

# The Enhancement of Low Quality Fingerprint Based on Fractional Calculus Mask

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## Abstract

*The performance of automatic fingerprint recognition systems highly depends on the quality of the fingerprint images. So it is important to implement a fingerprint image enhancement process before feature extraction and matching stages. In this paper, we propose an effective filtering method based on a fractional calculus mask to enhance fingerprint image. First an eight direction image is obtained in a convenient way. Then based on fractional calculus, a mask has been got on the eight direction of negative x-coordinate, positive x-coordinate, negative y-coordinate, positive y-coordinate, left downward diagonal, left upward diagonal, right downward diagonal, and right upward diagonal. Finally the mask is used to filter and enhance fingerprint images according to eight direction image. The proposed algorithm has been evaluated on the databases of fingerprint verification competition 2004DB3 (FVC2004). The experimental results confirm that the proposed algorithm is effective for low quality fingerprint image enhancement.*

## 1. Introduction

Fingerprint recognition is one of the most popular and reliable biometric techniques and is widely used in many important applications such as electronic personal identification card, e-commerce. Fingerprint recognition has been studied for many years and various recognition techniques, including acquisition, classification, enhancement and matching are rapidly developed and advanced. However, there are still many challenging tasks in this field and the enhancement of low quality fingerprint is one of the important issues [1]. The quality of fingerprint image degrades due to the factors such as scars, non-uniform contact with the sensors, and the environmental condition during the capturing process, etc. Fig. 1 shows three examples of the low

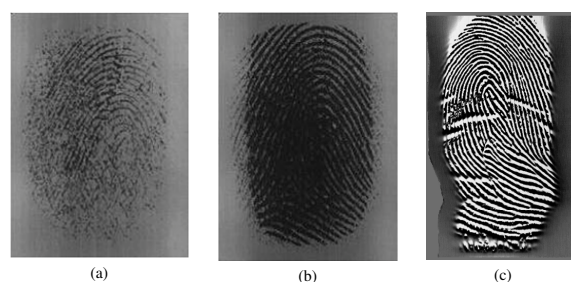


Figure 1. Three examples of low quality fingerprints. (a) too dry, (b) too wet, (c) with many scars.

quality fingerprint images. The poor quality fingerprints seriously affect the whole recognition system's performance.

During the past decades, Several methods have been proposed for the enhancement of fingerprint images, which can be broadly categorized as spatial filtering-based [2][3], and frequency filtering-based [4][5][6] [7][8][9]. Spatial filtering deals with the pixels of image directly. In a pixel-wise image processing operation the new value of each pixel only depends on its previous value and some global parameters (but not on the value of the neighboring pixels). It has the advantage of simple, intuitive and easy to analysis, but it ignores the frequency information of fingerprint and does not produce satisfying and definitive results for fingerprint image enhancement. Frequency filtering is the most widely used technique for fingerprint image enhancement. The Hong, Wan, and Jain [4] proposed an effective method based on Gabor filters. Gabor filters have both frequency-selective and orientation-selective properties and have optimal joint resolution in both spatial and frequency domains. Yang et al. [5] developed a modified Gabor filter to enhance fingerprint, the parameters were computed through some principles instead of experience. Wang et al. [6] suggested replacing standard Gabor filter with Log-Gabor filter to overcome the drawbacks that the maximum bandwidth

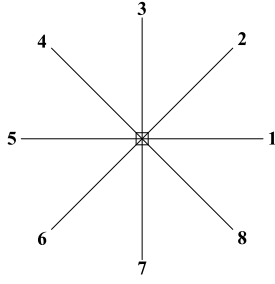


Figure 2. Illustration of eight direction.

of a Gabor filter is limited to approximately one octave and Gabor filters are not optimal if one is seeking broad spectral information with maximal spatial localization. Jiangang and Jie [7] used the scale space theory in the computer vision to enhance the fingerprint. Chikkerur, Cartwright, and Govindaraju [8] proposed an efficient implementation of frequency filtering based on short-time Fourier transform (STFT) that requires partitioning the image into small overlapping blocks and performing Fourier analysis separately on each block. Fronthaler et al. [9] used a Laplacian like image-scale pyramid to decompose the original fingerprint into three smaller images corresponding to different frequency bands. Each image was then processed through frequency filtering. Frequency filtering had noise disturbance in calculating the orientation of ridges and frequency and its precision is low. In this paper, we proposed a new fingerprint image enhancement method based on fractional calculus mask. Compared with tradition frequency filtering technology of image processing, this method has many advantages, such as not requiring estimation of the ridge frequency component and Gaussian parameters, simple and easy application, high performance and good practicality [10].

