WCCI 2010 IEEE World Congress on Computational Intelligence
July, 18-23, 2010 - CCIB, Barcelona, Spain FUZZ-IEEE

Stability Analysis of SIRMs Based Type-2 Fuzzy Logic Control Systems

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Abstract— **This paper tries to provide a stability analysis approach for the single input rule modules (SIRMs) based type-2 fuzzy logic control systems. First, in the neighbor of the equilibrium point, the closed-form input-output mappings of type-2 SIRMs (T2SIRMs) are explored, and the derivatives of T2SIRMs at the equilibrium point are computed. Then, how to compute the Jacobian matrix of the SIRMs based type-2 fuzzy logic control systems, which is a fundamental step for local stability analysis, is presented. At last, two examples on stabilization control of the TORA system and the inverted pendulum system are given. The results in both examples demonstrate that the stability analysis results agree completely with the control results.**

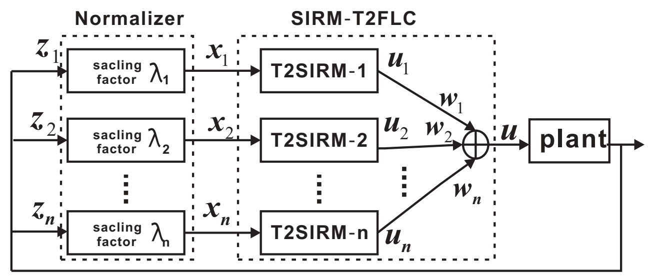

Fig. 1. Block diagram of the SIRMs based type-2 fuzzy logic control system

I. Introduction

In control applications, stability analysis is an important issue. Stability analysis is not an easy task for fuzzy logic control systems, especially for type-2 fuzzy logic control systems. Most results on stability analysis are explored for special fuzzy logic control systems, in which the controlled plants are described using TS type fuzzy rules [1-3]. For this kind of type-2 fuzzy logic control systems, Lam et al. [4] and Biglarbegian et al. [5] have proved some useful stability theorems when the controlled plant and the fuzzy controller both are described by TS type fuzzy rules. But, for general type-2 fuzzy logic control systems which have non-TS type fuzzy rules, global stability analysis is quite difficult, as such type-2 fuzzy logic control systems are seriously nonlinear. What is more, in this case, even local stability analysis may be impossible, for it is hard to obtain the closed-form input-output mappings of the type-2 fuzzy logic controllers (T2FLCs).

Single input rule modules (SIRMs) based fuzzy logic control scheme is first proposed by J. Yi, et al. [6-8] to simplify the design process of conventional type-1 fuzzy logic controllers (T1FLCs), and then studied by H. Seki, et al. [9-11]. Compared with the conventional T1FLCs, the number of rules of the SIRMs based control scheme can be reduced greatly. And, this control scheme has been applied to many control problems, such as, stabilization control of parallel-type double inverted pendulum systems [6], anti-swing and positioning control of overhead traveling crane [8], etc. Recently, we have extended the SIRMs based fuzzy logic control scheme to the type-2 case [12]. Moreover, we have also utilized this SIRMs based type-2 fuzzy logic controller (SIRM-T2FLC) to the stabilization control of the TORA system [13] and the inverted pendulum [14].

However, till now, no stability analysis has been done for the designed SIRM-T2FLCs which have non-TS type fuzzy rules. As it is difficult to make global stability analysis for the SIRMs based type-2 fuzzy logic control system, in this paper, we try to provide an approach for its local stability analysis. To achieve this goal, we first explore the closed-form input-output mappings of the type-2 SIRMs (T2SIRMs), which are necessary components of SIRM-T2FLCs, and then, compute the derivatives of such T2SIRMs at their equilibrium points. On this basis, we present how to compute the closed-loop Jacobian matrix of the SIRMs based type-2 fuzzy logic control system. This is a fundamental step for local stability analysis of the closed-loop system. At last, two examples are given to show the effectiveness of the proposed stability analysis approach.

The organization of this paper is as follows. In Section II, the SIRMs based type-2 fuzzy logic control system is introduced briefly. In Section III, the input-output mappings and derivatives of T2SIRMs are studied and how to compute the Jacobian matrix of the closed-loop system is demonstrated. In Section IV, two examples are given. At last, conclusions are drawn in Section V.

II. SIRMs Based Type-2 Fuzzy Logic Control System

In this section, we will give a brief introduction of the SIRMs based type-2 fuzzy logic control system. For simplicity, the controlled plant is assumed to have n state variables and only one control input. The results can be readily extended to controlled plants which have multiple control inputs.

