Letter

Lyapunov Conditions for Finite-Time Input-to-State Stability of Impulsive Switched Systems

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Dear Editor,

This letter studies finite-time input-to-state stability (FTISS) for impulsive switched systems. A set of Lyapunov-based conditions are established for guaranteeing FTISS property. When constituent modes governing continuous dynamics are FTISS and discrete dynamics involving impulses are destabilizing, the FTISS can be retained if impulsive-switching signals satisfy an average dwell-time (ADT) condition. When some or even all constituent modes governing continuous dynamics are not FTISS and discrete dynamics involving impulses are stabilizing, the FTISS can be achieved if impulsive-switching signals satisfy a reverse ADT condition. Examples are presented to illustrate the efficiency of proposed results.

Introduction: Many practical systems can be modeled by hybrid systems which involve both discrete-time and continuous-time behaviors [1], [2]. Switched systems and impulsive systems are two general classes of hybrid systems. Switched systems involve a finite number of constituent modes and a switching signal orchestrating the switching between them [3], while impulsive systems depict real world processes that generate instantaneous state resets at discrete times [4]. Impulsive switched system, as a more comprehensive dynamical system, involves impulses and switching in a single framework [5]. Such system, as it is known, does not retain the property of constituent mode. Besides, the impulses governing the instantaneous state changes often divide into two classes: destabilizing impulses and stabilizing impulses [6], [7]. In that scenario, a suitable choice of impulsive-switching signal plays an important role in guaranteeing stability or robustness for impulsive switched systems.

Input-to-state stability (ISS) characterizes an asymptotic convergence behavior of solutions with external inputs. Roughly speaking, ISS includes an asymptotic stability of the solutions in the absence of external inputs and an asymptotic gain property with respect to external inputs [8]–[10]. Finite-time stability, having a faster rates of convergence time than asymptotic stability, requires that the solutions reach to equilibrium point during a finite time interval [11]–[14]. The settling time, however, typically is unknown and depends on the initial conditions. Combining the properties of ISS and finite-time stability, [15] introduced a concept of FTISS for continuous-time systems. The theory of FTISS has proved very useful not only in the analysis of input systems, but also in the design controllers and observers of control systems. Surprisingly, there has been little FTISS results on dynamic systems up to now, see [16]–[18]. Notice that although the impulse and switching are considered in [17], [18], those results only considered the case of stable continuous dynamics with destabilizing impulses. More importantly, a detailed analysis of the settling time has not been carried out in these existing works.

This letter focuses on FTISS of impulsive switched systems. Some sufficient conditions, which rely on a relation among continuous dynamics, impulsive actions, and external input, are presented. The main contributions are threefold: 1) The constituent modes governing continuous dynamics, which may or may not be FTISS, are seriously taken into account; 2) Regarding the impulsive actions in discrete dynamics, two classes of impulses including destabilizing impulse and stabilizing impulse are considered, respectively; 3) A precise estimation of settling time, whenever external inputs are absent, can be deduced under certain impulsive-switching signals.

Problem statement: Consider the impulsive switched system

$$\begin{cases} \dot{x}(t) = f_{\sigma(t)}(x(t), \omega(t)), & \text{for } t \notin \Gamma \\ x(t) = g_{\sigma(t)}(x(t^-), \omega(t^-)), & \text{for } t \in \Gamma \end{cases} \tag{1}$$

where $t \geq 0$, $x \in \mathbb{R}^n$ is the state of system, $\omega \in \mathbb{R}^m$ is an external input. $\Gamma = \{t_k, k \in \mathbb{Z}_+\}$, abbreviated as $\{t_k\}$, is a strictly increasing impulse-switching time sequence satisfying $0 = t_0 < t_1 < t_2 < \cdots < t_k \to +\infty$. $\sigma : \mathbb{R}_+ \to \Omega$ is a switching signal. f_i, $g_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are continuous functions for $i \in \Omega$ and moreover $f_i(0,0) = g_i(0,0) = 0$. We assume that f_i and g_i satisfy certain conditions so system (1) possesses unique solutions in forward time, see [14], [16], [17]. Denote $(\{t_k\}, \sigma)$ as an impulsive-switching signal for later use. $N(t,s)$ denotes the number of the impulsive-switching times t_k during the interval (s,t).

