter technique to solve a limiting control problem, a decentralized control strategy is obtained, which is further verified to be an ϵ-Nash equilibrium of mean-field game. We also show a financial example with some numerical results.

We point out that the results established in this work are based on Definition 1, where the admissible control set $\bar{\mathcal{U}}_i$ depends on the common noise W. In reality, W may not be observed by all agents. That is to say, W is absent from $\bar{\mathcal{U}}_i$, which results in the unavailability of Lemma 1. In this case, how to solve Problem II will face many technical challenges. We will come back to this topic in our future work.

Appendix A
Proof of Lemma 1

Proof: Set

$$\tilde{\mathcal{X}} = (\mathcal{X}_1,\dots,\mathcal{X}_N)^\tau,\ u = (u_1,\dots,u_N)^\tau,\ \tilde{G} = (\bar{G},\dots,\bar{G})^\tau$$
$$\widehat{W} = (W_1,\dots,W_N)^\tau,\ \tilde{\mathcal{X}}_0 = (a_{10},\dots,a_{N0})^\tau$$
$$\tilde{A} = AI_N,\ \tilde{B} = BI_N,\ \hat{G} = \frac{G}{N}\mathbf{1}_{N\times N},\ \bar{\Xi} = (\bar{\sigma},\dots,\bar{\sigma})^\tau$$

where I_N and $\mathbf{1}_{N\times N}$ represent $N\times N$ identity matrix and $N\times N$ matrix with all entries equal to 1, respectively. Then the first equation of (1) is

$$\begin{cases} d\tilde{\mathcal{X}} = \left[(\tilde{A}+\hat{G})\tilde{\mathcal{X}} + \tilde{B}u + \tilde{G}\right]dt + \sigma d\hat{W} + \bar{\Xi}dW \\ \tilde{\mathcal{X}}(0) = \tilde{\mathcal{X}}_0 \end{cases} \tag{60}$$

which is uniquely solvable with $\tilde{\mathcal{X}} \in \mathcal{L}^2_{\mathcal{F}}\left(0,T;\mathbb{R}^N\right)$. Similarly, by virtue of [37] and (60), the second equation of (1) and (2) produce unique solutions $(\mathcal{Y}_i,\mathcal{Z}_{i1},\dots,\mathcal{Z}_{iN},\mathcal{Z}_i) \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^{N+2})$ and $Y_i \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R})$, respectively. ■

Appendix B
Proof of Lemma 2

Proof: We first prove that $J_i(u_i)$ $(i\in\mathcal{N})$ is strictly convex. For simplicity, we drop the subscript i here. For any two different controls $v', v \in \bar{\mathcal{U}}$ and any $\lambda_1,\lambda_2 \in (0,1)$ with $\lambda_1+\lambda_2 = 1$, we have

$$\vartheta^{\lambda_1 v' + \lambda_2 v} = \lambda_1\vartheta^{v'} + \lambda_2\vartheta^{v}$$

where $\vartheta = x, y$. Then

$$\begin{aligned} &J(\lambda_1 v' + \lambda_2 v) - \lambda_1 J(v') - \lambda_2 J(v) \\ &= -\frac{1}{2}\lambda_1\lambda_2\mathbb{E}\bigg\{\int_0^T\left[Q_1\left(x^{v'}-x^{v}\right)^2 + R(v'-v)^2\right]dt \\ &\quad + K_1(x^{v'}(T)-x^{v}(T))^2 + K_2(y^{v'}(0)-y^{v}(0))^2\bigg\} < 0 \end{aligned}$$

which implies that $J(v)$ is strictly convex. Moreover,

$$J(v) \geq \frac{1}{2}\left\{\mathbb{E}\int_0^T\left[R\left(v+\frac{r}{R}\right)^2 - \frac{\bar{Q}_1^2}{Q_1} - \frac{r^2}{R}\right]dt - \frac{\bar{K}_1^2}{K_1} - \frac{\bar{K}_2^2}{K_2}\right\}$$

which shows the coercive property of $J(v)$. Thus, by convex analysis theory, Problem II is uniquely solvable. ■

Appendix C
Proof of Theorem 1

Proof: i) Noting the fourth equality of (18), we assume that

$$q_i^* = \alpha p_i^* + \beta x_i^* + \gamma, \quad i\in\mathcal{N} \tag{61}$$

where $\alpha(T) = -H$, $\beta(T) = K_1$, $\gamma(T) = \bar{K}_1$. Using Itô's formula to (61), we arrive at

$$\begin{aligned} dq_i^* = \Big[&(\dot{\alpha}+C_1\alpha)p_i^* + (\dot{\beta}+A\beta)x_i^* + R^{-1}B\beta(D-B\alpha)\hat{p}_i^* \\ &- R^{-1}B^2\beta^2\hat{x}_i^* + \dot{\gamma} - R^{-1}B^2\beta\gamma - R^{-1}B\beta r + \beta G x_0 \\ &+ \beta\bar{G}\Big]dt + \beta\sigma dW_i + \left(\alpha C_2 p_i^* + \beta\bar{\sigma}\right)dW. \end{aligned}$$