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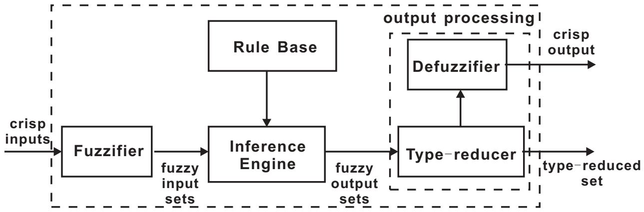

Fig. 2. The inference structure of T2SIRM [15-19]

Figure 1 shows a block diagram of the SIRMs based type-2 fuzzy logic control system. In our following discussion, we suppose that the controlled plant is depicted as:

$$\dot{\boldsymbol{z}}(t) = \boldsymbol{f}(\boldsymbol{z}(t)) + \boldsymbol{g}(\boldsymbol{z}(t))u(\boldsymbol{z}(t)), \tag{1}$$

where $\boldsymbol{z}(t) = (z_1(t), z_2(t), ..., z_n(t))^{\mathrm{T}} \in \mathbb{R}^n$ is the state vector of the system, $\boldsymbol{f}(\boldsymbol{z}(t)) = (f_1(\boldsymbol{z}(t)),\ f_2(\boldsymbol{z}(t)),\ ...,\ f_n(\boldsymbol{z}(t)))^{\mathrm{T}} \in \mathbb{R}^{n\times 1}$, $\boldsymbol{g}(\boldsymbol{z}(t)) = (g_1(\boldsymbol{z}(t)),\ g_2(\boldsymbol{z}(t)),\ ...,\ g_n(\boldsymbol{z}(t)))^{\mathrm{T}} \in \mathbb{R}^{n\times 1}$, and $u(\boldsymbol{z}(t)) \in \mathbb{R}$ is the control input to the system from a SIRM-T2FLC.

A SIRM-T2FLC for the controlled plant which has n states has n input items x_1, x_2, ..., x_n, which are normalized values of z_1, z_2, ..., z_n, i.e. $x_i = \lambda_i z_i$, where λ_i is the scaling factor for the variable z_i. A SIRM-T2FLC with n input items is composed of n type-2 SIRMs (T2SIRMs). The T2SIRM for the input item x_i (T2SIRM-i) can be expressed as [12-14]

$$\text{T2SIRM-}i: \quad \{R_i^j: \text{ if } x_i \text{ is } \widetilde{A}_i^j, \text{ then } u_i \text{ is } \widetilde{C}_i^j\}_{j=1}^{m_i} \tag{2}$$

where $\widetilde{A}_i^j$s are interval type-2 fuzzy sets (IT2FSs), $\widetilde{C}_i^j$s are interval weighting factors, and $\widetilde{C}_i^j = [\underline{c}_i^j, \overline{c}_i^j]$. IT2FSs $\widetilde{A}_i^j$s and interval weighting factors $[\underline{c}_i^j, \overline{c}_i^j]$s can be formed by blurring type-1 fuzzy sets A_i^js and crisp values c_i^js, respectively [15-20].

A T2SIRM can be seen as a single-input-single-output type-2 fuzzy logic system (T2FLS) [15-20]. Therefore, the inference structure of T2SIRMs is the same as the inference structure of T2FLSs. The inference structure of T2SIRMs is shown in Fig. 2 and it consists of a fuzzifier, an inference engine, a rule base, a type-reducer and a defuzzifier [15-20].

Once a crisp input $x_i = \lambda_i z_i$ is applied to T2SIRM-i, through the singleton fuzzifier and the inference process, the interval firing strength of the jth rule can be expressed as

$$F_i^j(x_i) = [\underline{f}_i^j(x_i), \overline{f}_i^j(x_i)], \tag{3}$$

where

$$\underline{f}_i^j(x_i) = \underline{\mu}_{\widetilde{A}_i^j}(x_i), \quad \overline{f}_i^j(x_i) = \overline{\mu}_{\widetilde{A}_i^j}(x_i), \tag{4}$$

in which $\underline{\mu}(), \overline{\mu}()$ denote the grades of the lower and upper membership functions (MFs) of IT2FSs.

To generate a crisp output from T2SIRM-i, the outputs of the inference engine should be type-reduced and then defuzzified. Here, we adopt the most widely used center-of-sets (COS) type-reduction method which is based on Karnik-Mendel algorithms [15-19]. With the COS type-reduction method [15-19], the output of the type-reducer in T2SIRM-i can be expressed as

$$U_i(x_i) = [u_{il}(x_i), u_{ir}(x_i)], \tag{5}$$

where the left end point $u_{il}(x_i)$ and the right end point $u_{ir}(x_i)$ can be expressed as

$$\begin{aligned} u_{il}(x_i) &= \min\left\{\frac{\sum_{j=1}^{m_i} f_i^j c_i^j}{\sum_{j=1}^{m_i} f_i^j} | f_i^j \in F_i^j(x_i), c_i^j \in [\underline{c}_i^j, \overline{c}_i^j]\right\} \\ &= \frac{\sum_{j=1}^{L_i} \overline{f}_i^j(x_i)\underline{c}_i^j + \sum_{j=L_i+1}^{m_i} \underline{f}_i^j(x_i)\underline{c}_i^j}{\sum_{j=1}^{L_i} \overline{f}_i^j(x_i) + \sum_{j=L_i+1}^{m_i} \underline{f}_i^j(x_i)} \end{aligned} \tag{6}$$

$$\begin{aligned} u_{ir}(x_i) &= \max\left\{\frac{\sum_{j=1}^{m_i} f_i^j c_i^j}{\sum_{j=1}^{m_i} f_i^j} | f_i^j \in F_i^j(x_i), c_i^j \in [\underline{c}_i^j, \overline{c}_i^j]\right\} \\ &= \frac{\sum_{j=1}^{R_i} \underline{f}_i^j(x_i)\overline{c}_i^j + \sum_{j=R_i+1}^{m_i} \overline{f}_i^j(x_i)\overline{c}_i^j}{\sum_{j=1}^{R_i} \underline{f}_i^j(x_i) + \sum_{j=R_i+1}^{m_i} \overline{f}_i^j(x_i)} \end{aligned} \tag{7}$$

where L_i and R_i are switch points that can be computed by the iterative Karnik-Mendel algorithms [15-19].