For given class of impulsive-switching signals $\mathcal{S}$, system (1) is UFTISS over $\mathcal{S}$ if there exist $\beta \in \mathcal{GKL}$ and $\gamma \in \mathcal{K}_\infty$, independent of the choice of the signal $(\{t_k\}, \sigma)$ in $\mathcal{S}$, such that for $x_0 \in \mathbb{R}^n$ and $\omega \in \mathbb{R}^m$, the resulting state $x(t) = x(t, x_0, \omega)$ satisfies

$$|x(t)| \leq \beta(|x_0|, t) + \gamma(\|\omega\|_{[0,t]}), \ \ t \geq 0. \tag{2}$$

System (1) is said to be *UFTISS* w.r.t. $\mathcal{B}_\varsigma$ over $\mathcal{S}$ if for $x_0 \in \mathcal{B}_\varsigma$, the resulting state $x(t)$ of system (1) satisfies (2).

Main results: This section presents some Lyapunov-based conditions for UFTISS of impulsive switched systems. Denote $\Omega_u \subseteq \Omega$ and $\Omega_s \subseteq \Omega$ as the set of unstable and stable constituent modes. We shall investigate two cases: $\Omega_u = \emptyset$, i.e., each constituent mode is individually stable; $\Omega_u \neq \emptyset$, i.e., some (perhaps even all) constituent modes are unstable. For convenience, we say that continuous function $V : \mathbb{R}^n \to \mathbb{R}_+$ is of class $\mathcal{F}_\alpha$ if 1) $V(x)$ is locally Lipschitz in x and $V(0) = 0$; 2) $\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$ for $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$.

Theorem 1: Assume that there exist constants $\eta \in (0,1)$, $a, b, d, \lambda, \mu \in \mathbb{R}_+$, functions $V_i \in \mathcal{F}_\alpha$, $\varphi \in \mathcal{K}_\infty$ such that

$$|x| \geq \varphi(|\omega|) \Rightarrow \begin{cases} D^+ V_i(x) \leq -a V_i^\eta(x) - b V_i(x), \ \ i \in \Omega & \text{(3a)} \\ V_i(g_i(x,\omega)) \leq e^{\frac{d}{1-\eta}} V_j(x), \ \ i,j \in \Omega. & \text{(3b)} \end{cases}$$

Then system (1) is UFTISS over $\mathcal{S}_+[\lambda,\mu]$, where $\mathcal{S}_+[\lambda,\mu]$ denote a class of impulsive-switching signals $(\{t_k\}, \sigma)$ satisfying

$$dN(t,s) + (b(\eta - 1) + \lambda)(t - s) \leq \mu, \ \ 0 \leq s \leq t. \tag{4}$$

Proof: Define $z(t) = \varphi(\|\omega\|_{[0,t]})$, $\psi(t) = \alpha_1^{-1}(e^{\frac{\mu}{1-\eta}} \alpha_2(\alpha_1^{-1}(e^{\frac{d}{1-\eta}} \alpha_2 \times (z(t)))))$, and $v(t) = V_{\sigma(t)}(x(t))$. Assuming that $|x(t)| \geq z(t) \geq \varphi(|\omega(t)|)$ holds for $t \in [\hat{\xi}, \check{\xi})$, it then follows from (3a) and (3b) that:

$$v^{1-\eta}(t) \leq e^{-\lambda(t-\hat{\xi})+\mu}\left(v^{1-\eta}(\hat{\xi}) + \tfrac{a}{b}\right) - \tfrac{a}{b}, \ t \in [\hat{\xi}, \check{\xi}). \tag{5}$$

Let $\check{t}_1 = \inf\{t \geq 0 : |x(t)| \leq z(t)\}$. It follows from (5) that $v^{1-\eta}(t) \leq e^{-\lambda t+\mu}(\alpha_2^{1-\eta}(|x_0|) + \frac{a}{b}) - \frac{a}{b} = \beta(|x_0|, t)$, $t \in [0, \check{t}_1)$. It indicates that

$$|x(t)| \leq \alpha^{-1}(\beta^{\frac{1}{1-\eta}}(|x_0|, t)) = \beta_1(|x_0|, t), \ \ t \in [0, \check{t}_1). \tag{6}$$