Comparing the above equality with the second equation of (18), it yields that

$$\begin{aligned} &k_{ii}^* = \beta\sigma,\ k_i^* = \alpha C_2 p_i^* + \beta\bar{\sigma} \\ &(\dot{\alpha}+C_1\alpha+A\alpha-C_3)p_i^* + R^{-1}B\beta(D-B\alpha)\hat{p}_i^* + \dot{\gamma} \\ &\quad + (\dot{\beta}+2A\beta+Q_1)x_i^* - R^{-1}B^2\beta^2\hat{x}_i^* + (A-R^{-1}B^2\beta)\gamma \\ &\quad + (\beta G - Q_1)x_0 + \beta\bar{G} - R^{-1}B\beta r + \bar{Q}_1 = 0. \end{aligned} \tag{62}$$

Taking $\mathbb{E}[\cdot]$ on both sides of (62), we get

$$\begin{aligned} &(\dot{\alpha}+C_1\alpha+A\alpha-C_3+R^{-1}BD\beta-R^{-1}B^2\alpha\beta)\mathbb{E}p_i^* + \dot{\gamma} \\ &\quad + (\dot{\beta}+2A\beta-R^{-1}B^2\beta^2+Q_1)\mathbb{E}x_i^* + (A-R^{-1}B^2\beta)\gamma \\ &\quad + (\beta G - Q_1)\mathbb{E}x_0 + \beta\bar{G} + \bar{Q}_1 - R^{-1}B\beta r = 0 \end{aligned} \tag{63}$$

which implies (20)–(22).

ii) We conjecture that

$$y_i^* = \pi_1 p_i^* + \pi_2 x_i^* + \pi_3, \quad i\in\mathcal{N} \tag{64}$$

where $\pi_1(T) = 0$, $\pi_2(T) = H$, $\pi_3(T) = 0$. Then we obtain

$$\begin{aligned} dy_i^* = \Big[&(\dot{\pi}_1+\pi_1 C_1)p_i^* + (\pi_2 BR^{-1}D - \pi_2 R^{-1}B^2\alpha)\hat{p}_i^* \\ &+ (\dot{\pi}_2+\pi_2 A)x_i^* - \pi_2 R^{-1}B^2\beta\hat{x}_i^* + \dot{\pi}_3 \\ &- \pi_2 R^{-1}B^2\gamma - \pi_2 BR^{-1}r + \pi_2 G x_0 + \pi_2\bar{G}\Big]dt \\ &+ \pi_2\sigma dW_i + (\pi_1 C_2 p_i^* + \pi_2\bar{\sigma})dW. \end{aligned}$$

Comparing the above equality with the second equation of (17), we have

$$z_{ii}^* = \pi_2\sigma,\ z_i^* = \pi_1 C_2 p_i^* + \pi_2\bar{\sigma} \tag{65}$$

$$\begin{aligned} &(\dot{\pi}_1+2C_1\pi_1)p_i^* + R^{-1}(B\pi_2+D)(D-B\alpha)\hat{p}_i^* + \dot{\pi}_3 \\ &\quad + [\dot{\pi}_2+(A+C_1)\pi_2+C_3]x_i^* - R^{-1}B\beta(\pi_2 B+D)\hat{x}_i^* \\ &\quad - R^{-1}B(\pi_2 B+D)\gamma - R^{-1}(\pi_2 B+D)r + (G\pi_2+F)x_0 \\ &\quad + \pi_2\bar{G} + C_1\pi_3 + \bar{F} + C_2 z_i^* = 0. \end{aligned} \tag{66}$$

Putting z_i^* given by (65) into (66) and taking $\mathbb{E}[\cdot]$ on both sides of (66), we obtain (23)–(25).

iii) Set

$$p_i^* = -\pi_4 y_i^* + \pi_5, \quad i\in\mathcal{N} \tag{67}$$

where $\pi_4(0) = K_2, \pi_5(0) = -\bar{K}_2$. Taking $\mathbb{E}[\cdot]$ on both sides of (67),

$$\mathbb{E}p_i^* = -\pi_4\mathbb{E}y_i^* + \pi_5. \tag{68}$$

Differentiating on (68), we get

$$\begin{aligned} d\mathbb{E}p_i^* = \Big\{&[-\dot{\pi}_4 - (C_2^2\pi_1 + R^{-1}D(D-B\alpha))\pi_4^2 \\ &+ C_1\pi_4]\mathbb{E}y_i^* + \dot{\pi}_5 + \pi_4[\pi_2 C_2\bar{\sigma} - R^{-1}D(r+B\gamma) + \bar{F}] \\ &+ (\pi_1 C_2^2 + R^{-1}D^2 - R^{-1}DB\alpha)\pi_4\pi_5 \\ &+ \pi_4(C_3 - R^{-1}DB\beta)\mathbb{E}x_i^* + \pi_4 F\mathbb{E}x_0\Big\}dt. \end{aligned} \tag{69}$$

Utilizing Theorem 4.2 in [38], it holds

$$x_0 = \lim_{N\to+\infty}\frac{1}{N}\sum_{i=1}^N x_i^* = \mathbb{E}\left(x_i^*\middle|\mathcal{F}_t^W\right).$$

