

## Letter

## A Novel Trajectory Tracking Control of AGV Based on Udwadia-Kalaba Approach

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Dear Editor,

This letter is about an automated guided vehicle (AGV) trajectory tracking control method based on Udwadia-Kalaba (U-K) approach. This method provides a novel, concise and explicit motion equation for constrained mechanical systems with holonomic and/or nonholonomic constraints as well as constraints that may be ideal or non-ideal. In this letter, constraints are classified into structural and performance constraints. The structural constraints are established without considering the trajectory control, and the dynamic model is built based on them. Then the expected trajectories are taken as the performance constraints, and the constraint torque is solved by analyzing the U-K equation. It is shown by numerical simulations that the proposed method can solve the trajectory tracking control problem of AGV well.

**Introduction:** In recent years, AGV-A kind of mobile robot, which is a nonholonomic mechanical system and a typical model of uncertain complex system, gradually plays a more and more important role. Around the modeling and control problems of AGV, domestic and foreign scholars have done a lot of research work, such as PID control [1], robust control [2], adaptive control [3], sliding mode control [4], fuzzy control [5], neural network control [6] and other control methods [7].

The problem of solving equations of motion for nonholonomic mechanical systems has been energetically and continuously worked on by many scientists since constrained motion was initially described by Lagrange (1787) [8]. For example, Gauss [9] introduced Gauss's principle, Gibbs [10] and Appell [11] obtained Gibbs-Appell equations, Poincaré [12] generalized Lagrange's equations, and Dirac [13] provided an algorithm to obtain the Lagrangian multipliers. However, they are all based on the D'Alembert principle, that is the virtual displacement principle. For constrained mechanical systems, Lagrangian multipliers can be used effectively for constrained calculation. However, the application of this method is not easy, as it is often very difficult to find Lagrangian multipliers to obtain the explicit equations of motion for systems, especially for systems with a large number of degrees of freedom and non-integrable constraints. The D'Alembert principle works well in many situations, but it is not applicable when the constraints are non-ideal [14]. Thus, Udwadia and Kalaba obtained a novel, concise and explicit equation of motion for constrained mechanical systems that may not satisfy the D'Alembert principle. It can be applied to constrained mechanical systems

with holonomic and/or nonholonomic constraints as well as constraints may be ideal or non-ideal, which is called U-K theory [15]. It leads to a new and fundamental understanding of constrained mechanical system and a new view of Lagrangian mechanics.

Compared with the conventional methods in the trajectory tracking control of AGV, the proposed method has essential differences. In this letter, the dynamic model of the AGV with a driving and turning front wheel is first established based on the U-K theory. Second, constraints are classified in a new way, which include structural constraints and performance constraints. The structural constraints are set up regardless of trajectories, and the dynamic model is built based on the structural constraints. The AGV's desired trajectory is set as performance constraints. Third, the constraint torque can be obtained based on the U-K approach to realize the trajectory tracking control of the AGV. It is shown by numerical simulations that, by the proposed method, the AGV's movements has high accuracy.

**Theoretical description:** Using the marvelous U-K theory, the novel, concise and explicit equation of motion for constrained mechanical systems can be derived in three steps [14].

1) Consider an unconstrained mechanical system with  $n$  generalized coordinates ( $q := [q_1, q_2, \dots, q_n]^T$ ) as  $M(q, t)\ddot{q} = Q(q, \dot{q}, t)$ , where  $M(q, t)$  denotes an  $n \times n$  positive definite inertia matrix,  $\dot{q}$  denotes the  $n \times 1$  velocity vector,  $\ddot{q}$  denotes the  $n \times 1$  acceleration vector, and  $Q(q, \dot{q}, t)$  denotes the "known"  $n$ -vector of impressed (also called "given") force [16] which is imposed on the system.

