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The Monotonicity and Convexity of Unnormalized Interval Type-2 TSK Fuzzy Logic Systems

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***Abstract*— This paper applies prior knowledge - monotonicity and convexity - to a Single-Input-Single-Output (SISO) unnormalized interval type-2 Takagi-Sugeno-Kang (TSK) Fuzzy Logic System (FLS). Sufficient conditions are provided to guarantee its monotonicity and convexity with respect to its input, respectively. The derived monotonic conditions focus on a zeroth-order TSK fuzzy model. Also, the corresponding proofs for the convex conditions of both the zeroth-order and first-order TSK fuzzy models are given, respectively. For the zeroth-order fuzzy systems, simulation examples demonstrate the validity of the theorems.**

I. Introduction

Since the late 1990s as a result of Prof. Jerry Mendel and his groups' works on type-2 fuzzy sets and systems [1,2], more and more researchers are paying close attention to type-2 fuzzy sets and systems. At present, interval type-2 fuzzy sets have received the most attention, and interval type-2 fuzzy sets and systems are being actively researched by an ever-growing number of researchers around the world [3]. Interval Type-2 Fuzzy Logic Systems (IT2FLSs) have obvious advantages for handling different sources of uncertainty, reducing the number of fuzzy rules, etc. Due to the merits, there exist a lot of literatures about the applications of IT2FLSs, for example, nonlinear channel equalization [5] and nonlinear system identification [6].

Until now, we have always constructed the fuzzy models of complex systems only using sample data. Since noisy data can not be avoided in practical applications, the information contained in sample data is always insufficient. Fortunately, some prior knowledge of plants or systems, such as bounded range, symmetry, monotonicity and convexity, etc., can compensate the insufficiency of the information from sample data [4,7]. Recently, lots of the work has been done to incorporate prior knowledge into neural networks [12], type-1 or traditional FLSs [8-10], etc. In [8], Won has presented sufficient conditions of monotonicity for the traditional normalized TSK fuzzy systems. In [9], Kim has introduced a constructive manner for the design of the convex type-1 TSK fuzzy systems. In [6,7], Li has proposed sufficient conditions to ensure that the prior knowledge of symmetry, bounded range and monotonicity can be incorporated into normalized interval type-2 FLSs.

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Normalizing the output of a TSK FLS has been shown to be unnecessary in some cases because it increases its computational complexity [11]. In spite of the fact that an Unnormalized Interval Type-2 TSK FLS (UIT2FLS) is simple, it can provide fast inference, reduce the computational complexity, and may achieve the same performance as the normalized TSK in some specific applications [1,5].

At present, to the authors' knowledge, there are no literatures that incorporate prior knowledge into UIT2FLSs. In this paper, we investigate two kinds of prior knowledge — monotonicity and convexity for SISO UIT2FLSs. The paper is organized as follows: Section II briefly introduces SISO UIT2FLSs. Based on Section II, in Section III and Section IV, sufficient conditions for SISO UIT2FLSs are given to ensure the monotonic and convex input-output relationships, respectively. And, simulation examples verify the validity of Theorems 3.1 and 4.1, respectively. Finally, Section V makes conclusions.

II. SISO UIT2FLS

Some preliminary knowledge about UIT2FLSs is introduced in this section.

A. Single-input first-order UIT2FLS

A single-input first-order UIT2FLS is depicted by M fuzzy IF-THEN rules. The ith rule R^i is denoted as

R^i: IF x is $\widetilde{A}^i$, THEN $Y^i = C_0^i + C_1^i x$,

where $i = 1, 2, \ldots, M$; x is the input variable; $\widetilde{A}^i$s are antecedent triangular IT2FSs which can be completely depicted by its the lower and upper membership functions — $\underline{\mu}_{\widetilde{A}^i}(x)$ and $\overline{\mu}_{\widetilde{A}^i}(x)$; and C_p^is $(p = 0, 1)$ are consequent parameters that are interval sets, i.e. $C_p^i = [c_p^i - s_p^i, c_p^i + s_p^i]$, in which c_p^i denotes the center of C_p^i and s_p^i denotes the spread of C_p^i. The firing set $F^i(x)$ of rule R^i is an interval type-1 set, i.e., $F^i(x) = [\underline{f}^i(x), \overline{f}^i(x)]$, in which $\underline{f}^i(x) = \underline{\mu}_{\widetilde{A}^i}(x)$ and $\overline{f}^i(x) = \overline{\mu}_{\widetilde{A}^i}(x)$. The consequent Y^i of rule R^i is also an interval set, i.e., $Y^i = [y_l^i(x), y_r^i(x)]$, where

$$y_l^i(x) = c_1^i x + c_0^i - |x|\, s_1^i - s_0^i, \tag{1}$$
$$y_r^i(x) = c_1^i x + c_0^i + |x|\, s_1^i + s_0^i. \tag{2}$$

According to Theorem 13-2 in [1], the final output of the unnormalized interval type-2 TSK model is inferred as

$$Y(x) = \frac{1}{2}\sum_{i=1}^{M}(\underline{f}^i(x) y_l^i(x) + \overline{f}^i(x) y_r^i(x)). \tag{3}$$

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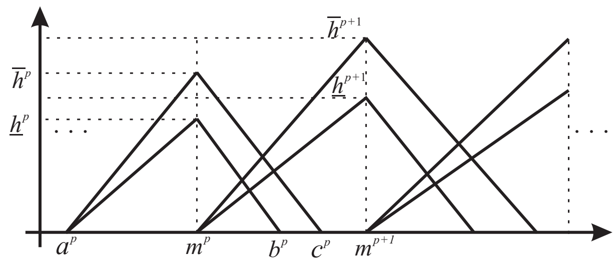

Fig. 1. Interval type-2 fuzzy partition in Section III.