Next, we show that $|x(t)| \leq \psi(t)$, $t \geq \check{t}_1$. Let $\hat{t}_1 = \inf\{t \geq \check{t}_1 : |x(t)| > z(t)\}$. It is obvious that $|x(t)| \leq z(t)$, $t \in [\check{t}_1, \hat{t}_1)$. If $\hat{t}_1 = \infty$, it holds that $|x(t)| \leq \psi(t)$, $t \geq \check{t}_1$. If $\hat{t}_1 < \infty$, it follows from (3b) that $v(\hat{t}_1) \leq e^{\frac{d}{1-\eta}} \times v(\hat{t}_1^-) \leq e^{\frac{d}{1-\eta}} \alpha_2(|z(\hat{t}_1)|)$. It implies that $|x(\hat{t}_1)| \leq \alpha_1^{-1}(e^{\frac{d}{1-\eta}} \alpha_2(|z(\hat{t}_1)|))$. Furthermore, for all $s \geq \hat{t}_1$, such that $|x(s)| > z(s)$, let $\hat{t} = \sup\{t \leq s : |x(t)| \leq z(t)\}$. Similarly, we can conclude that $|x(\hat{t})| \leq \alpha_1^{-1} \times$

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$(e^{\frac{d}{1-\eta}}\alpha_2(|z(\hat{t})|))$. It follows from (3a) and (3b) that $v^{1-\eta}(t) \le e^{-\lambda(t-\hat{t})+\mu} \times v^{1-\eta}(\hat{t}) \le e^{\mu}\alpha_2^{1-\eta}(|x(\hat{t})|)$, $t\in[\hat{t},s]$. It then follows from that $|x(t)| \le \alpha_1^{-1} \times (e^{\frac{\mu}{1-\eta}}\alpha_2(|x(\hat{t})|)) \le \alpha_1^{-1}(e^{\frac{\mu}{1-\eta}}\alpha_2(\alpha_1^{-1}(e^{\frac{d}{1-\eta}}\alpha_2(|z(\hat{t})|)))) = \psi(\hat{t}) \le \psi(t)$. From the above discussion, for $t \ge \check{t}_1$, we have

$$|x(t)| \le \psi(t) = \alpha_1^{-1}(e^{\frac{\mu}{1-\eta}}\alpha_2(\alpha_1^{-1}(e^{\frac{d}{1-\eta}}\alpha_2(\|\omega\|_{[0,t]})))) = \gamma(\|\omega\|_{[0,t]}). \tag{7}$$

We then conclude from (6) and (7) that $|x(t)| \le \beta_1(|x_0|,t)+\gamma(\|\omega\|_{[0,t]})$, $t\ge 0$, thus, system (1) is UFTISS over $\mathcal{S}_+[\lambda,\mu]$. ■

Remark 1: It was shown in [15], [16] that the function V_i satisfying condition (3a) is a sufficient condition for the i-th mode governing continuous dynamics to be FTISS. Condition (3b) indicates that the discrete dynamics involving impulses are destabilizing. Condition (4) imposes an upper bound constraint to the number of impulsive-switching times. Specifically, for $\lambda = b(1-\eta)$, it only holds when the number of impulsive-switching times is no larger than $N_0 = \mu/d$. And for $\lambda < b(1-\eta)$, it can be rewritten as

$$N(t,s) \le N_0 + \frac{t-s}{\tau_a},\ \ 0\le s\le t \tag{8}$$

where $\tau_a, N_0 \in \mathbb{R}_+$ are appropriately constants. It is equivalent to the ADT condition for switched systems in [1] and impulsive systems in [9]. With the ADT condition given by (4), Theorem 1 shows that if switching (or impulse) does not occur too frequently, the UFTISS of system (1) can be retained successfully.