Replacing $\mathbb{E}x_i^*$ by $\mathbb{E}x_0$ in (69), we have

$$\begin{aligned} d\mathbb{E}p_i^* = \Big\{&[-\dot{\pi}_4 - (C_2^2\pi_1 + R^{-1}D(D-B\alpha))\pi_4^2 \\ &+ C_1\pi_4]\mathbb{E}y_i^* + \dot{\pi}_5 + \pi_4[\pi_2 C_2\bar{\sigma} - R^{-1}D(r+B\gamma) \\ &+ \bar{F}] + (\pi_1 C_2^2 + R^{-1}D^2 - R^{-1}DB\alpha)\pi_4\pi_5 \\ &+ \pi_4(C_3 - R^{-1}DB\beta + F)\mathbb{E}x_0\Big\}dt \\ = (&-C_1\pi_4\mathbb{E}y_i^* + C_1\pi_5)dt, \end{aligned}$$

which suggests (27) and (28).

iv) Finally, we proceed to give the relationship between y_i^* and x_i^*. By virtue of ii) and iii), we have

$$y_i^* = \pi_1(-\pi_4 y_i^* + \pi_5) + \pi_2 x_i^* + \pi_3$$

which implies

$$y_i^* = (1+\pi_1\pi_4)^{-1}\left(\pi_2 x_i^* + \pi_3 + \pi_1\pi_5\right).$$

■

Appendix D
Proof of Lemma 4

To prove Lemma 4, we first present a lemma (Lemma B1). Let

$$\begin{aligned} &\Theta_1 = A - R^{-1}B^2\beta, \quad \Theta_2 = \beta G - Q_1 \\ &\Theta_3 = \beta\bar{G} + \bar{Q}_1 - R^{-1}B\beta r \\ &\Theta_4 = (-R^{-1}Br + \bar{G} + C_2\bar{\sigma})\pi_2 + \bar{F} - R^{-1}Dr \\ &\Theta_5 = G\pi_2 + F, \quad \Theta_6 = -R^{-1}B(B\pi_2 + D) \\ &\Theta_7 = -(A + G + B\mathbb{A}_2) \\ &\Theta_8 = C_2^2\pi_1\pi_4 + R^{-1}D\pi_4(D - B\alpha) - C_1 \\ &\Theta_9 = \left(C_2\bar{\sigma}\pi_2 - R^{-1}Dr + \bar{F}\right)\pi_4 \\ &\Theta_{10} = -R^{-1}DB\pi_4, \quad \Theta_{11} = (F + C_3 - R^{-1}DB\beta)\pi_4 \\ &\Theta_{14} = (\Theta_3, \Theta_4)^\tau, \quad \Theta_{15} = (\bar{K}_1, 0)^\tau \\ &\Theta_{18} = (\Theta_9, R^{-1}Br - \bar{G})^\tau, \quad \Theta_{19} = (-\bar{K}_2, a_0)^\tau \\ &\Phi = (\gamma, \pi_3)^\tau, \quad \phi = (\pi_5, \mathbb{E}x_0)^\tau \\ &\Theta_{12} = \begin{pmatrix} \Theta_1 & 0 \\ \Theta_6 & C_1 \end{pmatrix}, \quad \Theta_{13} = \begin{pmatrix} 0 & \Theta_2 \\ 0 & \Theta_5 \end{pmatrix} \\ &\Theta_{16} = \begin{pmatrix} \Theta_8 & \Theta_{11} \\ -B\mathbb{A}_3 & \Theta_7 \end{pmatrix}, \quad \Theta_{17} = \begin{pmatrix} \Theta_{10} & 0 \\ R^{-1}B^2 & -B\mathbb{A}_1 \end{pmatrix} \end{aligned}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{H} = \begin{pmatrix} -\Theta_{12} & -\Theta_{13} \\ -\Theta_{17} & -\Theta_{16} \end{pmatrix}$$

and let $\mathbf{0}$ be a zero matrix (or vector). Introduce

$$\dot{\tilde{P}} + \Theta_{16}\tilde{P} - \tilde{P}\Theta_{12} - \tilde{P}\Theta_{13}\tilde{P} + \Theta_{17} = \mathbf{0}, \ \tilde{P}(0) = \mathbf{0} \tag{70}$$

$$\dot{\tilde{Q}} + (\Theta_{16} - \tilde{P}\Theta_{13})\tilde{Q} - \tilde{P}\Theta_{14} + \Theta_{18} = \mathbf{0}, \ \tilde{Q}(0) = \Theta_{19}. \tag{71}$$

Inspired by Theorem 5.12 in [39], we introduce

$$\begin{pmatrix} \dot{U}(t) \\ \dot{V}(t) \end{pmatrix} = \tilde{H}(t)\begin{pmatrix} U(t) \\ V(t) \end{pmatrix}, \quad \begin{pmatrix} U(0) \\ V(0) \end{pmatrix} = \begin{pmatrix} I \\ \mathbf{0} \end{pmatrix}$$

whose unique solution is

$$\begin{pmatrix} U \\ V \end{pmatrix} = e^{\int_0^t \tilde{H}(r)dr}\begin{pmatrix} I \\ \mathbf{0} \end{pmatrix}.$$

Lemma B1: Under Assumptions 1–3, (70) and (71) admit unique solutions.