The remainder of this paper is organized as follows: Section 2 indicates the details of the enhancement of fingerprint image. Section 3 provides the experimental results of our algorithm. We summarize our work in section 4.

## 2. Proposed enhancement method

The Implementation of the proposed enhancement algorithm works in three stages: eight direction image computation, generation of fractional calculus mask and finally fingerprint filtering.

### 2.1. Eight direction image computation

The fingerprint direction image represents the local orientation of the ridges. The estimation of the direction image is critical to fingerprint image enhancement. To estimate the direction field, we divide the ridge direction of a pixel into 8 directions (Fig. 2).

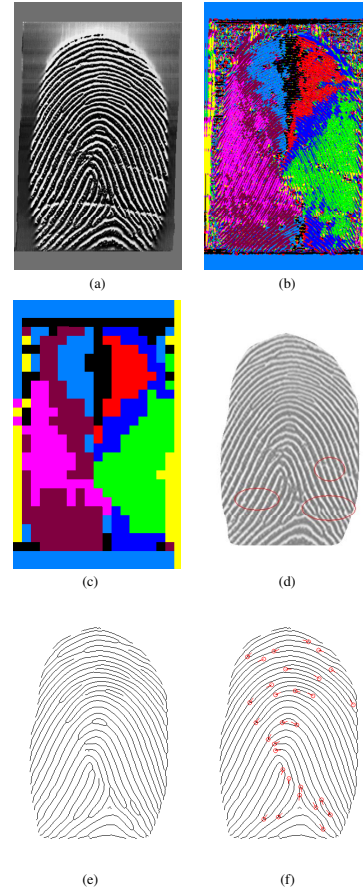


Figure 3. A fingerprint image and its enhancement results. (a)original fingerprint, (b) pixel eight direction image, (c) block eight direction image, (d) enhancement image of (a), (e)thinning image and (f) minutiae image.

The direction  $D(i, j)$  at a pixel  $(i, j)$  in a image is computed as follows [11]. First, we compute  $S_d$ , the average gray values of differences in direction  $d$  ( $d = 1, 2, \dots, 8$  means one of the 8 directions).

$$S_d = \frac{1}{N} \sum_{k=1, N} |f(i, j) - f_d(i_k, j_k)| \quad (1)$$

where  $f(i, j)$  and  $f_d(i_k, j_k)$  are the gray values at pixels  $(i, j)$  and  $(i_k, j_k)$  respectively.  $f_d(i_k, j_k)$  is the  $k$ th pixel in direction  $d$  from  $(i, j)$ ,  $N$  ( $N = 16$  in our method) is the number of pixels chosen for this computation. The direction  $D(i, j)$  at a pixel  $(i, j)$  is the direction for which  $S_d$  is minimum. The variation of the gray values described in the above expression is expected to be smallest in the direction of ridges, and to be the largest along the orthogonal to the ridge direction. Thus, the direction  $D(i, j)$  at a point  $(i, j)$  indicates the direction of maximum gray level uniformity in the image. The direction image  $D$  computed above is called the pixel-wise direction image. Fig. 3(a) is an fin-

gerprint image, whose pixel-wise eight direction image is given in Fig. 3(b).

To reduce the effect of noise, we partition the image into small blocks of size  $(16 \times 16)$  in our method and set the ridge direction of each pixel in one block as the direction of that block, that is, the mean direction of all the pixels in the block. With a given block, we calculate the number of pixels in the block where ridge direction is estimated as  $d$  ( $d = 1, 2, \dots, 8$ ). The direction which occurs maximum number of times is chosen as the direction of the block. The image so constructed from the block directions is called the block direction image.