B. Single-input zeroth-order UIT2FLS

Consider a single-input zeroth-order UIT2FLS whose fuzzy rule base consists of M fuzzy rules which can be regarded as a particular form of the previous rules R^is. The ith fuzzy rule is

$$\widehat{R}^i\text{: IF } x \text{ is } \widetilde{A}^i, \text{ THEN } Y^i = C_0^i,$$

where the characters in rule $\widehat{R}^i$ have the same meanings as before. For the sake of convenience, let $C_0^i = [\underline{w}^i, \overline{w}^i]$, i.e., $c_0^i - s_0^i = \underline{w}^i, c_0^i + s_0^i = \overline{w}^i (i = 1, 2, \ldots, M)$. Accordingly, $y_l^i(x) = \underline{w}^i$ and $y_r^i(x) = \overline{w}^i$.

III. MONOTONIC UIT2FLS

In this section, without loss of generality, we only consider the monotonic increase. The definition of single-input monotonic fuzzy systems is given as follows.

Definition 3.1: Let x be an input of a fuzzy system defined on $U = [a, b]$, where $U \subset \mathbf{R}$, and $y = F(x)$ be the output of the fuzzy system in the set $V \subset \mathbf{R}$. Then, $F : U \mapsto V$ is said to be monotonically increasing if $a \leq x_1 \leq x_2 \leq b$ implies $F(x_1) \leq F(x_2)$ [8].

A. Single-input zeroth-order monotonic UIT2FLS

This section focuses on the single-input zeroth-order FLS described by $\widehat{R}^p$s $(p = 1, 2, \ldots, M)$ in which the lower and upper membership functions of $\widetilde{A}^p$ can be written as

$$\underline{\mu}_{\widetilde{A}^p}(x) = \begin{cases} \underline{h}^p \frac{x-a^p}{m^p-a^p}, & a^p < x \leq m^p \\ \underline{h}^p \frac{b^p-x}{b^p-m^p}, & m^p < x \leq b^p \\ 0, & x \leq a^p \text{ or } x > b^p \end{cases}, \tag{4}$$

and

$$\overline{\mu}_{\widetilde{A}^p}(x) = \begin{cases} \overline{h}^p \frac{x-a^p}{m^p-a^p}, & a^p < x \leq m^p \\ \overline{h}^p \frac{c^p-x}{c^p-m^p}, & m^p < x \leq c^p \\ 0, & x \leq a^p \text{ or } x > c^p \end{cases}, \tag{5}$$

where $a^p < m^p < b^p \leq c^p$ and $\underline{h}^p \leq \overline{h}^p \leq 1$. Fig. 1 is a specific case of the membership functions.

Theorem 3.1: Assume that the input domain $U = [u_l, u_r]$ is partitioned by M triangular IT2FSs $\widetilde{A}^1, \widetilde{A}^2, \ldots, \widetilde{A}^M$ as shown in Fig. 1. Then, the UIT2FLS is monotonically increasing with respect to x, if the following conditions are satisfied:

1) No more than two fuzzy rules are fired, i.e. $a^1 = u_l$ or $a^1 = m^2 = u_l$, $m^M = b^M = c^M = u_r$, $a^{p+1} = m^p$, $a^p < m^p < b^p \leq c^p \leq m^{p+1}$;
2) $\underline{k}_+^{p+1}\underline{w}^{p+1} + \overline{k}_+^{p+1}\overline{w}^{p+1} - \underline{k}_-^p\underline{w}^p - \overline{k}_-^p\overline{w}^p \geq 0$;
3) $\underline{w}^{p+1} \geq \underline{w}^p$ and $\overline{w}^{p+1} \geq \overline{w}^p (p = 1, 2, \ldots, M-1)$;
4) $\underline{v}_{\max}^p\underline{w}^{p+1} + \overline{v}_{\max}^p\overline{w}^{p+1} - \underline{v}_{\min}^p\underline{w}^p - \overline{v}_{\min}^p\overline{w}^p \geq 0$ and $(\underline{v}_{\min}^{p+1} - \underline{v}_{\max}^p)\underline{w}^{p+1} + (\overline{v}_{\min}^{p+1} - \overline{v}_{\max}^p)\overline{w}^{p+1} \geq 0$;

where

$$\begin{aligned} &\overline{k}_+^p = \overline{h}^p/(m^p - a^p), \underline{k}_+^p = \underline{h}^p/(m^p - a^p), \\ &\overline{k}_-^p = \overline{h}^p/(c^p - m^p), \underline{k}_-^p = \underline{h}^p/(b^p - m^p), \\ &(p = 1, 2, \ldots, M), \\ \underline{v}_{\max}^p &= \sup\{\underline{\mu}_{\widetilde{A}^p}(x) + \underline{\mu}_{\widetilde{A}^{p+1}}(x) : x \in S^p\}, \\ \underline{v}_{\min}^p &= \inf\{\underline{\mu}_{\widetilde{A}^p}(x) + \underline{\mu}_{\widetilde{A}^{p+1}}(x) : x \in S^p\}, \\ \overline{v}_{\max}^p &= \sup\{\overline{\mu}_{\widetilde{A}^p}(x) + \overline{\mu}_{\widetilde{A}^{p+1}}(x) : x \in S^p\}, \\ \overline{v}_{\min}^p &= \inf\{\overline{\mu}_{\widetilde{A}^p}(x) + \overline{\mu}_{\widetilde{A}^{p+1}}(x) : x \in S^p\}, \\ &S^p = [m^p, m^{p+1}] \quad (p = 1, 2, \ldots, M-1). \end{aligned}$$

Proof: Assume that no more than two fuzzy IT2FSs can be fired and the order numbers of the two fired rules are p and $p+1$. Substituting $\underline{f}^i(x) = \underline{\mu}_{\widetilde{A}^i}(x)$, $\overline{f}^i(x) = \overline{\mu}_{\widetilde{A}^i}(x)$, $y_l^i(x) = \underline{w}^i$ and $y_r^i(x) = \overline{w}^i$ into (3), the final output can be rewritten as

$$Y(x) = \frac{1}{2}\sum_{k=p}^{p+1}(\underline{\mu}_{\widetilde{A}^k}(x)\underline{w}^k + \overline{\mu}_{\widetilde{A}^k}(x)\overline{w}^k). \tag{6}$$

As illustrated in Fig. 1, the theorem will be proved in three cases as follows.