Corollary 1: Let $\mathcal{S}_+^{\rm avg}[\tau_a,N_0]$ denote a class of ADT impulsive-switching signals satisfying (8). Assume that there exist constants $\eta\in(0,1)$, $a,b,d,\tau_a,N_0,\varsigma,T_\varsigma \in \mathbb{R}_+$, functions $V_i \in \mathcal{F}_\alpha$, $\alpha_1,\alpha_2,\varphi \in \mathcal{K}_\infty$ such that (3a), (3b) hold and $b(1-\eta)T_\varsigma - dN_0 + \ln(\frac{b\alpha_2^{1-\eta}(\varsigma)+a}{a}) \ge \frac{d}{\tau_a}T_\varsigma$. Then system (1) is UFTISS w.r.t. $\mathcal{B}_\varsigma$ over $\mathcal{S}_+^{avg}[\tau_a,N_0]$. Moreover, the settling time, whenever external inputs are absent, is bounded by $T(x_0,\mathcal{S}_+^{avg}[\tau_a,N_0]) \le T_\varsigma$, $x_0\in\mathcal{B}_\varsigma$.

Theorem 2: Assume that there exist constants $\eta\in(0,1)$, $a,b,d,\lambda,\mu \in \mathbb{R}_+$, functions $V_i \in \mathcal{F}_\alpha$, $\varphi\in\mathcal{K}_\infty$ such that

$$|x| \ge \varphi(|\omega|) \Rightarrow \begin{cases} D^+V_i(x) \le -aV_i^{\eta}(x)+bV_i(x), & i\in\Omega \quad (9a)\\ V_i(g_i(x,\omega)) \le e^{\frac{-d}{1-\eta}}V_j(x), & i,j\in\Omega. \quad (9b)\end{cases}$$

Then system (1) is UFTISS over $\mathcal{S}_-[\lambda,\mu]$, where $\mathcal{S}_-[\lambda,\mu]$ denote a class of impulsive-switching signals $(\{t_k\},\sigma)$ satisfying

$$-dN(t,s)+(b(1-\eta)+\lambda)(t-s) \le \mu,\ \ 0\le s\le t. \tag{10}$$

Proof: Define $\check{t}_1$ and $v(t)$ as in Theorem 1. Moreover, define $\vartheta = \alpha_2^{\eta-1}(a/b)$, $\Xi_1(|x_0|,t) = e^{b(1-\eta)t}(\alpha_2^{1-\eta}(|x_0|)-\frac{a}{b})+\frac{a}{b}$, and $\Xi_2(|x_0|,t) = \begin{cases} e^{-\lambda t+\mu}\alpha_2^{1-\eta}(|x_0|), & t\in[0,T_\rho),\\ e^{b(1-\eta)(t-T_\rho)}\frac{a}{b}(\rho-1)+\frac{a}{b}, & t\ge T_\rho,\end{cases}$ where $\rho\in(0,1)$ and $T_\rho = \inf\{t\ge 0: e^{-\lambda t+\mu}\alpha_2^{1-\eta}(|x_0|) \le \frac{a}{b}\rho\}$. We firstly show that for $t\in[0,\check{t}_1)$,

$$v^{1-\eta}(t) \le \begin{cases}\Xi_1(|x_0|,t), & x_0\in\mathcal{B}_\vartheta\\ \Xi_2(|x_0|,t), & x_0\notin\mathcal{B}_\vartheta\end{cases} = \beta(|x_0|,t). \tag{11}$$

When $x_0\in\mathcal{B}_\vartheta$, it follows from (9a) and (9b) that $v^{1-\eta}(t) \le e^{b(1-\eta)t}(v^{1-\eta}(0)-\frac{a}{b})+\frac{a}{b} \le \Xi_1(|x_0|,t)$, $t\in[0,\check{t}_1)$. While $x_0\notin\mathcal{B}_\vartheta$, we can conclude from (9a) and (9b) that for $t\in[0,\check{t}_1)$,

$$v^{1-\eta}(t) \le e^{-dN(t,0)+b(1-\eta)t}v^{1-\eta}(0) \le e^{-\lambda t+\mu}v^{1-\eta}(0). \tag{12}$$