After the COS type-reduction process, the defuzzified crisp output from T2SIRM-i is the average of $u_{il}(x_i)$ and $u_{ir}(x_i)$, i.e.

$$u_i(x_i) = \frac{1}{2}(u_{il}(x_i) + u_{ir}(x_i)). \tag{8}$$

Once the fuzzy inference result $u_i(x_i)$ of T2SIRM-i is already calculated, to express clearly the different role of each input item on system performance, the output $u(\boldsymbol{z})$ of SIRM-T2FLC will be computed as the summation of the products of the fuzzy inference result $u_i(x_i)$ of T2SIRM-i and the importance degree w_i of the ith input item x_i. In other words, the output $u(\boldsymbol{z})$ of SIRM-T2FLC can be calculated as [12-14]

$$u(\boldsymbol{z}) = \sum_{i=1}^{n} w_i u_i(x_i) = \sum_{i=1}^{n} w_i u_i(\lambda_i z_i). \tag{9}$$

As discussed in [12-14], the total number of rules in SIRM-T2FLCs increases only linearly with the number of input items and SIRM-T2FLCs can deal with the rule explosion problem.

III. Stability Analysis of SIRM Based Type-2 Fuzzy Logic Control System

It is quite difficult, maybe impossible, to analyze the global stability of the closed-loop systems in (1) controlled by SIRM-T2FLCs. Therefore, in this section, we will only consider the local stability in the neighbor of the equilibrium points of the SIRMs based type-2 fuzzy logic control systems. Below, we will first explore the input-output mappings and the derivatives of T2SIRMs, and then study the local stability of the SIRMs based type-2 fuzzy logic control systems.

For simplicity, in this paper, we suppose that the equilibrium point of the SIRMs based type-2 fuzzy logic control systems is the origin $\mathbf{0}$. Similar results can be obtained for the equilibrium points in other locations.

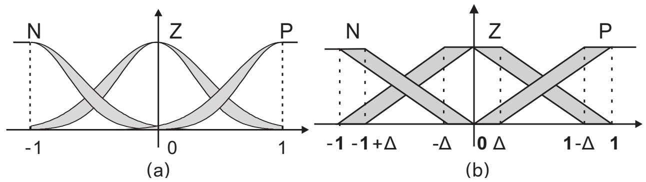

Fig. 3. Type-2 fuzzy partition of the input domain $[-1, 1]$: (a) Gaussian IT2FSs, (b) triangular IT2FSs

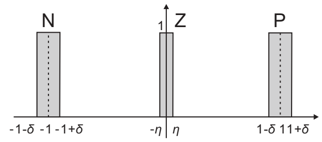

Fig. 4. Interval partition of the output domain

A. Input-Output Mapping and Derivatives of T2SIRM

Usually, the IT2FSs adopted in T2FLCs are Gaussian or triangular. If Gaussian or triangular IT2FSs are utilized in T2SIRM, the input domain $[-1, 1]$ of the normalized variable x_i can be partitioned as shown in Fig. 3(a) or Fig. 3(b). For the output variable u, the partition of its domain is often chosen as shown in Fig. 4. In SIRM-T2FLCs, we always choose the following two kinds of T2SIRMs as its components (denoted as T2SIRM-I and T2SIRM-II for short):

$$\text{T2SIRM-I}: \begin{cases} x = \mathbf{N} & \rightarrow \quad u = \mathbf{N} \\ x = \mathbf{Z} & \rightarrow \quad u = \mathbf{Z} \\ x = \mathbf{P} & \rightarrow \quad u = \mathbf{P} \end{cases} \tag{10}$$

$$\text{T2SIRM-II}: \begin{cases} x = \mathbf{N} & \rightarrow \quad u = \mathbf{P} \\ x = \mathbf{Z} & \rightarrow \quad u = \mathbf{Z} \\ x = \mathbf{P} & \rightarrow \quad u = \mathbf{N} \end{cases} \tag{11}$$

In the following, we study the input-output mappings and derivatives of T2SIRM-I and T2SIRM-II in two situations.