1) When $x \in S^p$, $x' \in S^p$, $x' > x$,

$$\begin{aligned} &Y(x') - Y(x) \\ =&\frac{1}{2}[(\underline{\mu}_{\widetilde{A}^p}(x') - \underline{\mu}_{\widetilde{A}^p}(x))\underline{w}^p + (\overline{\mu}_{\widetilde{A}^p}(x') \\ &- \overline{\mu}_{\widetilde{A}^p}(x))\overline{w}^p + (\underline{\mu}_{\widetilde{A}^{p+1}}(x') - \underline{\mu}_{\widetilde{A}^{p+1}}(x))\underline{w}^{p+1} \\ &+ (\overline{\mu}_{\widetilde{A}^{p+1}}(x') - \overline{\mu}_{\widetilde{A}^{p+1}}(x))\overline{w}^{p+1}]. \end{aligned} \tag{7}$$

According to the different regions that x and x' belong to, we can discuss the first case as follows.

a) When $x \leq b^p$, $x' \leq b^p$ and $x \leq x'$, (7) can be rewritten as

$$\begin{aligned} &Y(x') - Y(x) \\ =&\frac{1}{2}[-\Delta x\underline{k}_-^p\underline{w}^p - \Delta x\overline{k}_-^p\overline{w}^p \\ &+ \Delta x\underline{k}_+^{p+1}\underline{w}^{p+1} + \Delta x\overline{k}_+^{p+1}\overline{w}^{p+1}] \\ =&\frac{1}{2}\Delta x[-\underline{k}_-^p\underline{w}^p - \overline{k}_-^p\overline{w}^p \\ &+ \underline{k}_+^{p+1}\underline{w}^{p+1} + \overline{k}_+^{p+1}\overline{w}^{p+1}], \end{aligned}$$

where $\Delta x = x' - x$ and $\Delta x > 0$.

b) When $x \leq b^p$, $b^p < x' \leq m^{p+1}$ and $x \leq x'$, we denote $\underline{\delta} = x' - b^p$, if $b^p < x' \leq m^{p+1}$; $\overline{\delta} = 0$, if $b^p < x' \leq c^p$; and $\overline{\delta} = x' - c^p$, if $c^p < x' \leq m^{p+1}$. So we can easily find $\underline{\delta} > 0$

and $\overline{\delta} \geq 0$. Furthermore, we have

$$\begin{aligned} &Y(x') - Y(x) \\ =&\frac{1}{2}[-(\Delta x - \underline{\delta})\underline{k}_{-}^{p}\underline{w}^{p} - (\Delta x - \overline{\delta})\overline{k}_{-}^{p}\overline{w}^{p} \\ &+ \Delta x \underline{k}_{+}^{p+1}\underline{w}^{p+1} + \Delta x \overline{k}_{+}^{p+1}\overline{w}^{p+1}] \\ \geq&\frac{1}{2}[-\Delta x \underline{k}_{-}^{p}\underline{w}^{p} - \Delta x \overline{k}_{-}^{p}\overline{w}^{p} \\ &+ \Delta x \underline{k}_{+}^{p+1}\underline{w}^{p+1} + \Delta x \overline{k}_{+}^{p+1}\overline{w}^{p+1}]. \end{aligned}$$

c) When $b^p < x \leq c^p$, $b^p < x' \leq m^{p+1}$ and $x \leq x'$, we let $\overline{\delta} = x' - c^p$, if $c^p < x' \leq m^{p+1}$. Since $\overline{\delta} > 0$, we get

$$\begin{aligned} &Y(x') - Y(x) \\ =&\frac{1}{2}[0 - (\Delta x - \overline{\delta})\overline{k}_{-}^{p}\overline{w}^{p} \\ &+ \Delta x \underline{k}_{+}^{p+1}\underline{w}^{p+1} + \Delta x \overline{k}_{+}^{p+1}\overline{w}^{p+1}] \\ \geq&\frac{1}{2}[-\Delta x \underline{k}_{-}^{p}\underline{w}^{p} - \Delta x \overline{k}_{-}^{p}\overline{w}^{p} \\ &+ \Delta x \underline{k}_{+}^{p+1}\underline{w}^{p+1} + \Delta x \overline{k}_{+}^{p+1}\overline{w}^{p+1}]. \end{aligned}$$

d) When $b^p < x \leq m^{p+1}$, $b^p < x' \leq m^{p+1}$ and $x \leq x'$, (7) can be rewritten as

$$\begin{aligned} &Y(x') - Y(x) \\ =&\frac{1}{2}[0 + 0 + \Delta x \underline{k}_{+}^{p+1}\underline{w}^{p+1} + \Delta x \overline{k}_{+}^{p+1}\overline{w}^{p+1}] \\ \geq&\frac{1}{2}[-\Delta x \underline{k}_{-}^{p}\underline{w}^{p} - \Delta x \overline{k}_{-}^{p}\overline{w}^{p} \\ &+ \Delta x \underline{k}_{+}^{p+1}\underline{w}^{p+1} + \Delta x \overline{k}_{+}^{p+1}\overline{w}^{p+1}]. \end{aligned}$$

Based on the four cases a), b), c), and d), we can derive $Y(x') \geq Y(x)$ on S^p if the second condition of Theorem 3.1 holds.