Proof of Lemma B1: Let $\tilde{P} = VU^{-1}$, which satisfies

$$\dot{\tilde{P}} = -\Theta_{17} - \Theta_{16}\tilde{P} + \tilde{P}\Theta_{12} + \tilde{P}\Theta_{13}\tilde{P}.$$

Then (70) is solvable. Assume that $\tilde{P}_1, \tilde{P}_2$ are two solutions of (70). Set $\hat{\tilde{P}} = \tilde{P}_1 - \tilde{P}_2$. Then $\hat{\tilde{P}}$ satisfies

$$\begin{cases} \dot{\hat{\tilde{P}}} + (\Theta_{16} - \tilde{P}_2\Theta_{13})\hat{\tilde{P}} - \hat{\tilde{P}}(\Theta_{12} + \Theta_{13}\tilde{P}_1) = \mathbf{0} \\ \hat{\tilde{P}}(0) = \mathbf{0}. \end{cases}$$

Gronwall's inequality implies $\hat{\tilde{P}} \equiv \mathbf{0}$. This proves the uniqueness for (70). Hence, (70) is uniquely solvable. Based on this, (71) is uniquely solvable. ■

Proof of Lemma 4: With the above notations, (22), (25), (28) and (35) are written as

$$\begin{cases} \dot{\Phi} + \Theta_{12}\Phi + \Theta_{13}\phi + \Theta_{14} = \mathbf{0}, & \Phi(T) = \Theta_{15} \\ \dot{\phi} + \Theta_{16}\phi + \Theta_{17}\Phi + \Theta_{18} = \mathbf{0}, & \phi(0) = \Theta_{19}. \end{cases} \tag{72}$$

Consider

$$\begin{cases} \dot{\bar{\Phi}} + \Theta_{12}\bar{\Phi} + \Theta_{13}(\tilde{P}\bar{\Phi} + \tilde{Q}) + \Theta_{14} = \mathbf{0} \\ \bar{\Phi}(T) = \Theta_{15} \end{cases} \tag{73}$$

where $\tilde{P}$ and $\tilde{Q}$ are given by (70) and (71). Then (73) admits a unique solution $\bar{\Phi}$. Define $\bar{\phi} = \tilde{P}\bar{\Phi} + \tilde{Q}$. Then $\bar{\phi}$ satisfies

$$\begin{cases} \dot{\bar{\phi}} + \Theta_{16}\bar{\phi} + \Theta_{17}\bar{\Phi} + \Theta_{18} = \mathbf{0} \\ \bar{\phi}(0) = \Theta_{19}. \end{cases}$$

Thus, our claims follow. ■

Appendix E
Proof of Lemma 5

Proof: It follows from (34) that:

$$\begin{aligned} \mathbb{E}|x_0(t)|^2 \le c\mathbb{E}\Big\{&|a_0|^2 + \int_0^T \Big(|B(\mathbb{A}_1\pi_3 + \mathbb{A}_3\pi_5)|^2 \\ &+ \left|R^{-1}B(B\gamma + r)\right|^2 + \left|\bar{G}\right|^2 + |\bar{\sigma}|^2\Big)dt\Big\} \le c. \end{aligned} \tag{74}$$

Here c represents a constant independent of N. Similarly, we get

$$\mathbb{E}\left|x_i^*(t)\right|^2 \leq c\mathbb{E}\left\{|a_{i0}|^2 + \int_0^T \left(\left|\hat{x}_i^*\right|^2 + |x_0|^2\right)dt\right\} \tag{75}$$

and

$$\mathbb{E}\left|\hat{x}_i^*(t)\right|^2 \leq c\mathbb{E}\left\{|a_0|^2 + \int_0^T \left(\left|x_i^*\right|^2 + |x_0|^2\right)dt\right\}. \tag{76}$$

Based on (74)–(76), we derive

$$\mathbb{E}\left[\left|x_i^*(t)\right|^2 + \left|\hat{x}_i^*(t)\right|^2\right] \leq c\mathbb{E}\left[|a_{i0}|^2 + |a_0|^2 + \int_0^T \left(\left|x_i^*\right|^2 + \left|\hat{x}_i^*\right|^2\right)dt\right].$$

Gronwall's inequality indicates that $\sup_{t\in[0,T]}\mathbb{E}|x_i^*(t)|^2$, $\sup_{t\in[0,T]}\mathbb{E}|\hat{x}_i^*(t)|^2$ and $\mathbb{E}\int_0^T |u_i^*(t)|^2 dt$ are bounded. Further, applying the basic estimates of BSDE to the second equation of (37), we obtain

$$\mathbb{E}\left[\left|y_i^*(t)\right|^2 + \int_0^T \left(\left|z_{ii}^*\right|^2 + \left|z_i^*\right|^2\right)dt\right] \leq c\mathbb{E}\left\{|x_i^*(T)|^2 + \int_0^T \left(\left|x_i^*\right|^2 + |u_i^*|^2 + |x_0|^2 + \left|\bar{F}\right|^2\right)dt\right\}.$$

On account of the $\mathcal{L}^2$-boundedness of x_i^*, u_i^*, x_0,

$$\mathbb{E}\left[\left|y_i^*(t)\right|^2 + \int_0^T \left(\left|z_{ii}^*\right|^2 + \left|z_i^*\right|^2\right)dt\right] \leq c.$$