However, some of the block directions are grossly incoherent with the surrounding block directions. This happens due to the presence of noisy regions. It is observed that the natural block directions do not change abruptly. To minimize the effect of such noisy block directions, the block direction image is smoothed. This is done by examining the  $3 \times 3$  neighborhoods of each of the block directions. So, any violation of this rule is a noise and is corrected by replacing the center block by the majority direction in the neighborhood. The block direction smoothing provides an additional handle to restore the fingerprint images at block level, unlike many smoothing techniques at pixel level. This results in improving the quality of fingerprint images substantially. The block direction image is given in Fig. 3(c).

## 2.2. Generation of fractional calculus mask

In recent years, the fractional calculus has become a hotspot of research at home and abroad. As we known, there are two different definitions: Riemann-Liouville definition and Grunwald-Letnikova definition [13]. The fractional calculus functions have two obvious features. One is that for major functions, it is a power function, and the other is that for the rest functions. Hence high frequency signals are pulled largely, while low frequency signals are affected little. In other words, it can enhance details of an image and make the texture clearer. Here we use Grunwald-Letnikova definition for fingerprint enhancement.

$v$ -order Grunwald-Letnikova-based fractional calculus can be expressed by:

$$\begin{aligned} D^v s(x) &= \frac{d^v}{[d(x-\alpha)]^v} s(x) \\ &= \lim_{N \rightarrow \infty} \left\{ \frac{\left(\frac{x-\alpha}{N}\right)^{-v}}{\Gamma(-v)} \sum_{k=0}^{N-1} \frac{\Gamma(k-v)}{\Gamma(k+1)} \right. \\ &\quad \left. \times s\left(x - k\left(\frac{x-\alpha}{N}\right)\right) \right\} \end{aligned} \quad (3)$$

where the duration of signal  $s(x)$  is  $[\alpha, x]$ , and  $v$  is any real number.  $D^v s(x)$  denotes Grunwald-Letnikova-based fractional calculus operator and  $v$  is the order. Without losing generality, suppose  $\alpha = 0$ . Hence the duration belongs to  $[0, x]$ . We divide the duration of  $s(x)$  into  $N$  equal intervals. Thus, there are  $N + 1$  nodes which can be given by  $s(0), s(x/N), \dots, s(x - kx/n), \dots, s(x)$ .