2) Assume that the system is constrained by  $h$  holonomic constraints  $\varphi_i(q, t) = 0$ ,  $i = 1, 2, \dots, h$ , and  $s$  nonholonomic constraints  $\psi_i(q, \dot{q}, t) = 0$ ,  $i = 1, 2, \dots, s$ . Furthermore, assume that the initial conditions of the system  $q_0$  and  $\dot{q}_0$  satisfy these constraint equations at time  $t = 0$ , i.e.,  $\varphi_i(q_0, t) = 0$ , and  $\psi_i(q_0, \dot{q}_0, t) = 0$ . On the assumption that the  $h + s$  constraints are sufficiently smooth, the non-holonomic constraints and holonomic constraints are differentiated by time once and twice, respectively, to obtain a set of constraints in the matrix form

$$A(q, \dot{q}, t)\ddot{q} = b(q, \dot{q}, t) \quad (1)$$

where  $A(q, \dot{q}, t)$  (may or may not be full rank) denotes an  $m \times n$  constraint matrix and  $b(q, \dot{q}, t)$  denotes an  $m \times 1$  column vector.

3) The additional generalized forces are imposed on the system to ensure that the desired trajectory is satisfied. Due to the existence of additional generalized forces, the actual explicit equation of motion of the constrained system can be described as

$$M(q, t)\ddot{q} = Q(q, \dot{q}, t) + Q^c(q, \dot{q}, t) \quad (2)$$

where  $Q^c(q, \dot{q}, t)$  denotes the additional  $n \times 1$  "constraint forces", that are imposed on the unconstrained system. Our aim is to determine  $Q^c(q, \dot{q}, t)$  at any time  $t$ , for the  $Q$  is known.

In Lagrangian mechanics,  $Q^c(q, \dot{q}, t)$  is governed by the usual D'Alembert Principle which indicates that constraint forces do zero work under virtual displacement and therefore is considered ideal constraints. However, constraints can also be non-ideal, while non-ideal constraints generate non-ideal constraint forces, such as friction force, electro-magnetic force, etc. If there are both ideal and non-ideal constraints in the system,  $Q^c(q, \dot{q}, t)$  can be expressed as

$$Q^c(q, \dot{q}, t) = Q_{id}^c(q, \dot{q}, t) + Q_{nid}^c(q, \dot{q}, t) \quad (3)$$

where  $Q_{id}^c(q, \dot{q}, t)$  denotes the ideal constraint force and  $Q_{nid}^c(q, \dot{q}, t)$  denotes the non-ideal one.

Udwadia and Kalaba have proved the general "explicit" equation of motion at time  $t$  for any constrained mechanical system. The ideal constraint force ( $Q_{id}^c$ ) and the non-ideal constraint force ( $Q_{nid}^c$ ) are, respectively

$$Q_{id}^c = M^{\frac{1}{2}} B^+ (b - A M^{-1} Q) \quad (4)$$

$$Q_{nid}^c = M^{\frac{1}{2}} (I - B^+ B) M^{-\frac{1}{2}} c \quad (5)$$

where the matrix  $B = A M^{-1/2}$ , the superscript "+" denotes the Moore-Penrose generalized inverse, and  $c$  is a suitable  $n$ -vector that

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is determined by the mechanical system.

From (2)–(5), the general “explicit” equation of motion (including both ideal and non-ideal constraints) is

$$M\ddot{q} = Q + M^{\frac{1}{2}}B^+(b - AM^{-1}Q) + M^{\frac{1}{2}}(I - B^+B)M^{-\frac{1}{2}}c. \quad (6)$$

If the work done by constraint forces under the virtual displacement is zero, then  $Q_{\text{nid}}^c = 0$ . Equation (6) reduces to obey the D'Alembert principle, which means the general “explicit” equation of motion is

$$M\ddot{q} = Q + M^{\frac{1}{2}}B^+(b - AM^{-1}Q). \quad (7)$$

Thus, at each instant of time  $t$ , the constrained system subject to an additional “constraint force”  $Q^c(t)$  is given by

$$Q^c(t) = M^{\frac{1}{2}}B^+(b - AM^{-1}Q). \quad (8)$$

**Trajectory tracking control of AGV:** Suppose that the absolute coordinate system XOY is fixed on the planar of the AGV with three wheels, consisting of one front wheel and two rear wheels. The front wheel can not only drive, but also turn. The model is shown in Fig. 1. The front wheel is independently driven by two DC servo motors for driving and turning respectively. Two rear wheels only have the role of supporting and guiding. The wheels contact with the ground is characterized by pure rolling and no slipping. The variables and parameters of AGV are shown in Table 1.