If $T_\rho \ge \check{t}_1$, it follows from (12) that $v^{1-\eta}(t) \le \Xi_2(|x_0|,t)$ for $t\in[0,\check{t}_1)$. If $T_\rho < \check{t}_1$, we can conclude from (12) that $v^{1-\eta}(t) \le e^{-\lambda t+\mu} \times \alpha_2^{1-\eta}(|x_0|)$ for $t\in[0,T_\rho)$ and $v^{1-\eta}(T_\rho) \le \frac{a}{b}\rho$. Then, it follows that $v^{1-\eta}(t) \le e^{b(1-\eta)(t-T_\rho)}(v^{1-\eta}(T_\rho)-\frac{a}{b})+\frac{a}{b} \le e^{b(1-\eta)(t-T_\rho)}\frac{a}{b}(\rho-1)+\frac{a}{b}$ for $t\in[T_\rho,\check{t}_1)$. We conclude that (11) holds. It then follows from (11) that $|x(t)| \le \alpha_1^{-1}(\beta^{\frac{1}{1-\eta}}(|x_0|,t)) = \beta_1(|x_0|,t)$, $t\in[0,\check{t}_1)$. Next, using similar arguments as Theorem 1, we can conclude that $|x(t)| \le \alpha_1^{-1}(e^{\frac{\mu}{1-\eta}}\alpha_2(\varphi(\|\omega\|_{[0,t]}))) = \gamma(\|\omega\|_{[0,t]})$, $t\ge\check{t}_1$. Then it follows that $|x(t)| \le \beta_1(|x_0|,t)+\gamma(\|\omega\|_{[0,t]})$, $t\ge 0$, thus, system (1) is UFTISS over $\mathcal{S}_-[\lambda,\mu]$. ■

Remark 2: It was shown in [13], [14] that the function V_i satisfying condition (9a) indicates that the i-th mode governing continuous dynamics can be unstable. Condition (9b) indicates that the discrete dynamics involving impulses are stabilizing. Subsequently, condition (10) imposes a lower bound constraint to the number of impulsive-switching times for ensuring UFTISS. It can be rewritten as

$$-N_0 + \frac{t-s}{\tau_\alpha} \le N(t,s),\ \ 0\le s\le t \tag{13}$$

where $\tau_\alpha, N_0\in\mathbb{R}_+$ are appropriately constants. It is equivalent to the reverse ADT condition for impulsive systems in [9]. With the reverse ADT condition given by (10), Theorem 2 shows that if there are no overly long intervals between two consecutive impulses (or switching), the UFTISS of system (1) can be achieved accordingly.

Corollary 2: Let $\mathcal{S}_-^{\rm avg}[\tau_\alpha,N_0]$ denote a class of reserve ADT impulsive-switching signals satisfying (13). Assume that there exist constants $\rho,\kappa,\eta\in(0,1)$, $a,b,d,\tau_\alpha,N_0,\varsigma,T_\varsigma\in\mathbb{R}_+$, functions $V_i\in\mathcal{F}_\alpha$, $\alpha_1,\alpha_2,\varphi\in\mathcal{K}_\infty$ such that (9a), (9b) hold and $dN_0+\ln(\frac{b\alpha_2^{1-\eta}(\varsigma)}{\rho a}) \le (\frac{d}{\tau_\alpha} - b(1-\eta))\kappa T_\varsigma$, $\frac{1}{b(\eta-1)}\ln(1-\rho) \le (1-\kappa)T_\varsigma$. Then system (1) is UFTISS w.r.t. $\mathcal{B}_\varsigma$ over $\mathcal{S}_-^{\rm avg}[\tau_\alpha,N_0]$. Moreover, the settling time, whenever external inputs are absent, is bounded by $T(x_0,\mathcal{S}_-^{\rm avg}[\tau_\alpha,N_0]) \le T_\varsigma$, $x_0\in\mathcal{B}_\varsigma$.

We further investigate the case: $\Omega_s \ne \emptyset$ and $\Omega_s \subsetneq \Omega$ i.e., some constituent modes may be stable. Denote $T^u(t,s)$ as the activation time of constituent modes in Ω_u during the interval $[s,t)$. We assume that there exist constants $\varrho\in\mathbb{R}_+$, γ_1, $\gamma_2\in[0,1]$ such that

$$\gamma_1(t-s)-\varrho \le T^u(t,s) \le \gamma_2(t-s)+\varrho,\ \ 0\le s\le t. \tag{14}$$

Notice that, for the case of $\gamma_2 = 0$, it only holds when the activation time of constituent modes in Ω_u is no larger than ϱ. For the case of $\gamma_1 = 1$, moreover, it only holds when the activation time of constituent modes in Ω_s is no larger than ϱ.