Then, we draw the desired conclusion. ∎

Appendix F
Proof of Lemma 6

Proof: By (36)–(38), we get

$$dX^{*(N)} = \Big\{(A+G)X^{*(N)} + B\Big[\mathbb{A}_2\widehat{x^*}^{(N)} + \mathbb{A}_1\pi_3 + \mathbb{A}_3\pi_5 - R^{-1}(B\gamma + r)\Big] + \bar{G}\Big\}dt + \sigma\frac{1}{N}\sum_{i=1}^N dW_i + \bar{\sigma}dW, \quad X^{*(N)}(0) = a_0^{(N)} = \frac{1}{N}\sum_{i=1}^N a_{i0}$$

$$dx^{*(N)} = \Big\{Ax^{*(N)} + B\Big[\mathbb{A}_2\hat{x}^{*(N)} + \mathbb{A}_1\pi_3 + \mathbb{A}_3\pi_5 - R^{-1}(B\gamma + r)\Big] + Gx_0 + \bar{G}\Big\}dt + \sigma\frac{1}{N}\sum_{i=1}^N dW_i + \bar{\sigma}dW, \quad x^{*(N)}(0) = a_0^{(N)}$$

and

$$d\hat{x}^{*(N)} = \Big\{(A+B\mathbb{A}_2)\hat{x}^{*(N)} + B\Big[\mathbb{A}_1\pi_3 + \mathbb{A}_3\pi_5 - R^{-1}(B\gamma + r)\Big] + Gx_0 + \bar{G} + (\sigma + fP)f\left(x^{*(N)} - \hat{x}^{*(N)}\right)\Big\}dt + (\sigma + fP)\frac{1}{N}\sum_{i=1}^N dW_i + \bar{\sigma}dW, \quad \hat{x}^{*(N)}(0) = a_0.$$

Recalling (34), it yields that

$$\begin{cases} d\left(X^{*(N)} - x_0\right) = \Big[(A+G)\left(X^{*(N)} - x_0\right) + B\mathbb{A}_2\left(\hat{x}^{*(N)} - x_0\right)\Big]dt + \sigma\frac{1}{N}\sum_{i=1}^N dW_i \\ X^{*(N)}(0) - x_0(0) = a_0^{(N)} - a_0 \end{cases} \tag{77}$$

$$\begin{cases} d\left(x^{*(N)} - x_0\right) = \Big[A\left(x^{*(N)} - x_0\right) + B\mathbb{A}_2\left(\hat{x}^{*(N)} - x_0\right)\Big]dt + \sigma\frac{1}{N}\sum_{i=1}^N dW_i \\ x^{*(N)}(0) - x_0(0) = a_0^{(N)} - a_0 \end{cases} \tag{78}$$

and

$$\begin{cases} d\left(\hat{x}^{*(N)} - x_0\right) = \Big[(A+B\mathbb{A}_2)\left(\hat{x}^{*(N)} - x_0\right) + (\sigma + fP)f\left(x^{*(N)} - \hat{x}^{*(N)}\right)\Big]dt + (\sigma + fP)\frac{1}{N}\sum_{i=1}^N dW_i \\ \hat{x}^{*(N)}(0) - x_0(0) = 0. \end{cases} \tag{79}$$

Taking squares and mathematical expectations on both sides of (77)–(79) in integral forms, we arrive at

$$\mathbb{E}\left|X^{*(N)}(t) - x_0(t)\right|^2 \leq c\left[\mathbb{E}\int_0^t\left(\left|X^{*(N)} - x_0\right|^2 + \left|\hat{x}^{*(N)} - x_0\right|^2\right)ds + \mathbb{E}\left|a_0^{(N)} - a_0\right|^2 + \Gamma_t\right] \tag{80}$$

$$\mathbb{E}\left|x^{*(N)}(t) - x_0(t)\right|^2 \leq c\left[\mathbb{E}\int_0^t\left(\left|\hat{x}^{*(N)} - x_0\right|^2 + \left|x^{*(N)} - x_0\right|^2\right)ds + \mathbb{E}\left|a_0^{(N)} - a_0\right|^2 + \Gamma_t\right] \tag{81}$$

$$\mathbb{E}\left|\hat{x}^{*(N)}(t) - x_0(t)\right|^2 \leq c\left[\mathbb{E}\int_0^t\left(\left|\hat{x}^{*(N)} - x_0\right|^2 + \mathbb{E}\left|x^{*(N)} - x_0\right|^2\right)ds + \Gamma_t\right] \tag{82}$$

where

$$\Gamma_t\mathbb{E}\left|\int_0^t \frac{1}{N}\sum_{i=1}^N dW_i\right|^2 = O\left(\frac{1}{N}\right)$$

$$\mathbb{E}\left|a_0^{(N)} - a_0\right|^2 = O\left(\frac{1}{N}\right). \tag{83}$$