- **The Antecedent IT2FSs in T2SIRM-I and T2SIRM-II are Gaussian** (as Shown in Fig. 3(a))

In order to reduce the number of the parameters in T2SIRM-I and T2SIRM-II, we suppose that the antecedent Gaussian IT2FSs **N, Z, P** in T2SIRM-I and T2SIRM-II are symmetric with respect to the origin 0, and they have the same uncertain width $[\underline{\sigma}^2, \overline{\sigma}^2]$. In the same way, we also assume that the consequent intervals **N, Z, P** in T2SIRM-I and T2SIRM-II are symmetric with respect to the origin 0, i.e. $\mathbf{N} = [-1-\delta, -1+\delta] = -\mathbf{P}$, and $\mathbf{Z} = [-\eta, \eta]$.

If more than two fuzzy rules can be fired at one point in the input domain, then the outputs of T2SIRM-I and T2SIRM-II at this point should be computed using the iterative Karnik-Mendel algorithms [15-19]. As a result, no closed-form expressions can be obtained for the input-output mappings of T2SIRM-I and T2SIRM-II at this point. If the antecedent IT2FSs in T2SIRM-I and T2SIRM-II are Gaussian, then, in the neighbor of the origin 0, the three fuzzy rules in T2SIRM-I and T2SIRM-II can always be fired. Hence, it is impossible to obtain the closed-form input-output mappings of T2SIRM-I and T2SIRM-II and compute the derivatives of T2SIRM-I and T2SIRM-II at the origin 0. However, if the consequent interval $\mathbf{Z}$ satisfies that $\mathbf{Z} = [-\eta, \eta] = 0$, the closed-form input-output mappings of T2SIRM-I and T2SIRM-II in the neighbor of the origin 0 can be obtained.

In this case, for T2SIRM-I, its input-output mapping in the neighbor of the origin 0 can be expressed as (details are omitted)

$$u_I(x) = \frac{1}{2}(u_{lI}(x) + u_{rI}(x)), \tag{12}$$

$$u_{lI}(x) = \frac{(-1-\delta)e^{-\frac{(x+1)^2}{2\overline{\sigma}^2}} + (1-\delta)e^{-\frac{(x-1)^2}{2\underline{\sigma}^2}}}{e^{-\frac{(x+1)^2}{2\overline{\sigma}^2}} + e^{-\frac{x^2}{2\underline{\sigma}^2}} + e^{-\frac{(x-1)^2}{2\underline{\sigma}^2}}}, \quad |x| \leq h, \tag{13}$$

$$u_{rI}(x) = \frac{(-1+\delta)e^{-\frac{(x+1)^2}{2\underline{\sigma}^2}} + (1+\delta)e^{-\frac{(x-1)^2}{2\overline{\sigma}^2}}}{e^{-\frac{(x+1)^2}{2\underline{\sigma}^2}} + e^{-\frac{x^2}{2\underline{\sigma}^2}} + e^{-\frac{(x-1)^2}{2\overline{\sigma}^2}}}, \quad |x| \leq h, \tag{14}$$

where

$$h = \begin{cases} \min\{b - \sqrt{b^2 - c}, 1\}, & if \quad b^2 - c \geq 0 \\ 1, & if \quad b^2 - c < 0 \end{cases} \tag{15}$$

and

$$b = \frac{\overline{\sigma}^2 + \underline{\sigma}^2}{\overline{\sigma}^2 - \underline{\sigma}^2}, \tag{16}$$

$$c = 1 - \frac{2\overline{\sigma}^2\underline{\sigma}^2}{\overline{\sigma}^2 - \underline{\sigma}^2} \ln \frac{1-\delta}{1+\delta}. \tag{17}$$

From(12)-(14), we can compute the derivative of T2SIRM-I at the origin 0 as: (details are omitted)

$$\frac{\partial u_I(x)}{\partial x}|_{x=0} = \frac{1}{2}\left[\frac{\partial u_{lI}(x)}{\partial x}|_{x=0} + \frac{\partial u_{rI}(x)}{x}|_{x=0}\right], \tag{18}$$

$$\frac{\partial u_{lI}(x)}{\partial x}|_{x=0} = \frac{p_l'(0)q_l(0) - p_l(0)q_l'(0)}{(q_l(0))^2}, \tag{19}$$

$$\frac{\partial u_{rI}(x)}{\partial x}|_{x=0} = \frac{p_r'(0)q_r(0) - p_r(0)q_r'(0)}{(q_r(0))^2}, \tag{20}$$

where

$$p_l(0) = (-1-\delta)e^{-\frac{1}{2\overline{\sigma}^2}} + (1-\delta)e^{-\frac{1}{2\underline{\sigma}^2}}, \tag{21}$$

$$p_l'(0) = \frac{1+\delta}{\overline{\sigma}^2}e^{-\frac{1}{2\overline{\sigma}^2}} + \frac{1-\delta}{\underline{\sigma}^2}e^{-\frac{1}{2\underline{\sigma}^2}}, \tag{22}$$

$$q_l(0) = 1 + e^{-\frac{1}{2\overline{\sigma}^2}} + e^{-\frac{1}{2\underline{\sigma}^2}}, \tag{23}$$

$$q_l'(0) = -\frac{1}{\overline{\sigma}^2}e^{-\frac{1}{2\overline{\sigma}^2}} + \frac{1}{\underline{\sigma}^2}e^{-\frac{1}{2\underline{\sigma}^2}}, \tag{24}$$