2) When $x \in S^p$ and $x' \in S^{p+1}$, it is obvious that $x' > x$. If the third condition of Theorem 3.1 holds, we can obtain

$$\begin{aligned} Y(x) \geq&\frac{1}{2}[(\underline{\mu}_{\widetilde{A}^p}(x) + \underline{\mu}_{\widetilde{A}^{p+1}}(x))\underline{w}^p \\ &+ (\overline{\mu}_{\widetilde{A}^p}(x) + \overline{\mu}_{\widetilde{A}^{p+1}}(x))\overline{w}^p] \\ \geq&\frac{1}{2}[\underline{v}_{\min}^{p}\underline{w}^p + \overline{v}_{\min}^{p}\overline{w}^p] \end{aligned}$$

and

$$\begin{aligned} Y(x) \leq&\frac{1}{2}[(\underline{\mu}_{\widetilde{A}^p}(x) + \underline{\mu}_{\widetilde{A}^{p+1}}(x))\underline{w}^{p+1} \\ &+ (\overline{\mu}_{\widetilde{A}^p}(x) + \overline{\mu}_{\widetilde{A}^{p+1}}(x))\overline{w}^{p+1}] \\ \leq&\frac{1}{2}[\underline{v}_{\max}^{p}\underline{w}^{p+1} + \overline{v}_{\max}^{p}\overline{w}^{p+1}]. \end{aligned}$$

So, we can derive $\frac{1}{2}(\underline{v}_{\min}^{p}\underline{w}^p + \overline{v}_{\min}^{p}\overline{w}^p) \leq Y(x) \leq \frac{1}{2}(\underline{v}_{\max}^{p}\underline{w}^{p+1} + \overline{v}_{\max}^{p}\overline{w}^{p+1})$. If the second inequality of the fourth condition holds, the minimum value of $Y(x)$ on the interval S^{p+1} must be greater than the maximum value of $Y(x)$ on the interval S^p. Thus, there exists the inequality $Y(x') \geq Y(x)$ in this case.

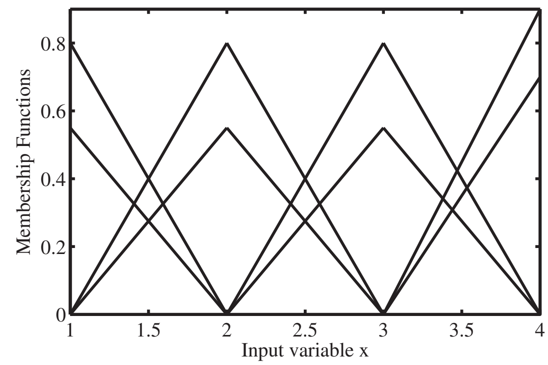

Fig. 2. Graph of the membership functions.

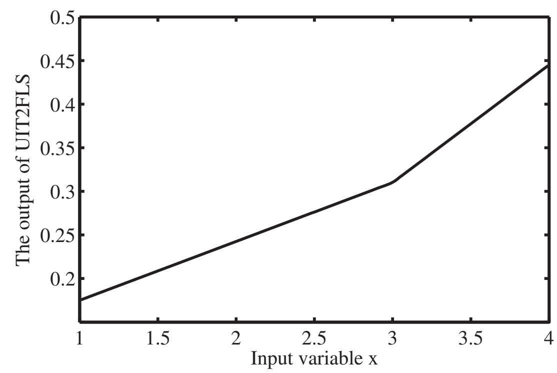

Fig. 3. Graph of the output of the zeroth-order monotonic UIT2FLS.

3) When $x \in S^p$, $x' \in S^q$ and $q > p$, $x' > x$ holds, where $p, q = 1, 2, ..., M$.
As mentioned above in 2), we can obtain
$$Y^{p+1}(x_1) \geq Y^p(x_2),$$
where $p = 1, 2, ..., M-1, x_1 \in S^{p+1}$ and $x_2 \in S^p$. Then, we have
$$Y(x') \geq Y(x).$$

Therefore, from the discussion above, we can conclude that the theorem holds.

B. Example of monotonic UIT2FLS

Consider the SISO zeroth-order UIT2FLS with four fuzzy rules as follows.

R^i: IF x is $\widetilde{A}^i$, THEN $Y^i = C_0^i$ $(i = 1, 2, 3, 4)$,

where $x \in U$, $U = [1, 4]$. Based on (4) and (5), the parameters of $\widetilde{A}^i$ $(i = 1, 2, 3, 4)$ are

$$\begin{aligned} &a^1 = 1, m^1 = 1, b^1 = 2, c^1 = 2, \\ &a^2 = 1, m^2 = 2, b^2 = 3, c^2 = 3, \\ &a^3 = 2, m^3 = 3, b^3 = 4, c^3 = 4, \\ &a^4 = 3, m^4 = 4, b^4 = 4, c^4 = 4, \\ &\overline{h}^i = 0.8, \underline{h}^i = 0.55 \ (i = 1, 2, 3), \\ &\overline{h}^4 = 0.9, \underline{h}^4 = 0.7. \end{aligned}$$

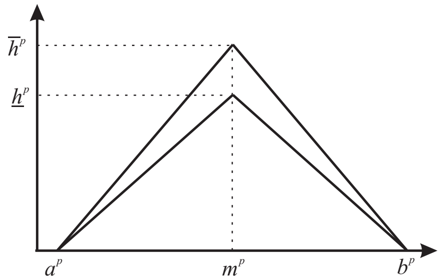

Fig. 4. Interval type-2 fuzzy membership function in Section IV.

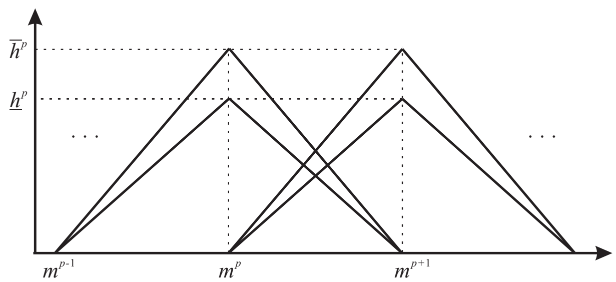

Fig. 5. Interval type-2 fuzzy partition in Section IV.

The consequent interval weights of the THEN-part are $C_0^1 = [0.2, 0.3]$, $C_0^2 = [0.3, 0.4]$, $C_0^3 = [0.4, 0.5]$ and $C_0^4 = [0.5, 0.6]$. All these parameters meet the conditions of Theorem 3.1, therefore the output of the UIT2FLS should be monotonically increasing with respect to its input. The four membership functions are depicted in Fig. 2, and the output of the zeroth-order monotonic UIT2FLS is shown in Fig. 3. It is obvious that the simulation result is consistent with Theorem 3.1.

IV. CONVEX UIT2FLS

In this part, we study sufficient conditions that make SISO UIT2FLSs become convex. The definition of single-input convex fuzzy systems is given below.