It follows from Gronwall's inequality and (80)–(83) that (39)–(41) hold. ∎

Appendix G
Proof of Lemma 7

Proof: In accordance with (36) and (37), we get

$$\begin{cases} d\left(\mathcal{X}_i^*-x_i^*\right)=\left[A\left(\mathcal{X}_i^*-x_i^*\right)+G\left(\mathcal{X}^{*(N)}-x_0\right)\right]dt \\ \mathcal{X}_i^*(0)-x_i^*(0)=0 \end{cases} \tag{84}$$

$$\begin{cases} -d\left(\mathcal{Y}_i^*-y_i^*\right)=\Big[C_1\left(\mathcal{Y}_i^*-y_i^*\right)+C_2\left(\mathcal{Z}_i^*-z_i^*\right) \\ \qquad +C_3\left(\mathcal{X}_i^*-x_i^*\right)+F\left(\mathcal{X}^{*(N)}-x_0\right)\Big]dt \\ \qquad -\sum_{j\neq i}^{N}\mathcal{Z}_{ij}^*dW_j-\left(\mathcal{Z}_{ii}^*-z_{ii}^*\right)dW_i-\left(\mathcal{Z}_i^*-z_i^*\right)dW \\ \mathcal{Y}_i^*(T)-y_i^*(T)=H\left[\mathcal{X}_i^*(T)-x_i^*(T)\right]. \end{cases} \tag{85}$$

It follows from (84) and Gronwall's inequality that (42) holds. Applying some estimate techniques of BSDE to (85), we obtain

$$\mathbb{E}\left|\mathcal{Y}_i^*(t)-y_i^*(t)\right|^2\leq c\mathbb{E}\Big[\left|\mathcal{X}_i^*(T)-x_i^*(T)\right|^2 \\ +\int_0^T\left(\left|\mathcal{X}_i^*-x_i^*\right|^2+\left|\mathcal{X}^{*(N)}-x_0\right|^2\right)dt\Big].$$

Since

$$\sup_{t\in[0,T]}\mathbb{E}\left|\mathcal{X}^{*(N)}(t)-x_0(t)\right|^2=O\left(\frac{1}{N}\right)$$

$$\sup_{i\in\mathcal{N}}\sup_{t\in[0,T]}\mathbb{E}\left|\mathcal{X}_i^*(t)-x_i^*(t)\right|^2=O\left(\frac{1}{N}\right).$$

Gronwall's inequality suggests that

$$\sup_{i\in\mathcal{N}}\sup_{t\in[0,T]}\mathbb{E}\left|\mathcal{Y}_i^*(t)-y_i^*(t)\right|^2=O\left(\frac{1}{N}\right).$$

■

Appendix H
Proof of Lemma 8

Proof:

$$\mathcal{J}_i(u_i^*,u_{-i}^*)-J_i(u_i^*)=\frac{1}{2}\mathbb{E}\Big\{\int_0^T\Big[Q_1\Big(\left(\mathcal{X}_i^*-\mathcal{X}^{*(N)}\right)^2 \\ -(x_i^*-x_0)^2\Big)+2\bar{Q}_1\Big(\left(\mathcal{X}_i^*-\mathcal{X}^{*(N)}\right)(x_i^*-x_0)\Big)\Big]dt \\ +K_1\left[\left(\mathcal{X}_i^*(T)\right)^2-(x_i^*(T))^2\right]+2\bar{K}_1\left[\mathcal{X}_i^*(T)-x_i^*(T)\right] \\ +K_2\left[\left(\mathcal{Y}_i^*(0)\right)^2-(y_i^*(0))^2\right]+2\bar{K}_2\left[\mathcal{Y}_i^*(0)-y_i^*(0)\right]\Big\}$$

where $i\in\mathcal{N}$. Utilizing (39) and (42) with Hölder's inequality, we have

$$\sup_{t\in[0,T]}\mathbb{E}\left|\left(\mathcal{X}_i^*-\mathcal{X}^{*(N)}\right)^2-(x_i^*-x_0)^2\right|=O\left(\frac{1}{\sqrt{N}}\right). \tag{86}$$

Evidently,

$$\sup_{t\in[0,T]}\mathbb{E}\left|\left(\mathcal{X}_i^*-\mathcal{X}^{*(N)}\right)(x_i^*-x_0)\right|=O\left(\frac{1}{\sqrt{N}}\right). \tag{87}$$

Similarly, (42) and (43) imply

$$\begin{cases} \mathbb{E}\left|\left(\mathcal{X}_i^*(T)\right)^2-(x_i^*(T))^2\right|=O\left(\frac{1}{\sqrt{N}}\right) \\ \mathbb{E}\left|\mathcal{X}_i^*(T)-x_i^*(T)\right|=O\left(\frac{1}{\sqrt{N}}\right) \\ \mathbb{E}\left|\left(\mathcal{Y}_i^*(0)\right)^2-(y_i^*(0))^2\right|=O\left(\frac{1}{\sqrt{N}}\right) \\ \mathbb{E}\left|\mathcal{Y}_i^*(0)-y_i^*(0)\right|=O\left(\frac{1}{\sqrt{N}}\right). \end{cases} \tag{88}$$