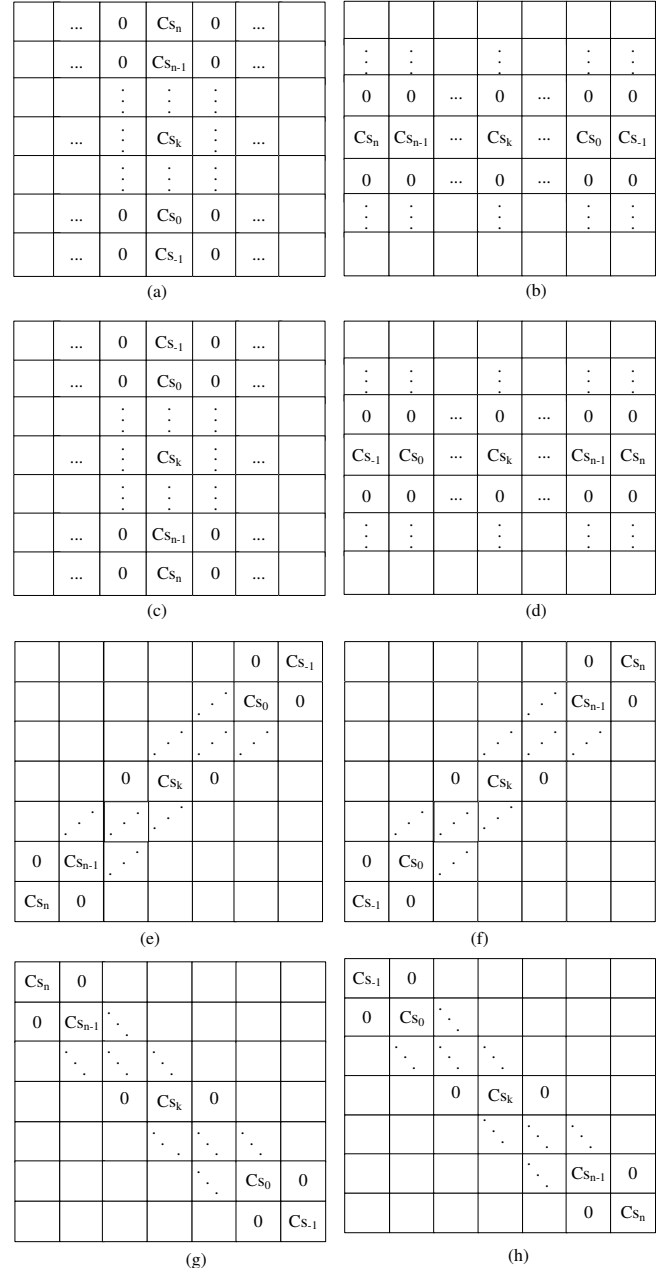


Figure 4. Fractional calculus mask on eight directions. (a)  $W_x^-$ , (b)  $W_y^-$ , (c)  $W_x^+$ , (d)  $W_y^+$ , (e)  $W_{LDD}$ , (f)  $W_{RUD}$ , (g)  $W_{LUD}$ , (h)  $W_{RDD}$ .

When  $N$  is big enough and to capture faster speed and precision of convergence, we can get rid of the limits symbol and rewrite (3) as (4) [13]:

$$\frac{d^v}{dx^v} s(x) \cong \frac{x^{-v} N^v}{\Gamma(-v)} \sum_{k=0}^{N-1} \frac{\Gamma(k-v)}{\Gamma(k+1)} s\left(x + \frac{vx}{2N} - \frac{kx}{N}\right) \quad (4)$$

It is the proximate expression, which simplifies fractional calculus to multiplication and add. For 2-D image

signal  $s(x, y)$ , it has the followings two expressions on x-coordinate and y-coordinate respectively [12]:

$$\begin{aligned} \frac{\partial^v s(x, y)}{\partial x^v} \cong & \left(\frac{v}{4} + \frac{v^2}{8}\right)s(x+1, y) \\ & + \left(1 - \frac{v^2}{2} - \frac{v^3}{8}\right)s(x, y) + \frac{1}{\Gamma(-v)} \\ & \times \sum_{k=1}^{n-2} \left[ \frac{\Gamma(k-v+1)}{(k+1)!} \cdot \left(\frac{v}{4} + \frac{v^2}{8}\right) + \frac{\Gamma(k-v)}{k!} \right. \\ & \cdot \left. \left(1 - \frac{v^2}{4}\right) + \frac{\Gamma(k-v-1)}{(k-1)!} \cdot \left(-\frac{v}{4} + \frac{v^2}{8}\right) \right] \\ & \times s(x-k, y) \\ & + \left[ \frac{\Gamma(n-v-1)}{(n-1)!\Gamma(-v)} \cdot \left(1 - \frac{v^2}{4}\right) + \frac{\Gamma(n-v-2)}{(n-2)!\Gamma(-v)} \right. \\ & \cdot \left. \left(-\frac{v}{4} + \frac{v^2}{8}\right) \right] s(x-n+1, y) \\ & + \frac{\Gamma(n-v-1)}{(n-1)!\Gamma(-v)} \cdot \left(-\frac{v}{4} + \frac{v^2}{8}\right) s(x-n, y) \end{aligned} \quad (5)$$

$$\begin{aligned} \frac{\partial^v s(x, y)}{\partial y^v} \cong & \left(\frac{v}{4} + \frac{v^2}{8}\right)s(x, y+1) \\ & + \left(1 - \frac{v^2}{2} - \frac{v^3}{8}\right)s(x, y) + \frac{1}{\Gamma(-v)} \\ & \times \sum_{k=1}^{n-2} \left[ \frac{\Gamma(k-v+1)}{(k+1)!} \cdot \left(\frac{v}{4} + \frac{v^2}{8}\right) + \frac{\Gamma(k-v)}{k!} \right. \\ & \cdot \left. \left(1 - \frac{v^2}{4}\right) + \frac{\Gamma(k-v-1)}{(k-1)!} \cdot \left(-\frac{v}{4} + \frac{v^2}{8}\right) \right] \\ & \times s(x, y-k) \\ & + \left[ \frac{\Gamma(n-v-1)}{(n-1)!\Gamma(-v)} \cdot \left(1 - \frac{v^2}{4}\right) + \frac{\Gamma(n-v-2)}{(n-2)!\Gamma(-v)} \right. \\ & \cdot \left. \left(-\frac{v}{4} + \frac{v^2}{8}\right) \right] s(x, y-n+1) \\ & + \frac{\Gamma(n-v-1)}{(n-1)!\Gamma(-v)} \cdot \left(-\frac{v}{4} + \frac{v^2}{8}\right) s(x, y-n) \end{aligned} \quad (6)$$