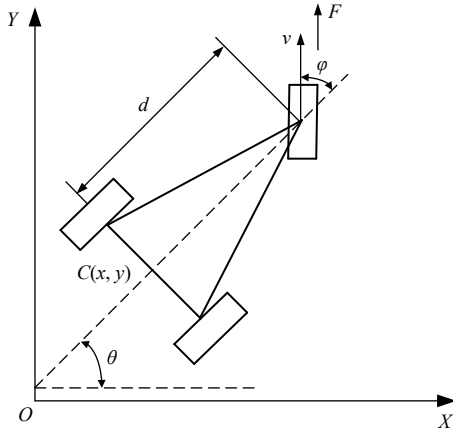


Fig. 1. AGV with a driving and turning front wheel.

According to Newton's law, we can get

$$F \cos \varphi = m \frac{d}{dt} (v \cos \varphi) = m(\dot{v} \cos \varphi - v \dot{\varphi} \sin \varphi). \quad (9)$$

The dynamic equation of the driving motor of the front wheel is

$$(k^2 J_1 + J_2) \dot{w}_1 + \left( k^2 B_1 + B_2 + \frac{k_m^2 k^2}{R_a} \right) w_1 = \frac{k_m k}{R_a} V_{a1} - r F \quad (10)$$

where  $k$  denotes the transmission systems gear ratio;  $k_m$  denotes electromagnetic torque constant of the driving/turning motor;  $R_a$  denotes armature resistance of the driving/turning motor;  $V_{a1}$  denotes input voltage of the driving motor;  $J_1$  denotes rotational inertia of the driving/turning motor;  $J_2$  denotes rotational inertia of the front wheel about the rolling axis;  $B_1$  denotes viscous friction coefficient of the driving/turning motor shaft;  $B_2$  denotes viscous friction coefficient of the front wheels axle; the torque of the driving motor is

$$u_d = \frac{k_m k}{R_a} V_{a1}. \quad (11)$$

The effects of non-ideal constraints are ignored. Then, the driving force  $F$  is dependent on  $u_d$  and  $r$ , then (10) is simplified as

$$F = \frac{u_d}{r}. \quad (12)$$

For this AGV model

$$\dot{x} = v \cos \varphi \cos \theta, \quad \dot{y} = v \cos \varphi \sin \theta. \quad (13)$$

Then, we can get

Table 1. Weighting/Optimal Gain/Minimum Cost

$m$	Mass of the AGV
$I$	Rotation inertia of the AGV's front wheel about the steering axis
$F$	Driving force
$v$	Velocity of the AGV
$\theta$	Orientation angle of the AGV
$\varphi$	Orientation angle of the AGV's front wheel
$r$	Radius of the front wheel
$d$	distance between the center of the front wheel and the point C
$u_d$	Driving torque of the driving motor
$u_t$	Turning torque of the turning motor
$w$	Orientation angular velocity of the AGV
$w_1$	Driving angular velocity of the front wheel
$w_2$	Turning angular velocity of the front wheel
$x$	$X$ component of the position of the point C of the AGV
$y$	$Y$ component of the position of the point C of the AGV
$\dot{x}$	$X$ component of the velocity of the point C of the AGV
$\dot{y}$	$Y$ component of the velocity of the point C of the AGV
$\ddot{x}$	$X$ component of the acceleration of the point C of the AGV
$\ddot{y}$	$Y$ component of the acceleration of the point C of the AGV

$$v = \dot{x} \frac{\cos \theta}{\cos \varphi} + \dot{y} \frac{\sin \theta}{\cos \varphi}. \quad (14)$$

Differentiating (14) with respect to  $t$  yields

$$\begin{aligned} \dot{v} = & \ddot{x} \frac{\sin \theta}{\cos \varphi} - \dot{x} \frac{\sin \theta}{\cos \varphi} \dot{\theta} + \dot{x} \cos \theta \frac{\tan \varphi}{\cos \varphi} \dot{\varphi} \\ & + \ddot{y} \frac{\sin \theta}{\cos \varphi} + \dot{y} \frac{\cos \theta}{\cos \varphi} \dot{\theta} + \dot{y} \sin \theta \frac{\tan \varphi}{\cos \varphi} \dot{\varphi}. \end{aligned} \quad (15)$$