Theorem 3: Assume that there exist constants $\eta\in(0,1)$, $\gamma_1,\gamma_2\in[0,1]$, $a,b,d,\varrho,\lambda,\mu\in\mathbb{R}_+$, functions $V_i\in\mathcal{F}_\alpha$, $\varphi\in\mathcal{K}_\infty$ such that (14) holds and

$$|x|\ge\varphi(|\omega|) \Rightarrow \begin{cases} D^+V_i(x) \le \begin{cases} -aV_i^{\eta}(x)-bV_i(x), & i\in\Omega_s\\ -aV_i^{\eta}(x)+bV_i(x), & i\in\Omega_u\end{cases}\\ V_i(g_i(x,\omega)) \le e^{\frac{-d}{1-\eta}}V_j(x), \ \ i,j\in\Omega.\end{cases}$$

Then system (1) is UFTISS over $\mathcal{S}_-^{\rm mix}[\lambda,\mu]$, where $\mathcal{S}_-^{\rm mix}[\lambda,\mu]$ denote a class of impulsive-switching signals $(\{t_k\},\sigma)$ satisfying

$$-dN(t,s)+(2b\gamma_2(1-\eta)+\lambda)(t-s) \le \mu,\ \ 0\le s\le t.$$

Remark 3: Recently, the work of [17] studies FTISS for nonlinear impulsive systems by means of ADT condition and Lyapunov method. However, the continuous dynamic potentially contributes to *FTISS* since it satisfies $D^+V(x) \le -aV^{\eta}(x)$ whenever $V(x)\ge\varphi(|\omega|)$. And the estimation of settling time remains unanswered. Considering both impulse and switching in a single framework, the UFTISS results in this paper considered two cases: stable modes with destabilizing impulses (Theorem 1) and unstable modes with stabilizing impulses (Theorems 2 and 3). More importantly, a precise formulation of settling time can be achieved under certain impulsive-switching signals satisfying ADT-like condition.

Numerical example: This section presents two examples with numerical simulations to illustrate the main results.

Example 1: Consider system (1) with $x,\omega\in\mathbb{R}^3$, and

$$\dot{x} = f_1(x,\omega) = \begin{cases} -0.4\sqrt{|x_1|}\text{sign}(x_1)-1.5x_1+\omega_1\\ -0.4\sqrt{|x_2|}\text{sign}(x_2)-1.5x_2+\omega_2\\ -0.4\sqrt{|x_3|}\text{sign}(x_3)-1.5x_3+\omega_3\end{cases}$$

$$\dot{x} = f_2(x,\omega) = \begin{cases} -0.2\sqrt{|x_1|}\text{sign}(x_1)-2x_1+0.5x_2+\omega_1\\ -0.2\sqrt{|x_2|}\text{sign}(x_2)-2x_2+0.5x_3+\omega_2\\ -0.2\sqrt{|x_3|}\text{sign}(x_3)-2x_3+0.5x_1+\omega_3\end{cases}$$

$$x = g_i(x,\omega) = \begin{bmatrix}0.7 & 0.3 & 0\\ 0 & 0.8 & 0.2\\ 0.1 & 0 & 0.9\end{bmatrix}x + \begin{bmatrix}0.3 & 0.2 & 0\\ 0.1 & 0 & 0.4\\ 0 & 0.2 & 0.3\end{bmatrix}\omega$$

Let $V_1(x) = V_2(x) = x^T x$ and $\varphi(r) = |r|$. It follows that (3a) and (3b) hold with $a = 0.2$, $b = 1$, $\eta = 3/4$, and $d = 0.25$. We conclude from Theorem 1 that system (1) is UFTISS over $\mathcal{S}_+[\lambda,\mu]$. Let $\omega_i(t) = \sin(0.5t)$, $i = 1,2,3$, $\lambda = 0.05$, $\mu = 0.25$, and $(\{t_k\},\sigma)$ is given $\{[0,1.25), 1\},\{[1.25,3),2\},\{[3,4.25),1\},\{[4.25,6),2\}\ldots$ The sate trajectories of system (1) with initial conditions $x_0^1 = (2,3,-3)^T$, $x_0^2 = (4,-2,2)^T$, and $x_0^3 = (-3,2,4)^T$ are shown in Fig. 1, respectively.