To speed up enhancement, a eight directions and symmetric fractional calculus masks which are respectively on the directions of negative x-coordinate, negative y-coordinate, positive x-coordinate, positive y-coordinate, left downward diagonal, right upward diagonal, left upward diagonal, and right downward diagonal are obtained [12] (see Fig. 4). From (5) and (6), we can get the mask coefficients of fractional calculus given by:

$$\begin{aligned} C_{s_{-1}} &= \frac{v}{4} + \frac{v^2}{8} \\ C_{s_0} &= 1 - \frac{v^2}{2} - \frac{v^3}{8} \\ &\dots \\ C_{s_k} &= \frac{1}{\Gamma(-v)} \left[ \frac{\Gamma(k-v+1)}{(k+1)!} \cdot \left(\frac{v}{4} + \frac{v^2}{8}\right) + \frac{\Gamma(k-v)}{k!} \cdot \left(1 - \frac{v^2}{4}\right) \right. \\ &\quad \left. + \frac{\Gamma(k-v-1)}{(k-1)!} \cdot \left(-\frac{v}{4} + \frac{v^2}{8}\right) \right] \\ C_{s_{n-1}} &= \frac{\Gamma(n-v-1)}{(n-1)!\Gamma(-v)} \cdot \left(1 - \frac{v^2}{4}\right) + \frac{\Gamma(n-v-2)}{(n-2)!\Gamma(-v)} \\ &\quad \cdot \left(-\frac{v}{4} + \frac{v^2}{8}\right) \\ C_{s_n} &= \frac{\Gamma(n-v-1)}{(n-1)!\Gamma(-v)} \cdot \left(-\frac{v}{4} + \frac{v^2}{8}\right) \end{aligned} \quad (7)$$

The mask coefficients of fractional calculus is a sparse matrix, whose coefficient has only  $n+2$  nonzero numbers. All the coefficients are the function of fraction calculus order  $v$ . In our method, we choose  $v = 0.85$  and  $n = 5$ .

### 2.3. Fingerprint filtering

In this stage, we do convoluting filtering using the above eight fractional calculus masks based on eight direction image. First we partition the image into small blocks of size

$(16 \times 16)$ . Then We enhance every block image following the ridge tendency. For example, if the block direction d is 2, the ridge tendency is right upward diagonal and we choose  $W_{RUD}$  to filter this block image(1 choosing  $W_y^+$ , 2 choosing  $W_{RUD}$ , 3 choosing  $W_x^-$ , 4 choosing  $W_{LUD}$ , 5 choosing  $W_y^-$ , 6 choosing  $W_{LDD}$ , 7 choosing  $W_x^+$  and 8 choosing  $W_{RDD}$ ).

The algorithms of  $W_x^-$ ,  $W_y^-$ ,  $W_x^+$ ,  $W_y^+$ ,  $W_{LDD}$ ,  $W_{RUD}$ ,  $W_{LUD}$ ,  $W_{RDD}$  based on eight direction image are respectively given by:

$$s_{x^-}(x_c, y_c) = \sum_{i=-2m}^1 \sum_{j=-m}^m W_x^-(i, j) s(x_c + i, y_c + j) \quad (8)$$

$$s_{x^+}(x_c, y_c) = \sum_{i=-1}^{2m} \sum_{j=-m}^m W_x^+(i, j) s(x_c + i, y_c + j) \quad (9)$$

$$s_{y^-}(x_c, y_c) = \sum_{i=-m}^m \sum_{j=-2m}^1 W_y^-(i, j) s(x_c + i, y_c + j) \quad (10)$$

$$s_{y^+}(x_c, y_c) = \sum_{i=-m}^m \sum_{j=-1}^{2m} W_y^+(i, j) s(x_c + i, y_c + j) \quad (11)$$