$$p_r(0) = (-1+\delta)e^{-\frac{1}{2\underline{\sigma}^2}} + (1+\delta)e^{-\frac{1}{2\overline{\sigma}^2}}, \tag{25}$$

$$p_r'(0) = \frac{1-\delta}{\underline{\sigma}^2}e^{-\frac{1}{2\underline{\sigma}^2}} + \frac{1+\delta}{\overline{\sigma}^2}e^{-\frac{1}{2\overline{\sigma}^2}}, \tag{26}$$

$$q_r(0) = 1 + e^{-\frac{1}{2\underline{\sigma}^2}} + e^{-\frac{1}{2\overline{\sigma}^2}}, \tag{27}$$

$$q_r'(0) = -\frac{1}{\underline{\sigma}^2}e^{-\frac{1}{2\underline{\sigma}^2}} + \frac{1}{\overline{\sigma}^2}e^{-\frac{1}{2\overline{\sigma}^2}}. \tag{28}$$

In the similar way, for T2SIRM-II, its input-output mapping in the neighbor of the origin 0 can be obtained as (details are omitted)

$$u_{II}(x) = -u_I(x). \tag{29}$$

Hence, the derivative of T2SIRM-II at the origin 0 can be computed as

$$\frac{\partial u_{II}(x)}{\partial x}|_{x=0} = -\frac{\partial u_I(x)}{\partial x}|_{x=0}. \tag{30}$$

- **The Antecedent IT2FSs in T2SIRM-I and T2SIRM-II are Triangular** (as shown in Fig. 3(b))

In this situation, no more than two fuzzy rules can be fired at each point in $[-1, 1]$, therefore, the closed-form input-output mappings of T2SIRM-I and T2SIRM-II in the neighbor of the origin 0 can be obtained.

For T2SIRM-I, (details are omitted)

$$u_I(x) = \begin{cases} x, & x \in [-1,1], \Delta = 0, \\ \frac{1}{2}\frac{(1-\delta-\eta)x}{1-\Delta}, & x \in [-\Delta, \Delta], \Delta \neq 0. \end{cases} \tag{31}$$

For T2SIRM-II, (details are omitted)

$$u_{II}(x) = \begin{cases} -x, & x \in [-1,1], \Delta = 0, \\ -\frac{1}{2}\frac{(1-\delta-\eta)x}{1-\Delta}, & x \in [-\Delta, \Delta], \Delta \neq 0. \end{cases} \tag{32}$$

From (31) and (32), the derivatives of T2SIRM-I and T2SIRM-II at the origin 0 can be computed as

$$\frac{\partial u_I(x)}{\partial x}|_{x=0} = \begin{cases} 1, & \Delta = 0 \\ \frac{1}{2}\frac{1-\delta-\eta}{1-\Delta}, & \Delta \neq 0 \end{cases} \tag{33}$$

$$\frac{\partial u_{II}(x)}{\partial x}|_{x=0} = \begin{cases} -1, & \Delta = 0 \\ -\frac{1}{2}\frac{1-\delta-\eta}{1-\Delta}, & \Delta \neq 0 \end{cases} \tag{34}$$

B. Stability Analysis

Suppose that $\mathbf{0}$ is an equilibrium of the autonomous system $\dot{\boldsymbol{z}}(t) = \boldsymbol{f}(\boldsymbol{z}(t))$, i.e. $\boldsymbol{f}(\mathbf{0}) = \mathbf{0}$, and that $\boldsymbol{f}(.), \boldsymbol{g}(.), u(.)$ are differentiable. Further, assume that the Jacobian matrix $J = \left[\frac{\partial(\boldsymbol{f}(\boldsymbol{z}(t))+\boldsymbol{g}(\boldsymbol{z}(t))u(\boldsymbol{z}(t)))}{\partial \boldsymbol{z}}\right]_{\boldsymbol{z}=\mathbf{0}}$ is bounded.

As well known [21], under above conditions, $\mathbf{0}$ is an exponentially stable equilibrium of (1), if: 1) $u(\mathbf{0})$=0, which assures that $\boldsymbol{z} = \mathbf{0}$ is an equilibrium of (1), 2) the real parts of the eigenvalues of its Jacobian matrix J are negative.

Therefore, in order to verdict the local stability of the closed-loop systems in (1), first, we should prove that the outputs of SIRM-T2FLCs satisfy that $u(\mathbf{0})$=0, and then, compute the closed-loop Jacobian matrix J to judge whether the real parts of its eigenvalues are negative.

- **The First Condition:**

For T2SIRM-I, if its antecedent IT2FSs are Gaussian, then, from (13) and (14), it is obvious that

$$u_{lI}(x) = -u_{rI}(-x). \tag{35}$$

Hence, in this case, the input-output mapping of T2SIRM-I at the origin 0 satisfies that

$$u_I(0) = \frac{u_{lI}(0) + u_{rI}(0)}{2} = \frac{-u_{rI}(0) + u_{rI}(0)}{2} = 0. \tag{36}$$

For T2SIRM-I, if its antecedent IT2FSs are triangular, then, from (31), it is obvious that $u_I(0) = 0$

From similar analysis, for T2SIRM-II, we can prove that $u_{II}(0) = 0$.