Definition 4.1: Let $x \in U \subset \mathbf{R}$ be an input of fuzzy system $y = F(x) \in V \subset \mathbf{R}$. Then, $F : U \mapsto V$ is said to be convex with respect to x, if the following condition is satisfied:

$$F(\lambda x_1 + (1-\lambda)x_2) \leq \lambda F(x_1) + (1-\lambda)F(x_2),$$

where $0 \leq \lambda \leq 1$ [10].

A. Single-input zeroth-order convex UIT2FLS

Consider a SISO UIT2FLS whose IF-THEN rules are the same as the rules in Section II-B. The lower and upper membership functions of the fuzzy sets $\widetilde{A}^p$s used here are expressed in (8) and (9), as illustrated in Fig. 4.

$$\underline{\mu}_{\widetilde{A}^p}(x) = \begin{cases} \underline{h}^p \frac{x-a^p}{m^p-a^p}, & a^p < x \leq m^p \\ \underline{h}^p \frac{b^p-x}{b^p-m^p}, & m^p < x \leq b^p \\ 0, & x \leq a^p \text{ or } x > b^p \end{cases} \quad (8)$$

$$\overline{\mu}_{\widetilde{A}^p}(x) = \begin{cases} \overline{h}^p \frac{x-a^p}{m^p-a^p}, & a^p < x \leq m^p \\ \overline{h}^p \frac{b^p-x}{b^p-m^p}, & m^p < x \leq b^p \\ 0, & x \leq a^p \text{ or } x > b^p \end{cases} \quad (9)$$

Some sufficient conditions which can ensure the convexity of the single-input zeroth-order UIT2FLSs are studied below.

Theorem 4.1: Assume that the input domain $U = (u_l, u_r)$ is partitioned by M triangular IT2FSs $\widetilde{A}^1, \widetilde{A}^2, \ldots, \widetilde{A}^M$ as shown in Fig. 5. Then, the UIT2FLS is convex with respect to x, if the following conditions are satisfied:

1) $m^p < m^q$ for $q > p$, $1 \leq p \leq M-1$ and $2 \leq q \leq M$;
2) $a^1 = m^1 = u_l$, $b^M = m^M = u_r$, $m^p = a^{p+1}$ and $b^p = m^{p+1}$ for $1 \leq p \leq M-1$;
3) $(m^{p+1} - m^p)(\underline{h}^{p-1}\underline{w}^{p-1} + \overline{h}^{p-1}\overline{w}^{p-1}) - (m^{p+1} - m^{p-1})(\underline{h}^p\underline{w}^p + \overline{h}^p\overline{w}^p) + (m^p - m^{p-1})(\underline{h}^{p+1}\underline{w}^{p+1} + \overline{h}^{p+1}\overline{w}^{p+1}) \geq 0$ for $2 \leq p \leq M-1$.

Proof: According to the first two conditions of Theorem 4.1, we can derive that only two interval type-2 fuzzy sets are fired at each point on the interval U as illustrated in Fig. 5. Hence, substituting

$$\underline{\mu}_{\widetilde{A}^p}(x) = \underline{h}^p(m^{p+1} - x)/(m^{p+1} - m^p), \quad (10)$$

$$\overline{\mu}_{\widetilde{A}^p}(x) = \overline{h}^p(m^{p+1} - x)/(m^{p+1} - m^p), \quad (11)$$

$$\underline{\mu}_{\widetilde{A}^{p+1}}(x) = \underline{h}^{p+1}(x - m^p)/(m^{p+1} - m^p), \quad (12)$$

$$\overline{\mu}_{\widetilde{A}^{p+1}}(x) = \overline{h}^{p+1}(x - m^p)/(m^{p+1} - m^p), \quad (13)$$

into (6), the final output can be rewritten as

$$Y(x) = \frac{1}{2(m^{p+1} - m^p)}[(m^{p+1} - x)(\underline{h}^p\underline{w}^p + \overline{h}^p\overline{w}^p) + (x - m^p)(\underline{h}^{p+1}\underline{w}^{p+1} + \overline{h}^{p+1}\overline{w}^{p+1})]. \quad (14)$$

After simple computation, we can obtain

$$Y(x) = a(p)x + b(p), \quad (15)$$

where

$$a(p) = \frac{\underline{h}^{p+1}\underline{w}^{p+1} + \overline{h}^{p+1}\overline{w}^{p+1} - \underline{h}^p\underline{w}^p - \overline{h}^p\overline{w}^p}{2(m^{p+1} - m^p)} \quad (16)$$

and

$$b(p) = \frac{m^{p+1}(\underline{h}^p\underline{w}^p + \overline{h}^p\overline{w}^p) - m^p(\underline{h}^{p+1}\underline{w}^{p+1} + \overline{h}^{p+1}\overline{w}^{p+1})}{2(m^{p+1} - m^p)}.$$

Once the fuzzy system has been constructed, $a(p)$ and $b(p)$ are constants. Since linear function is a kind of convex function, each segment of $Y(x)$ on open interval (m^p, m^{p+1}) $(p = 1, 2, \ldots, M)$ is convex. But in the neighborhood of the points m^p $(p = 2, 3, \ldots, M-1)$, $Y(x)$ may not be convex. In order to ensure that the fuzzy system is convex on the input domain U, the relationship between *slope*$(\overline{AB})$ and *slope*$(\overline{BC})$ must satisfy (17) [9], as illustrated in Fig. 6.

$$\textit{slope}(\overline{AB}) \leq \textit{slope}(\overline{BC}), \quad (17)$$

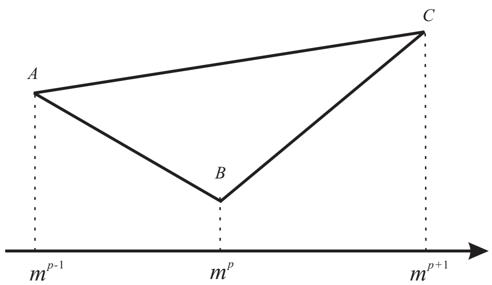

Fig. 6. Three vertices in three-cord property.