Based on (86)–(88), our result holds. ■

Appendix I
Proof of Lemma 9

Proof: It follows from the first equations of (49) and (50) that:

$$\mathbb{E}\left|l_i^0(t)\right|^2\leq c\mathbb{E}\Big[|a_{i0}|^2+\int_0^T\left(\left|l_i^0\right|^2+|u_i|^2+|x_0|^2\right)dt\Big]$$

and

$$\mathbb{E}\left|l_j^0(t)\right|^2\leq c\mathbb{E}\Big[|a_{j0}|^2+\int_0^T\left(\left|l_j^0\right|^2+\left|u_j^*\right|^2+|x_0|^2\right)dt\Big]$$

where $i,j\in\mathcal{N},\ j\neq i$. Then we obtain

$$\mathbb{E}\sum_{k=1}^{N}\left|l_k^0(t)\right|^2\leq c\mathbb{E}\Big\{N\max_{1\leq k\leq N}|a_{k0}|^2+\int_0^T\Big(\sum_{k=1}^{N}\left|l_k^0\right|^2 \\ +\sum_{k\neq i}^{N}\left|u_k^*\right|^2+|u_i|^2+N|x_0|^2\Big)dt\Big\}.$$

By the $\mathcal{L}^2$-boundedness of x_0, u_i, u_j^* $(j\neq i)$ and Gronwall's inequality, we arrive at

$$\mathbb{E}\sum_{j=1}^{N}\left|l_j^0(t)\right|^2=O(N)$$

and $\sup_{1\leq i\leq N}\sup_{t\in[0,T]}\mathbb{E}|l_i^0(t)|^2\leq c$. Analogously, by the estimates of BSDE, we derive that for any $i\in\mathcal{N}$, $\sup_{t\in[0,T]}\mathbb{E}\left|m_i^0(t)\right|^2$, $\mathbb{E}\int_0^T|n_{ii}^0(t)|^2dt$ and $\mathbb{E}\int_0^T|n_i^0(t)|^2dt$ are bounded. ■

References

[1] J. M. Lasry and P. L. Lions, "Mean field games," *Jpn. J. Math.*, vol. 2, pp. 229–260, Mar. 2007.

[2] M. Huang, P. E. Caines, and R. P. Malhamé, "Large-population cost-coupled LQG problems with non-uniform agents: Individual-mass behavior and decentralized ε-Nash equilibria," *IEEE Trans. Autom. Control*, vol. 52, pp. 1560–1571, Sept. 2007.

[3] R. Carmona, F. Delarue, and D. Lacker, "Mean field games with common noise," *Ann. Probab.*, vol. 44, no. 6, pp. 3740–3803, 2016.

[4] A. Aurell, R. Carmona, G. Dayanikli, and M. Lauriere, "Optimal incentives to mitigate epidemics: A Stackelberg mean field game approach," *SIAM J. Control Optim.*, vol. 60, no. 2, pp. 294–322, Apr. 2022.

[5] J. Moon and T. Başar, "Linear quadratic mean field Stackelberg differential games," *Automatica*, vol. 97, pp. 200–213, Dec. 2018.

[6] A. Bensoussan, K. C. Sung, S. C. Yam, and S. P. Yung, "Linear-quadratic mean field games," *J. Optim. Theory Appl.*, vol. 169, no. 2, pp. 496–529, 2016.

[7] J. Moon and T. Başar, "Linear quadratic risk-sensitive and robust mean field games," *IEEE Trans. Autom. Control*, vol. 62, no. 3, pp. 1062–1077, Mar. 2017.

[8] R. Xu and F. Zhang, "ϵ-Nash mean-field games for general linear-quadratic systems with applications," *Automatica*, vol. 114, p. 108835, Jan. 2020.

[9] M. Li, N. Li, and Z. Wu, "Linear-quadratic mean-field game for stochastic large-population systems with jump diffusion," *IET Control Theory Appl.*, vol. 14, no. 3, pp. 481–489, Feb. 2020.

[10] B. Wang and J. Zhang, "Distributed output feedback control of Markov jump multi-agent systems," *Automatica*, vol. 49, no. 5, pp. 1397–1402, May 2013.

[11] Y. Xu, Y. Yuan, Z. Wang, and X. Li, "Noncooperative model predictive game with Markov jump graph," *IEEE/CAA J. Autom. Sinica*, vol. 10, no. 4, pp. 1–14, Apr. 2022.

[12] Y. Achdou, P. Mannucci, C. Marchi, and N. Tchou, "Deterministic mean field games with control on the acceleration and state constraints," *SIAM J. Math. Anal.*, vol. 54, no. 3, pp. 3757–3788, Jun. 2022.

[13] Y. Hu, J. Huang, and T. Nie, "Linear-quadratic-Gaussian mixed mean-field games with heterogeneous input constraints," *SIAM J. Control Optim.*, vol. 56, no. 4, pp. 2835–2877, 2018.

[14] J. F. Bonnans, J. Gianatti, and L. Pfeiffer, "A Lagrangian approach for aggregative mean field games of controls with mixed and final constraints," *SIAM J. Control Optim.*, vol. 61, no. 1, pp. 105–134, Feb. 2023.

[15] M. Huang, P. Caines, and R. Malhamé, "Social optima in mean field LQG control: Centralized and decentralized strategies," *IEEE Trans. Autom. Control*, vol. 57, no. 7, pp. 1736–1751, Jul. 2012.

[16] B. Wang, H. Zhang, and J. Zhang, "Mean field linear quadratic control: Uniform stabilization and social optimality," *Automatica*, vol. 121, pp. 1–14, Nov. 2020.