$$s_{LDD}^v(x_c, y_c) = \sum_{i=-1}^{2m} \sum_{j=-2m}^1 W_{LDD}(i, j) s(x_c + i, y_c + j) \quad (12)$$

$$s_{RUD}^v(x_c, y_c) = \sum_{i=-2m}^1 \sum_{j=-1}^{2m} W_{RUD}(i, j) s(x_c + i, y_c + j) \quad (13)$$

$$s_{LUD}^v(x_c, y_c) = \sum_{i=-2m}^1 \sum_{j=-2m}^1 W_{LUD}(i, j) s(x_c + i, y_c + j) \quad (14)$$

$$s_{RDD}^v(x_c, y_c) = \sum_{i=-1}^{2m} \sum_{j=-1}^{2m} W_{RDD}(i, j) s(x_c + i, y_c + j) \quad (15)$$

Where  $v$  is the order;  $m = (n+1)/2$ ;  $(x_c, y_c)$  is the pixel centered at the block. Fig. 3(d) shows the enhancement image of Fig. 3(a). Then the fingerprint skeleton is obtained by thinning the binary ridges [14]. A series of postprocessing steps are produced to cut short branches, remove holes and delete spurious minutiae. The Thinning and minutiae images are displayed in Fig. 3(e) and Fig. 3(f) respectively.

## 3. Experimental results

Two experiments are carried to evaluate the performance of our method on FVC2004 DB3\_A[15] which contains 800 fingerprints from 100 different fingers. Experiment I aims to directly test the performance of our method in qualitative way; experiment II is designed to evaluate the influence of our method on matching performance. In both of the experiments, we compare the results of our method with the mostly widely used fingerprint enhancement method based on Gabor filters [4]. All the experiments are conducted on PC Intel Core2 E6550 @ 2.33 GHZ.

### 3.1. Experiment I

For comparing the two methods qualitatively, we randomly give out some comparative results between our method and the Gabor method [4] (as shown in Fig. 5 -

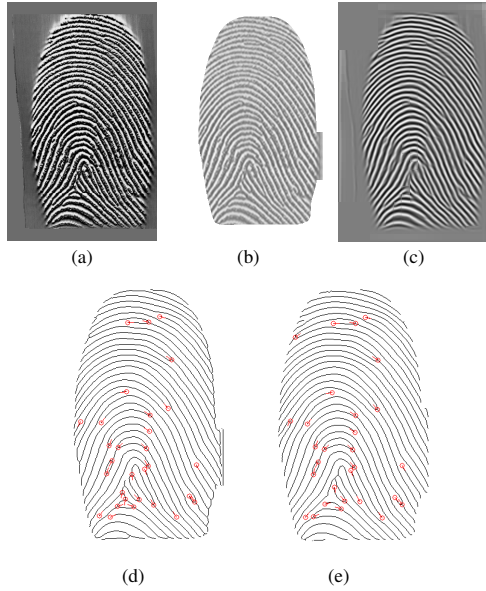


Figure 5. (a)original fingerprint, (b)enhancement image by the proposed method, (c) enhancement image by Gabor method, (d) minutiae image of (b), (e)minutiae image of (d).

Fig. 6). In the following, we will display the estimation results to demonstrate that the performance of our method is convincing.

Compared Fig. 5(d) with Fig. 5(e), we can find there are almost no differences in the enhancement performance between the two methods when the fingerprint image quality is high.

However, with the fingerprint image quality degradation, the proposed method is superior to Gabor method. Fig. 6(a) is a fingerprint image with many scars. Our method could repair the scars properly, while Gabor method failed. The enhancement performance of Gabor method highly depends on the accuracy of ridge orientation and frequency calculation, which can not be obtained properly in scars areas. In our method, to reduce the effect of noise, we partition the image into small blocks of size and estimate the block directions. The block directions is more robust than point directions. In addition, to minimize the effect of such noisy block directions the block direction image is smoothed by examining the  $3 \times 3$  neighborhoods of each of the block directions. The block direction smoothing provides an additional handle to restore the fingerprint images at block level, unlike many smoothing techniques at pixel level. This results in improving the quality of fingerprint images substantially which can be found in comparing Fig. 6(b) with Fig. 6(c).