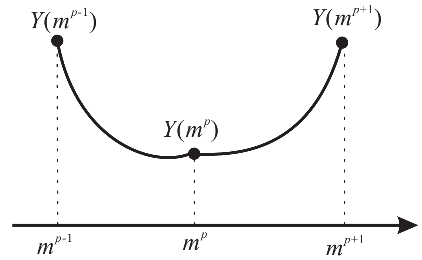

Fig. 7. Quadratic function in S^{p-1} and S^p.

where *slope*$(\overline{AB})$ and *slope*$(\overline{BC})$ denote the slopes of the straight-line segments. Moreover, from (15), we can infer that *slope*$(\overline{AB}) = a(p-1)$ and *slope*$(\overline{BC}) = a(p)$.

Based on (17), the following inequality can be derived

$$a(p-1) \leq a(p). \tag{18}$$

Substituting (16) into (18), we have

$$\begin{aligned}
&(m^{p+1} - m^p)(\underline{h}^{p-1}\underline{w}^{p-1} + \overline{h}^{p-1}\overline{w}^{p-1}) \\
&- (\underline{h}^p\underline{w}^p + \overline{h}^p\overline{w}^p)(m^{p+1} - m^{p-1}) \\
&+ (m^p - m^{p-1})(\underline{h}^{p+1}\underline{w}^{p+1} + \overline{h}^{p+1}\overline{w}^{p+1}) \geq 0.
\end{aligned}$$

According to (17), every three adjacent vertices can constitute a triangle which is a kind of convex polygons. Therefore, the whole polygon is also convex if the third condition of Theorem 4.1 holds.

B. Single-input first-order convex UIT2FLS

Consider an SISO UIT2FLS whose IF-THEN rules are the same as the rules in Section II-A. The lower and upper membership functions of the fuzzy sets $\widetilde{A}^p$s $(p = 1, 2, \ldots, M)$ used in this part are depicted in (8) and (9), as illustrated in Fig. 4. Then, we derive the following sufficient conditions which can guarantee the convexity of the single-input first-order UIT2FLSs.

Theorem 4.2: Assume that the input domain $U = (u_l, u_r)$ is partitioned by triangular IT2FSs $\widetilde{A}^1, \widetilde{A}^2, \ldots, \widetilde{A}^M$ as shown in Fig. 5. Then, the UIT2FLS is convex with respect to x, if the following conditions are satisfied:

1) $m^p < m^q$ for $q > p$, $1 \leq p \leq M-1$ and $2 \leq q \leq M$;
2) $a^1 = m^1 = u_l$, $b^M = m^M = u_r$, $m^p = a^{p+1}$ and $b^p = m^{p+1}$ for $1 \leq p \leq M-1$;
3) $(\overline{h}^{p+1} + \underline{h}^{p+1})c_1^{p+1} + (\overline{h}^{p+1} - \underline{h}^{p+1})\mathrm{sgn}(x)s_1^{p+1} - (\overline{h}^p + \underline{h}^p)c_1^p - (\overline{h}^p - \underline{h}^p)\mathrm{sgn}(x)s_1^p \geq 0$ for $x \in (m^p, m^{p+1})$;
4) $(\overline{h}^{p+1} + \underline{h}^{p+1})(m^p - m^{p-1})(m^p c_1^{p+1} + c_0^{p+1}) + (\overline{h}^{p+1} - \underline{h}^{p+1})(m^p - m^{p-1})(|m^p| s_1^{p+1} + s_0^{p+1}) - (\overline{h}^p + \underline{h}^p)(m^{p+1} - m^{p-1})(m^p c_1^p + c_0^p) - (\overline{h}^p - \underline{h}^p)(m^{p+1} - m^{p-1})(|m^p| s_1^p + s_0^p) + (\overline{h}^{p-1} + \underline{h}^{p-1})(m^{p+1} - m^p)(m^p c_1^{p-1} + c_0^{p-1}) + (\overline{h}^{p-1} - \underline{h}^{p-1})(m^{p+1} - m^p)(|m^p| s_1^{p-1} + s_0^{p-1}) \geq 0$ for $2 \leq p \leq M-1$.

Proof: According to the first two conditions of Theorem 4.2, we can derive that only two interval type-2 fuzzy sets are fired at each point on the interval U as illustrated in Fig. 5. Hence, substituting (1), (2), (10), (11), (12) and (13) into (6), the final output can be computed as

$$\begin{aligned}
Y(x) =& \frac{1}{2}(\underline{h}^p \frac{m^{p+1} - x}{m^{p+1} - m^p}(c_1^p x + c_0^p - |x|\, s_1^p - s_0^p) \\
&+ \overline{h}^p \frac{m^{p+1} - x}{m^{p+1} - m^p}(c_1^p x + c_0^p + |x|\, s_1^p + s_0^p) \\
&+ \underline{h}^{p+1} \frac{x - m^p}{m^{p+1} - m^p}(c_1^{p+1} x + c_0^{p+1} - |x|\, s_1^{p+1}) \\
&- s_0^{p+1} + \overline{h}^{p+1} \frac{x - m^p}{m^{p+1} - m^p}(c_1^{p+1} x \\
&+ c_0^{p+1} + |x|\, s_1^{p+1} + s_0^{p+1})).
\end{aligned} \tag{19}$$

We notice that $|x| = \mathrm{sgn}(x)x$, where $\mathrm{sgn}(x)$ is sign function. (19) can be rewritten as

$$\begin{aligned}
Y(x) =& \frac{1}{2}(\underline{h}^p \frac{m^{p+1} - x}{m^{p+1} - m^p}(c_1^p x + c_0^p - x\mathrm{sgn}(x) s_1^p - s_0^p) \\
&+ \overline{h}^p \frac{m^{p+1} - x}{m^{p+1} - m^p}(c_1^p x + c_0^p + x\mathrm{sgn}(x) s_1^p + s_0^p) \\
&+ \underline{h}^{p+1} \frac{x - m^p}{m^{p+1} - m^p}(c_1^{p+1} x + c_0^{p+1} - x\mathrm{sgn}(x) s_1^{p+1} \\
&- s_0^{p+1}) + \overline{h}^{p+1} \frac{x - m^p}{m^{p+1} - m^p}(c_1^{p+1} x + c_0^{p+1} \\
&+ x\mathrm{sgn}(x) s_1^{p+1} + s_0^{p+1})).
\end{aligned} \tag{20}$$