Therefore, the T2SIRM-Is/T2SIRM-IIs based type-2 fuzzy logic controllers satisfy the first condition.

- **The Second Condition: Computation of the Jacobian Matrix**

Below, we will discuss how to obtain the Jacobian matrix at the equilibrium $\mathbf{0}$ for the SIRMs based type-2 fuzzy logic control systems.

Denote the Jacobian matrix J as $J = [a_{ij}]_{n \times n}$, where

$$a_{i,j} = \frac{\partial f_i(\boldsymbol{z}(t))}{\partial z_j}|_{\boldsymbol{z}=\mathbf{0}} + \frac{\partial g_i(\boldsymbol{z}(t))u(\boldsymbol{z}(t))}{\partial z_j}|_{\boldsymbol{z}=\mathbf{0}}. \tag{37}$$

Note that

$$\begin{aligned} &\left[\frac{\partial g_i(\boldsymbol{z}(t))u(\boldsymbol{z}(t))}{\partial z_j}\right]_{\boldsymbol{z}=\mathbf{0}} \\ =&u(\mathbf{0})\frac{\partial g_i(\boldsymbol{z}(t))}{\partial z_j}|_{\boldsymbol{z}=\mathbf{0}} + g_i(\mathbf{0})\frac{\partial u(\boldsymbol{z}(t))}{\partial z_j}|_{\boldsymbol{z}=\mathbf{0}}. \end{aligned} \tag{38}$$

From above discussion, we know that

$$u(\mathbf{0}) = 0, \tag{39}$$

$$u(\boldsymbol{z}(t)) = \sum_{i=1}^{n} w_i u_i(x_i) = \sum_{i=1}^{n} w_i u_i(\lambda_i z_i). \tag{40}$$

Hence, from (38) (39) and (40), we can obtain that

$$\begin{aligned} \left[\frac{\partial g_i(\boldsymbol{z}(t))u(\boldsymbol{z}(t))}{\partial z_j}\right]_{\boldsymbol{z}=\mathbf{0}} &= g_i(\mathbf{0})\frac{\partial u(\boldsymbol{z}(t))}{\partial z_j}|_{\boldsymbol{z}=\mathbf{0}} \\ &= g_i(\mathbf{0}) w_j \frac{\partial u_j(\lambda_j z_j)}{\partial z_j}|_{z_j=0} \\ &= g_i(\mathbf{0}) w_j \left(\frac{\partial u_j(\lambda_j z_j)}{\partial(\lambda_j z_j)}\frac{\partial \lambda_j z_j}{\partial z_j}\right)|_{z_j=0} \\ &= g_i(\mathbf{0}) w_j \lambda_j \frac{\partial u_j(x_j)}{\partial x_j}|_{x_j=0}. \end{aligned} \tag{41}$$

In summary,

$$a_{i,j} = \frac{\partial f_i(\boldsymbol{z}(t))}{\partial z_j}|_{\boldsymbol{z}=\mathbf{0}} + g_i(\mathbf{0}) w_j \lambda_j \frac{\partial u_j(x_j)}{\partial x_j}|_{x_j=0}. \tag{42}$$

As discussed in the previous subsection, for the specific T2SIRMs – T2SIRM-I and T2SIRM-II, $\frac{\partial u_j(x_j)}{\partial x_j}|_{x_j=0}$ can be determined. Thus, we can obtain the Jacobian matrix at the equilibrium $\mathbf{0}$ for the SIRMs based type-2 fuzzy logic control systems. And, it is obvious that, for T2SIRM-I/II based fuzzy logic controllers, the scaling factors λ_i, the changeable width $\underline{\sigma}^2, \overline{\sigma}^2, \Delta$ and the importance degrees w_i all affect the stability of the equilibrium $\mathbf{0}$ of the SIRMs based type-2 fuzzy logic control systems.

In conclusion, as the closed-loop Jacobian matrix of the SIRMs based type-2 fuzzy logic control system can be obtained, we can judge whether the SIRMs based type-2 fuzzy logic control system is exponentially stable through observing the locations of the eigenvalues of the Jacobian matrix J.

IV. Examples

In this section, we use the following two examples to demonstrate the effectiveness of the stability analysis approach. The first example is the stabilization control of the TORA system, while the second example is the stabilization control of the inverted pendulum systems.

A. Control of the TORA System

Suppose that the disturbance $F = 0$, then, the dynamic of the TORA system can be given by the following equation [13,22]:

$$\dot{\boldsymbol{z}} = \boldsymbol{f}(\boldsymbol{z}) + \boldsymbol{g}(\boldsymbol{z})u, \tag{43}$$

where $\boldsymbol{z} = (z_1, z_2, z_3, z_4)^{\mathrm{T}}$, u is the torque applied to the eccentric mass m and obtained from a SIRM-T2FLC, and

$$\boldsymbol{f}(\boldsymbol{z}) = \begin{bmatrix} z_2 \\ \frac{-z_1+\varepsilon z_4^2 \sin z_3}{1-\varepsilon^2\cos^2 z_3} \\ z_4 \\ \frac{\varepsilon \cos z_3 (z_1 - \varepsilon z_4^2 \sin z_3)}{1-\varepsilon^2\cos^2 z_3} \end{bmatrix}, \tag{44}$$

$$\boldsymbol{g}(\boldsymbol{z}) = \begin{bmatrix} 0 \\ \frac{-\varepsilon \cos z_3}{1-\varepsilon^2\cos^2 z_3} \\ 0 \\ \frac{1}{1-\varepsilon^2\cos^2 z_3} \end{bmatrix}, \tag{45}$$

where the coupling parameter ε is set up to be 0.1.