[17] K. Du, J. Huang, and Z. Wu, "Linear quadratic mean-field-game of backward stochastic differential systems," *Math. Control Relat. Fields*, vol. 8, pp. 653–678, 2018.

[18] J. Huang, S. Wang, and Z. Wu, "Backward mean-field linear-quadratic-Gaussian (LQG) games: Full and partial information," *IEEE Trans. Autom. Control*, vol. 61, no. 12, pp. 3784–3796, Dec. 2016.

[19] M. Ye, D. Li, Q.-L. Han, and L. Ding, "Distributed Nash equilibrium seeking for general networked games with bounded disturbances," *IEEE/CAA J. Autom. Sinica*, vol. 10, no. 2, pp. 376–387, Feb. 2023.

[20] B. Djehiche, H. Tembine, and R. Tempone, "A stochastic maximum principle for risk-sensitive mean-field type control," *IEEE Trans. Autom. Control*, vol. 60, no. 10, pp. 2640–2649, Feb. 2015.

[21] M. Hafayed and S. Meherrem, "On optimal control of mean-field stochastic systems driven by Teugels martingales via derivative with respect to measures," *Internat. J. Control*, vol. 93, no. 5, pp. 1053–1062, 2020.

[22] Y. Ni, J. Zhang, and M. Krstic, "Time-inconsistent mean-field stochastic LQ problem: Open-loop time-consistent control," *IEEE Trans. Autom. Control*, vol. 63, no. 9, pp. 2771–2786, Sept. 2018.

[23] B. Yang and J. Wu, "Infinite horizon optimal control for mean-field stochastic delay systems driven by Teugels martingales under partial information," *Optim. Control Appl. Meth.*, vol. 41, pp. 1371–1397, 2020.

[24] J. Yong, "A linear-quadratic optimal control problem for mean-field stochastic differential equations," *SIAM J. Control Optim.*, vol. 51, no. 4, pp. 2809–2838, 2013.

[25] H. Wang, C. Zhang, K. Li, and X. Ma, "Game theory-based multi-agent capacity optimization for integrated energy systems with compressed air energy storage," *Energy*, vol. 221, p. 119777, Apr. 2021.

[26] S. Wu, "Partially-observed maximum principle for backward stochastic differential delay equations," *IEEE/CAA J. Autom. Sinica*, pp. 1–6, Mar. 2017. DOI: 10.1109/JAS.2017.7510472.

[27] H. Ma and B. Liu, "Linear-quadratic optimal control problem for partially observed forward-backward stochastic differential equations of mean-field type," *Asian J. Control*, vol. 18, no. 6, pp. 2146–2157, May 2016.

[28] M. Wang, Q. Shi, and Q. Meng, "Optimal control of forward-backward stochastic jump-diffusion differential systems with observation noises: Stochastic maximum principle," *Asian J. Control*, vol. 23, no. 1, pp. 241–254, 2021.

[29] Y. Wang and L. Wang, "Forward-backward stochastic differential games for optimal investment and dividend problem of an insurer under model uncertainty," *Appl. Math. Model.*, vol. 58, pp. 254–269, Jun. 2018.

[30] S. Zhang, J. Xiong, and X. Liu, "Stochastic maximum principle for partially observed forward-backward stochastic differential equations with jumps and regime switching," *Sci. China Inf. Sci.*, vol. 61, no. 7, p. 070211, Jul. 2018.

[31] S. Wang and H. Xiao, "Individual and mass behavior in large population forward-backward stochastic control problems: Centralized and Nash equilibrium solutions," *Optim. Control Appl. Meth.*, vol. 42, pp. 1269–1292, Apr. 2021.

[32] A. Bensoussan, X. Feng, and J. Huang, "Linear-quadratic-Gaussian mean-field-game with partial observation and common noise," *Math. Control Relat. Fields*, vol. 11, no. 1, pp. 23–46, 2021.

[33] P. Huang, G. Wang, W. Wang, and Y. Wang, "A linear-quadratic mean-field game of backward stochastic differential equation with partial information and common noise," *Appl. Math. Comput.*, vol. 446, p. 127899, 2023.

[34] G. Wang, Z. Wu, and J. Xiong, "A linear-quadratic optimal control problem of forward-backward stochastic differential equations with partial information," *IEEE Trans. Autom. Control*, vol. 60, no. 11, pp. 2904–2916, Nov. 2015.

[35] G. Wang, Z. Wu, and J. Xiong, *An Introduction to Optimal Control of FBSDE With Incomplete Information*. New York: Springer-Verlag, 2018.

[36] A. E. B. Lim and X. Zhou, "Linear-quadratic control of backward stochastic differential equations," *SIAM J. Control Optim.*, vol. 40, no. 2, pp. 450–474, 2001.

[37] E. Pardoux, and S. Peng, "Adapted solution of backward stochastic equation," *Systems Control Lett.*, vol. 14, no. 1, pp. 55–61, Jan. 1990.

[38] D. Majerek, W. Nowak, and W. Zieba, "Conditional strong law of large number," *Int. J. Pure Appl. Math.*, vol. 20, no. 2, pp. 143–156, 2005.

[39] J. Engwerda, *LQ Dynamic Optimization and Differential Games*. Chichester, West Sussex, England: John Wiley and Sons Ltd, 2005.

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