Simplifying (20) and combining the similar terms, we get

$$Y(x) = \widehat{a}(p)x^2 + \widehat{b}(p)x + \widehat{c}(p), \tag{21}$$

where

$$\begin{aligned}
\widehat{a}(p) =& \frac{1}{2(m^{p+1} - m^p)}[(\overline{h}^{p+1} + \underline{h}^{p+1})c_1^{p+1} + (\overline{h}^{p+1} - \underline{h}^{p+1}) \\
&\times \mathrm{sgn}(x) s_1^{p+1} - (\overline{h}^p + \underline{h}^p)c_1^p - (\overline{h}^p - \underline{h}^p)\mathrm{sgn}(x) s_1^p], \\
\widehat{b}(p) =& \frac{1}{2(m^{p+1} - m^p)}[(\overline{h}^{p+1} + \underline{h}^{p+1})c_0^{p+1} + (\overline{h}^{p+1} - \underline{h}^{p+1}) \\
&\times s_0^{p+1} - (\overline{h}^p + \underline{h}^p)c_0^p - (\overline{h}^p - \underline{h}^p)s_0^p + (\overline{h}^p + \underline{h}^p) \\
&\times m^{p+1} c_1^p + (\overline{h}^p - \underline{h}^p) m^{p+1} \mathrm{sgn}(x) s_1^p - (\overline{h}^{p+1} \\
&+ \underline{h}^{p+1}) m^p c_1^{p+1} - (\overline{h}^{p+1} - \underline{h}^{p+1}) m^p \mathrm{sgn}(x) s_1^{p+1}], \\
\widehat{c}(p) =& \frac{1}{2(m^{p+1} - m^p)}[(\overline{h}^p + \underline{h}^p) m^{p+1} c_0^p + (\overline{h}^p - \underline{h}^p) m^{p+1} s_0^p \\
&- (\overline{h}^{p+1} + \underline{h}^{p+1}) m^p c_0^{p+1} - (\overline{h}^{p+1} - \underline{h}^{p+1}) m^p s_0^{p+1}] \\
&\text{for} \quad x \in S^p.
\end{aligned}$$

Once the fuzzy system has been constructed, $\widehat{a}(p)$, $\widehat{b}(p)$ and $\widehat{c}(p)$ are constants. $Y(x)$ can be a quadratic function with respect to x when $x \in S^p$. Therefore, if $m^{p+1} > m^p$ and $\widehat{a}(p) \geq 0$, that is, the third condition of Theorem 4.2 holds, then the segments are convex when $x \in (m^p, m^{p+1})$ $(p = 1, 2, \ldots, M-1)$, as displayed in Fig. 7. But in the neighborhood of the points m^p $(p = 2, 3, \ldots, M-1)$, the fuzzy system output $Y(x)$ may not be convex. In order to ensure that $Y(x)$ is convex on the input domain U, the following inequality (22) should be satisfied [10]:

$$\lim_{x \to m^{p-}} \frac{dY(x)}{dx} \leq \lim_{x \to m^{p+}} \frac{dY(x)}{dx}, \tag{22}$$

where $\frac{dY(x)}{dx} = 2\widehat{a}(p)x + \widehat{b}(p)\big|_{x=m_p}$.

On the basis of (22), we have

$$2\widehat{a}(p-1)m^p + \widehat{b}(p-1) \leq 2\widehat{a}(p)m^p + \widehat{b}(p). \tag{23}$$

Substituting $\widehat{a}(p-1), \widehat{a}(p), \widehat{b}(p-1)$ and $\widehat{b}(p)$ into (23), we can obtain

$$\begin{aligned}
&2\frac{1}{(m^p - m^{p-1})}[(\overline{h}^p + \underline{h}^p)c_1^p + (\overline{h}^p - \underline{h}^p)\mathrm{sgn}(m^p)s_1^p \\
&- (\overline{h}^{p-1} + \underline{h}^{p-1})c_1^{p-1} - (\overline{h}^{p-1} - \underline{h}^{p-1})\mathrm{sgn}(m^p)s_1^{p-1}]m^p \\
&+ \frac{1}{(m^p - m^{p-1})}[(\overline{h}^p + \underline{h}^p)c_0^p + (\overline{h}^p - \underline{h}^p)s_0^p \\
&- (\overline{h}^{p-1} + \underline{h}^{p-1})c_0^{p-1} - (\overline{h}^{p-1} - \underline{h}^{p-1})s_0^{p-1} \\
&+ (\overline{h}^{p-1} + \underline{h}^{p-1})m^p c_1^{p-1} + (\overline{h}^{p-1} - \underline{h}^{p-1})m^p\mathrm{sgn}(m^p)s_1^{p-1} \\
&- (\overline{h}^p + \underline{h}^p)m^{p-1}c_1^p - (\overline{h}^p - \underline{h}^p)m^{p-1}\mathrm{sgn}(m^p)s_1^p] \\
\leq &2\frac{1}{(m^{p+1} - m^p)}[(\overline{h}^{p+1} + \underline{h}^{p+1})c_1^{p+1} + (\overline{h}^{p+1} - \underline{h}^{p+1}) \\
&\times \mathrm{sgn}(m^p)s_1^{p+1} - (\overline{h}^p + \underline{h}^p)c_1^p - (\overline{h}^p - \underline{h}^p)\mathrm{sgn}(m^p)s_1^p]m^p \\
&+ \frac{1}{(m^{p+1} - m^p)}[(\overline{h}^{p+1} + \underline{h}^{p+1})c_0^{p+1} + (\overline{h}^{p+1} - \underline{h}^{p+1})s_0^{p+1} \\
&- (\overline{h}^p + \underline{h}^p)c_0^p - (\overline{h}^p - \underline{h}^p)s_0^p + (\overline{h}^p + \underline{h}^p)m^{p+1}c_1^p \\
&+ (\overline{h}^p - \underline{h}^p)m^{p+1}\mathrm{sgn}(m^p)s_1^p - (\overline{h}^{p+1} + \underline{h}^{p+1})m^p c_1^{p+1} \\
&- (\overline{h}^{p+1} - \underline{h}^{p+1})m^p\mathrm{sgn}(m^p)s_1^{p+1}].
\end{aligned} \tag{24}$$