### 3.2. Experiment II

In this experiment, we are going to evaluate the influence of our method on matching performance. The matching

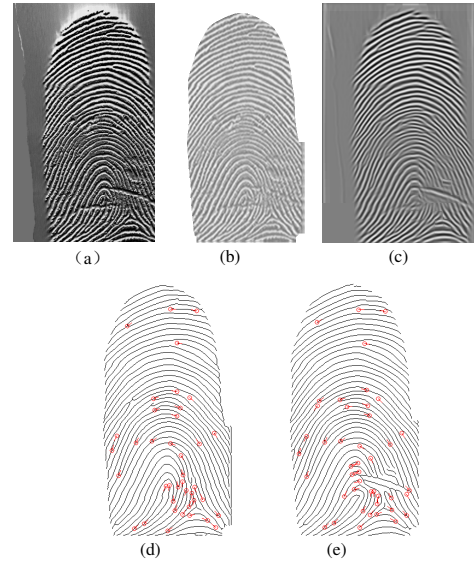


Figure 6. (a)original fingerprint, (b)enhancement image by the proposed method, (c) enhancement image by Gabor method, (d) minutiae image of (b), (e)minutiae image of (d).

system is implemented according to a matching algorithm based on minutiae. Since minutiae extraction depends heavily on the enhanced ridges, the matching results can reflect the accuracy of enhancement. In order to validate the performance of the proposed method, we compare with Gabor method[4]. We use the same matching system except the difference of enhancement method(using our method and Gabor method respectively).

The proposed method has been evaluated on FVC2004 DB3\_A[15] which contains 800 fingerprints from 100 different fingers. In our method, we choose  $v = 0.85$  and  $n = 5$ . The 7 mask coefficients are  $C_{s_{-1}} \cong 0.3028$ ,  $C_{s_0} \cong 0.5620$ ,  $C_{s_1} \cong -0.8380$ ,  $C_{s_2} \cong 0.04422$ ,  $C_{s_3} \cong -0.01621$ ,  $C_{s_4} \cong -0.01223$ ,  $C_{s_5} \cong 0.001605$ . It can find that the sum of the mask coefficients is not zero, which prove difference between fraction calculus and integral one. We trained the parameters based on FVC2004 DB3\_B. In [12], the authors claim that the strong points of  $v$  focus on the range  $[0.8, 0.9]$ . In our experiment, the parameter  $n$  is selected as  $[3, 11]$  (odd number) and  $v$  is extend to  $[0.70, 0.95]$  (every 0.05). We chose the parameters  $n$  and  $v$  which can get the best enhancement results. Tab. 1 illustrates the matching results by using the proposed method and the Gabor method respectively. From the results, we can find that the matching performance based on the fractional calculus mask by our method is better than Gabor method.

### 4. Conclusions and Discussions

It is a difficult and challenging task to enhance the low quality fingerprint image. In this paper, we propose an frac-



Table 1. The performance of enhancement based on the proposed method and Gabor method over FVC2004 DB3 database

Method	EER (%)	FMR100 (%)	FMR1000 (%)	ZeroFMR (%)
Our	2.1376	4.2500	9.5714	14.5000
Gabor	2.7969	5.0357	11.2869	16.6429

tional calculus mask to enhance the fingerprint image. Fractional calculus is an effective tool and has attracted people's attention in many fields. In our method, a eight direction image estimation method is applied to compute the fingerprint direction. To reduce the effect of noise, we estimate the block directions. The block direction is more robust than point directions. In addition, to minimize the effect of noisy block on directions, the block direction image is smoothed by examining the  $3 \times 3$  neighborhoods of each block directions. Then based on fractional calculus, a mask has been got on the direction of negative x-coordinate, positive x-coordinate, negative y-coordinate, positive y-coordinate, left downward diagonal, left upward diagonal, right downward diagonal, and right upward diagonal. Finally the mask is used to filter and enhance fingerprint image according to eight direction image. The proposed algorithm has been evaluated on the databases of fingerprint verification competition 2004DB3 (FVC2004). The experimental results confirm that the proposed algorithm is effective for fingerprint image enhancement. The further research can focus on how to extend the mask to more direction and get more robust fingerprint enhancement.

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