From (9), (12) and (15), we can obtain the first structural constraint

$$mr \frac{\cos \theta}{\cos \varphi} \ddot{x} + mr \frac{\sin \theta}{\cos \varphi} \ddot{y} = mr \frac{\sin \theta}{\cos \varphi} \dot{\theta} \dot{x} - mr \frac{\cos \theta}{\cos \varphi} \dot{\theta} \dot{y} + u_d. \quad (16)$$

The dynamic equation of the turning motor of the front wheel is

$$(k^2 J_1 + I) \dot{w}_2 + \left( k^2 B_1 + B_3 + \frac{k_m^2 k^2}{R_a} \right) w_2 = \frac{k_m k}{R_a} V_{a2} \quad (17)$$

where  $V_{a2}$  denotes input voltage of the turning motor;  $I$  denotes rotational inertia of the AGV's front wheel about the steering axis;  $B_3$  denotes viscous friction coefficient of the front turning axle; the torque of the turning motor is

$$u_t = \frac{k_m k}{R_a} V_{a2}. \quad (18)$$

Here, the effects of non-ideal constraints are ignored. Then, the turning angular acceleration  $\ddot{\varphi} = \dot{w}_2$  is dependent on  $I$  and  $u_t$ , upon using relation (18) in (17), (17) is simplified as (the second structural constraint)

$$I \ddot{\varphi} = u_t. \quad (19)$$

From (13), the constraint is given by  $\dot{x} \sin \theta - \dot{y} \cos \theta = 0$ . Differentiating it with respect to  $t$  (the third structural constraint)

$$-\sin \theta \ddot{x} + \cos \theta \ddot{y} = \cos \theta \dot{\theta} \dot{x} + \sin \theta \dot{\theta} \dot{y}. \quad (20)$$

Furthermore, orientation angle of the AGV and orientation angular velocity of the AGV has the relationship

$$\dot{\theta} = w = \frac{v \sin \varphi}{d}. \quad (21)$$

By combining (21) with (14), we can get

$$d \dot{\theta} = \dot{x} \cos \theta \tan \varphi + \dot{y} \sin \theta \tan \varphi. \quad (22)$$

Differentiating (22) with respect to  $t$  (the fourth structural constraint)

$$\begin{aligned} \cos \theta \tan \varphi \ddot{x} + \sin \theta \tan \varphi \ddot{y} - d \ddot{\theta} = & \sin \theta \tan \varphi \dot{\theta} \dot{x} \\ & - \cos \theta \frac{\dot{x} \dot{\varphi}}{(\cos \varphi)^2} - \cos \theta \tan \varphi \dot{\varphi} \dot{y} - \sin \theta \frac{\dot{y} \dot{\varphi}}{(\cos \varphi)^2}. \end{aligned} \quad (23)$$

Thus, (16), (19), (20) and (23) can be put into the form of (2), with

$$\begin{bmatrix} mr \frac{\cos \theta}{\cos \varphi} & mr \frac{\sin \theta}{\cos \varphi} & 0 & 0 \\ 0 & 0 & 0 & I \\ -\sin \theta & \cos \theta & 0 & 0 \\ \cos \theta \tan \varphi & \sin \theta \tan \varphi & -d & 0 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \\ \ddot{\varphi} \end{bmatrix} = \begin{bmatrix} mr \frac{\sin \theta}{\cos \varphi} \dot{\theta} \dot{x} - mr \frac{\cos \theta}{\cos \varphi} \dot{\theta} \dot{y} \\ 0 \\ \cos \theta \dot{\theta} \dot{x} + \sin \theta \dot{\theta} \dot{y} \\ \left( \sin \theta \tan \varphi \dot{x} \dot{\theta} - \cos \theta \frac{\dot{x} \dot{\varphi}}{(\cos \varphi)^2} \right) \\ -\cos \theta \tan \varphi \dot{y} \dot{\theta} - \sin \theta \frac{\dot{y} \dot{\varphi}}{(\cos \varphi)^2} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_d \\ u_t \end{bmatrix}. \quad (24)$$

Based on (2), the constraint torque can also be described as

$$\tau = M(q, t) \ddot{q} - Q(q, \dot{q}, t) \quad (25)$$

where

$$\tau = [\tau_1, \tau_2, 0, 0]^T = Q^c(q, \dot{q}, t) = \begin{bmatrix} \mathbf{I}_{2 \times 2} \\ \mathbf{0}_{2 \times 2} \end{bmatrix} \begin{bmatrix} u_d \\ u_t \end{bmatrix}.$$

Comparing (4) and (25), we can get

$$\tau = M^{\frac{1}{2}} (AM^{-\frac{1}{2}})^+ (b - AM^{-1}Q). \quad (26)$$

When the AGV needs to move along a desired trajectory, driving motor and turning motor should generate corresponding torques respectively which are denoted by  $\tau_1$  and  $\tau_2$ .