Simplifying (24) and combining similar terms, we have

$$\begin{aligned}
&(\overline{h}^{p+1} + \underline{h}^{p+1})(m^p - m^{p-1})(m^p c_1^{p+1} + c_0^{p+1}) \\
&+ (\overline{h}^{p+1} - \underline{h}^{p+1})(m^p - m^{p-1})(m^p\mathrm{sgn}(m^p)s_1^{p+1} + s_0^{p+1}) \\
&- (\overline{h}^p + \underline{h}^p)(m^{p+1} - m^{p-1})(m^p c_1^p + c_0^p) \\
&- (\overline{h}^p - \underline{h}^p)(m^{p+1} - m^{p-1})(m^p\mathrm{sgn}(m^p)s_1^p + s_0^p) \\
&+ (\overline{h}^{p-1} + \underline{h}^{p-1})(m^{p+1} - m^p)(m^p c_1^{p-1} + c_0^{p-1}) \\
&+ (\overline{h}^{p-1} - \underline{h}^{p-1})(m^{p+1} - m^p)(m^p\mathrm{sgn}(m^p)s_1^{p-1} + s_0^{p-1}) \geq 0.
\end{aligned}$$

Because the equation $|m^p| = \mathrm{sgn}(m^p)m^p$ holds, we can derive the fourth condition of Theorem 4.2.

In Theorem 4.2, the conditions 1)-3) ensure that the segment between m^p and m^{p+1} for $1 \leq p \leq M-1$ is convex and the condition 4) guarantees that the fuzzy system output is convex in the neighborhood of m^p $(p = 1, 2, \ldots, M)$. Consequently, these constraints can make Theorem 4.2 hold.

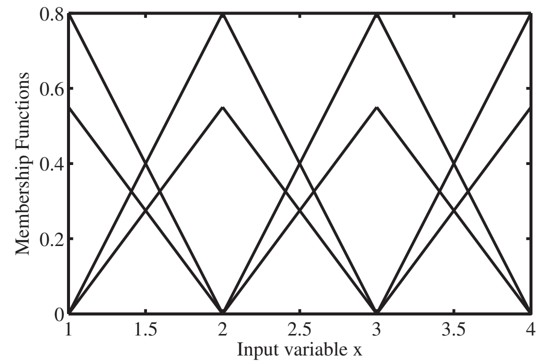

Fig. 8. Graph of the membership functions.

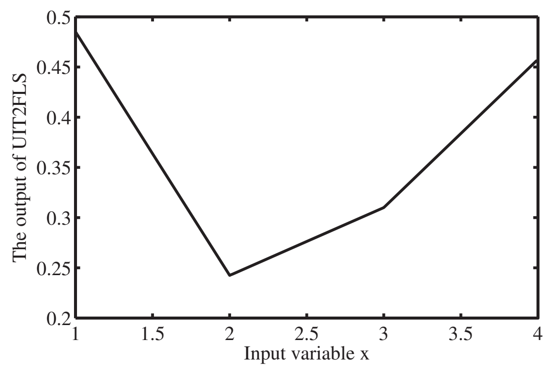

Fig. 9. Graph of the output of the zeroth-order convex UIT2FLS.

C. Example of convex UIT2FLS

Consider the SISO zeroth-order UIT2FLS with four fuzzy rules as follows.

R^i: IF x is $\widetilde{A}^i$, THEN $Y^i = C_0^i$ $(i = 1, 2, 3, 4)$,

where $x \in U$, $U = [1, 4]$. Based on (8) and (9), the parameters of $\widetilde{A}^i$ $(i = 1, 2, 3, 4)$ are

$$\begin{aligned}
&a^1 = 1, m^1 = 1, b^1 = 2, \\
&a^2 = 1, m^2 = 2, b^2 = 3, \\
&a^3 = 2, m^3 = 3, b^3 = 4, \\
&a^4 = 3, m^4 = 4, b^4 = 4, \\
&\overline{h}^i = 0.8,\ \underline{h}^i = 0.55\ (i = 1, 2, 3, 4).
\end{aligned}$$

The consequent interval weights of the THEN-part are $C_0^1 = [0.6, 0.8]$, $C_0^2 = [0.3, 0.4]$, $C_0^3 = [0.4, 0.5]$ and $C_0^4 = [0.5, 0.8]$. All these parameters meet the conditions of Theorem 4.1, therefore the output of the UIT2FLS should be convex with respect to its input. The four membership functions are depicted in Fig. 8, and the output of the zeroth-order convex UIT2FLS is shown in Fig. 9. It is obvious that the simulation result is consistent with Theorem 4.1.

In Theorem 4.2, since the inequalities that the parameters of the single-input first-order UIT2FLS need to meet are very complex, it is quite difficult to solve such inequalities and estimate these parameters. Therefore, we should find a feasible ways to identify the parameters.

V. Conclusions

In this paper, how to encode the prior knowledge of monotonicity and convexity into SISO UIT2FLSs has been studied. The derived sufficient conditions ensure that the prior knowledge can be incorporated. For the zeroth-order UIT2FLS, we have provided two simulation examples to verify the validity of Theorem 3.1 and Theorem 4.1. Simulation results have demonstrated the correctness of the two theorems. But, it is difficult to determine the parameters of SISO first-order UIT2FLSs discussed in Theorem 4.2. What is more, we have not discussed how to obtain UIT2FLSs' parameters to satisfy the conditions presented in the aforementioned three theorems. Therefore, we plan to find systematic optimization methods to identify these parameters to acquire satisfactory performance in system identification problems.

Acknowledgment

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