In [13], we have shown how to design the SRIM-T2FLC for this system. The T2SIRMs chosen for z_1, z_2, z_3 and z_4 are T2SIRM-I, T2SIRM-I, T2SIRM-II and T2SIRM-II, respectively. In these T2SIRMs, Gaussian IT2FSs (as shown in Fig. 3(a)) are adopted, and, the parameters in the consequent intervals are chosen as $\eta = 0, \delta = 0.1$. The other parameters of the SIRM-T2FLC for the TORA system are [13]:

$$\begin{aligned} &\lambda_1 = 4.27, \lambda_2 = 1.94, \lambda_3 = 0.58, \lambda_4 = 0.67 \\ &w_1 = 9.98, w_2 = 0.71, w_3 = 2.84, w_4 = 8.65 \\ &[\underline{\sigma}_1^2, \overline{\sigma}_1^2] = [0.48, 0.63], [\underline{\sigma}_2^2, \overline{\sigma}_2^2] = [0.82, 0.98], \\ &[\underline{\sigma}_3^2, \overline{\sigma}_3^2] = [0.21, 0.42], [\underline{\sigma}_4^2, \overline{\sigma}_4^2] = [0.25, 0.36]. \end{aligned}$$

Substituting these parameters into (18)-(30) and (42), we can obtain the Jacobian matrix at the equilibrium **0** for this closed-loop system as

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4.4840 & -0.0829 & 0.1353 & 0.5174 \\ 0 & 0 & 0 & 1 \\ 34.8403 & 0.8294 & -1.3533 & -5.1736 \end{bmatrix}.$$

The eigenvalues of this Jacobian matrix are: -4.1265, $-0.3817 + 0.8653\mathrm{i}$, $-0.3817 - 0.8653\mathrm{i}$, -0.3667. Thus, we know that this closed-loop system is stable in the neighbor of the equilibrium **0**.

One simulation result of the SIRM-T2FLC in this example is shown in Fig. 5.

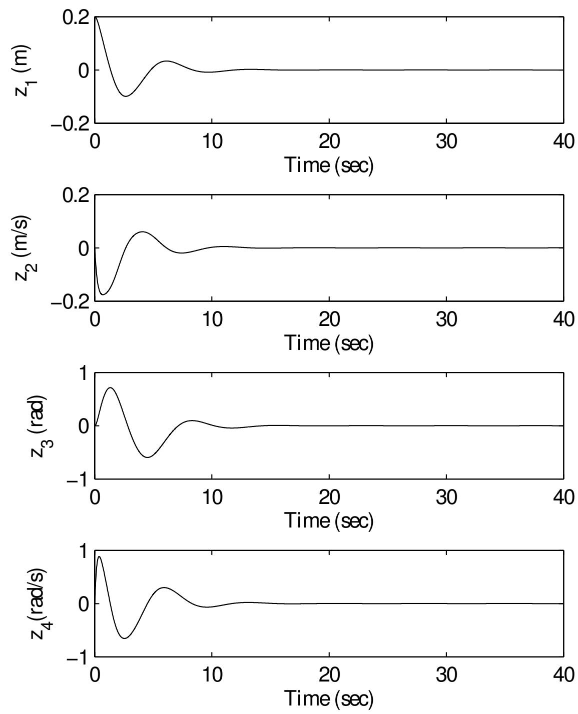

Fig. 5. Control result of the SIRM-T2FLC in example 1

B. Control of the Inverted Pendulum System

The dynamic equations of the inverted pendulum system can be expressed as [14,23]

$$\dot{z_1} = z_2, \tag{46}$$

$$\dot{z_2} = \frac{(M+m)g\sin z_1 - mlz_2^2 \sin z_1 \cos z_1 - u(t)\cos z_1}{\frac{4}{3}(M+m)l - ml\cos^2 z_1}, \tag{47}$$

$$\dot{z_3} = z_4, \tag{48}$$

$$\dot{z_4} = \frac{\frac{4}{3}mlz_2^2 \sin z_1 - mg\sin z_1 \cos z_1 + \frac{4}{3}u(t)}{\frac{4}{3}(M+m) - m\cos^2 z_1}, \tag{49}$$

where $z_1 = \theta$ is the pole angle, $z_2 = \dot{\theta}$ is the angular velocity of the pole, $z_3 = x$ is the position of the cart, and z_4 is the velocity of the cart. In our simulation, the mass of the cart M is 1kg, the mass of the pole m is 0.1kg, half of the pole length l is 1m, and g=9.81.