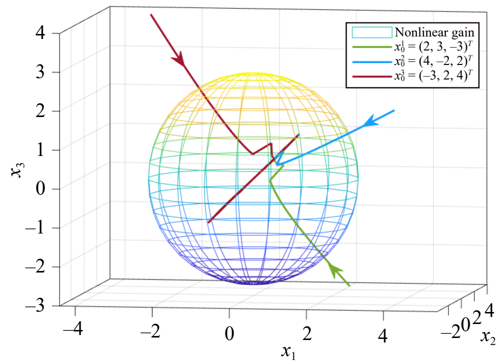

Fig. 1. Simulation results of Example 1.

Example 2: Consider the following switched system:

$$\begin{aligned} \sigma = 1: \ \dot{x}(t) &= -0.25x^{\frac{1}{3}}(t) + 0.4x(t) + \omega(t) \\ \sigma = 2: \ \dot{x}(t) &= -0.3x^{\frac{1}{3}}(t) + 0.5x(t) + \omega(t) \end{aligned} \tag{15}$$

involving impulsive actions at switching instant t_k

$$x(t_k) = \zeta x(t_k^-) + \omega(t_k^-) \tag{16}$$

where $t \geq 0$, $\zeta \in \mathbb{R}_+$ and $\{t_k\}$ satisfies (8) with $\tau_a = 0.25$ and $N_0 = 2$. Next, we shall design ζ to stabilize system (15) and (16) in FTISS sense. Let $V_1(x) = V_2(x) = |x|$ and $\varphi(r) = 4|r|$. It follows that (9a) and (9b) hold with $a = 0.25$, $b = 0.75$, $\eta = 1/3$, and $d \leq -3\ln(0.25+\zeta)/2$. Let $\varsigma = 1$, $T_\varsigma = 4.5$, $\rho = 1/4$, and $\kappa = 4/5$. We conclude from Corollary 2 that, when $\zeta \leq 0.3415$, system (15) and (16) is UFTISS over $\mathcal{S}_-^{\text{avg}}[0.25,2]$. And the settling time satisfies $T(x_0,\mathcal{S}_-^{\text{avg}}[0.25,2]) \leq 4.5$, $x_0 \in \mathcal{B}_1$. Let $x_0 = 0.95$, $\zeta = 0.34$, $\omega(t) = 0.15\sin(t)$, and $(\{t_k\},\sigma)$ is given $\{[0,0.3),1\},\{[0.3,0.5),2\},\{[0.5,0.8),1\},\{[0.8.1),2\},\ldots$ One can observe from Fig. 2(a) that system (15) and (16) is FTISS, and moreover, the unforced system is finite-time stable before $T_\varsigma = 4.5$. Under same conditions, if we take $\zeta = 0.88$, then it goes against the restrictions on ζ by Corollary 2. Notice that although the impulses may still stabilize system (15) and (16) in FTISS sense, the desired settling time of unforced system cannot be achieved, see Fig. 2(b).

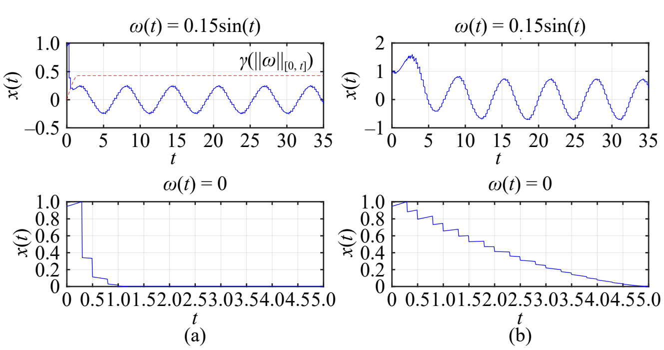

Fig. 2. Simulation results of Example 2. (a) Trajectories of system (15) and (16) with $\zeta = 0.34$; (b) Trajectories of system (15) and (16) with $\zeta = 0.88$.

Conclusion: This letter studied FTISS for impulsive switched systems involving external inputs affecting both constituent modes and impulsive dynamics. With the help of ADT-like condition, some Lyapunov-based conditions have been proposed with certain classes of impulsive-switching signals. Moreover, a precise formulation of settling time, whenever external inputs are absent, has been achieved under the designed impulsive-switching signals.

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