**Numerical simulation:** Suppose the AGV needs to track a circle:  $x^2 + y^2 = 1$ . That is

$$x = \sin(t), \quad y = -\cos(t). \quad (27)$$

Differentiating (27) twice with respect to  $t$  to obtain the performance constraints of form (1) with

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}, \quad \ddot{q} = [\ddot{x}, \ddot{y}, \ddot{\theta}, \ddot{\varphi}]^T.$$

From (2) and (24), we know

$$Q^c(q, \dot{q}, t) = \begin{bmatrix} \mathbf{I}_{2 \times 2} \\ \mathbf{0}_{2 \times 2} \end{bmatrix} \begin{bmatrix} u_d \\ u_t \end{bmatrix} = Cu$$

where

$$C = A^+ = \begin{bmatrix} \mathbf{I}_{2 \times 2} \\ \mathbf{0}_{2 \times 2} \end{bmatrix}, \quad u = \begin{bmatrix} u_d \\ u_t \end{bmatrix}.$$

Hence, the driving torque and the turning torque generated by the two motors respectively can be obtained by

$$u = C^+ M^{\frac{1}{2}} (AM^{-\frac{1}{2}})^+ (b - AM^{-1}Q). \quad (28)$$

The motion equation of AGV model is obtained through the above theoretical analysis, and then it is numerically simulated by ODE45 differential solver in Matlab, and the error tolerance is  $10^{-12}$ . Table 2 shows the parameter values used in the simulation.

Initial conditions are respectively given as:  $x(0) = 0$ ,  $y(0) = -1$ ,  $\theta(0) = 0$ ,  $\varphi(0) = \tan^{-1}(\frac{13}{10})$ ,  $\dot{x}(0) = 1$ ,  $\dot{y}(0) = 0$ ,  $\dot{\theta}(0) = 1$ ,  $\dot{\varphi}(0) = 0$ . The desired trajectory of the AGV is  $x_d = \sin(t)$ ,  $y_d = -\cos(t)$ , and simulation time is 30 s.

The simulation results are shown in Fig. 2, which shows that AGV can track a circular trajectory well by this method.

**Conclusion:** This letter proposes an AGV trajectory tracking control method based on U-K approach. In which, constraints are classified into structural constraints and performance constraints. The dynamic model is built based on the structural constraints, and expected trajectories are taken as the performance constraints. Through theoretical analysis and numerical simulation, it is proved that this method can solve the trajectory tracking control of AGV well.

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Table 2. The Symbol Definition of the AGV Variables and Parameters

Parameter	Value
Mass of the AGV	$m = 1000 \text{ kg}$
Radius of the front wheel	$r = 0.25 \text{ m}$
Distance from center of front wheel to point C	$d = 1.3 \text{ m}$
The moment of inertia of the front wheel	$I = 0.1 \text{ kg} \cdot \text{m}^2$

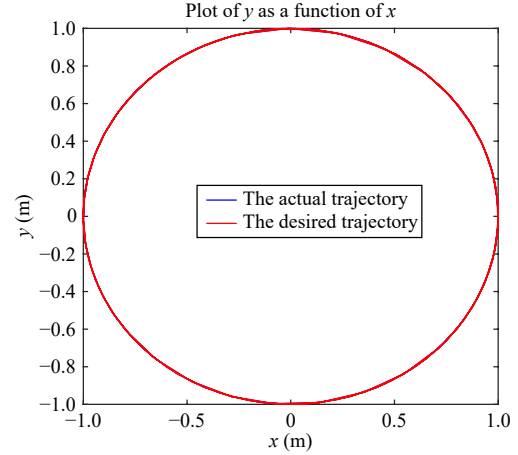


Fig. 2. The actual trajectory and the desired trajectory of the AGV.

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