In [14], we have shown how to design the SRIM-T2FLC for this system. The T2SIRMs chosen for z_1, z_2, z_3 and z_4 are all T2SIRM-I. In these T2SIRMs, triangular IT2FSs, whose widths are set up to be $\Delta = 0.2$, are adopted, and, the parameters in the consequent intervals are chosen as $\eta = 0, \delta = 0$. The other parameters of the SIRM-T2FLC for the inverted pendulum system are [14]:

$$\begin{aligned} &\lambda_1 = 3.82, \lambda_2 = 1.90, \lambda_3 = 0.5, \lambda_4 = 1, \\ &w_1 = 50, w_2 = 50, w_3 = 20, w_4 = 20. \end{aligned}$$

Substituting these parameters into (33), (34) and (42), we can obtain the Jacobian matrix at the equilibrium **0** for this

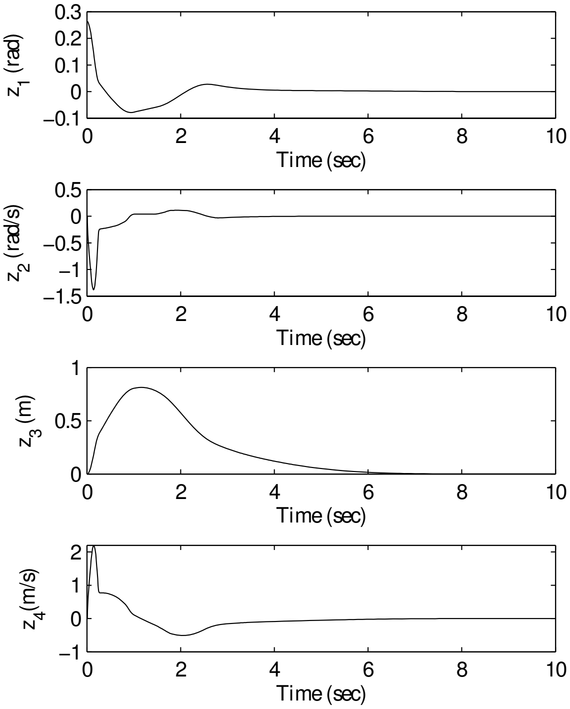

Fig. 6. Control result of the SIRM-T2FLC in example 2

closed-loop system as

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -79.4517 & -43.4451 & -4.5732 & -9.1463 \\ 0 & 0 & 0 & 1 \\ 185.6237 & 92.6829 & 9.7561 & 19.5122 \end{bmatrix}.$$

The eigenvalues of this Jacobian matrix are: -28.8101, $-0.7474 + 1.0435\text{i}$, $-0.7474 - 1.0435\text{i}$, -0.9451. Thus, we know that this closed-loop system is stable in the neighbor of the equilibrium $\mathbf{0}$.

One simulation result of the SIRM-T2FLC in this example is shown in Fig. 6. We also depict the stability domain (with respect to the pole angle and the angular velocity) of the inverted pendulum system controlled by the SIRM-T2FLC in Fig. 7, where t_s denotes the stabilizing time. From this figure, we can see that the stability domain of the proposed controller is large enough.

V. Conclusions

In our work, a stability analysis approach for the SIRMs based type-2 fuzzy logic control systems is provided. Although stability analysis of conventional non-TS type T2FLCs is difficult, stability analysis of the SIRMs based type-2 fuzzy logic control systems becomes possible, because the closed-form input-output mappings of T2SIRMs which are necessary components of SIRM-T2FLCs can be obtained in the neighbor of the equilibrium point under some reasonable assumptions. These mappings can be used to realize the fundamental step for local stability analysis – computation of the Jacobian matrix of the closed-loop system. These mappings also demonstrate that, in the neighbor

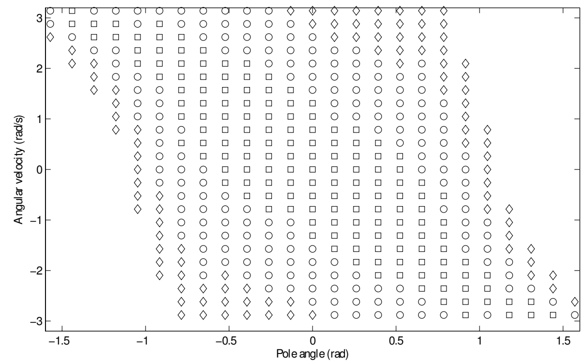

Fig. 7. Stability region of the pole angle and the angular velocity (□: $t_s \leq$ 10s, ○: $t_s \leq$ 15s, ◇:$t_s \leq$ 20s)

of the equilibrium point, the T2SIRM which uses Gaussian antecedent IT2FSs has nonlinear characteristics while the T2SIRM which uses triangular antecedent IT2FSs has linear characteristics. The proposed approach is applied to the stabilization control of the TORA system and the inverted pendulum system. The fact that the stability analysis results are consistent entirely with the control results shows the effectiveness of the proposed approach.

Acknowledgment

This work was partly supported by the NSFC Project under Grant No. 60975060, National 863 Program (No. 2007AA04Z239), China.

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