Nested Saturated Control of Uncertain Complex Cascade Systems Using Mixed Saturation Levels

Meng Li, *Member, IEEE*, and Zhigang Zeng, *Fellow, IEEE*

***Abstract*—This study addresses the problem of global asymptotic stability for uncertain complex cascade systems composed of multiple integrator systems and non-strict feedforward nonlinear systems. To tackle the complexity inherent in such structures, a novel nested saturated control design is proposed that incorporates both constant saturation levels and state-dependent saturation levels. Specifically, a modified differentiable saturation function is proposed to facilitate the saturation reduction analysis of the uncertain complex cascade systems under the presence of mixed saturation levels. In addition, the design of modified differentiable saturation function will help to construct a hierarchical global convergence strategy to improve the robustness of control design scheme. Through calculation of relevant inequalities, time derivative of boundary surface and simple Lyapunov function, saturation reduction analysis and convergence analysis are carried out, and then a set of explicit parameter conditions are provided to ensure global asymptotic stability in the closed-loop systems. Finally, a simplified system of the mechanical model is presented to validate the effectiveness of the proposed method.**

***Index Terms*—Differentiable saturation functions, global stabilization analysis, mixed saturation levels, nested saturated control, uncertain complex cascade systems.**

I. Introduction

CONSTRAINTS on control inputs are common in practical systems [1]. The stabilization issue of control systems subject to input constraints has gradually received a lot of attention [2]−[6]. At the same time, a saturated control method is developed, which takes the saturation factor into account in advance [7]−[11]. As the study of saturation stabilization has evolved, it has gradually been found that the application of nested saturation is suitable for dealing with feedforward nonlinear systems that are in an upper triangular form [12]−[14]. An elegant solution for the global stabilization problem of multiply integrators systems is provided in [15], where a family of state feedback control laws based on nested saturation is constructed for the first time. For multiply integrators systems subject to input saturation, a class of nested saturated controllers has been proposed in [16]. In particular, nested saturated controller can be used to deal with multiply integrator systems with uncertain parameters. In addition, the nested saturated controller has been utilized to globally stabilize the multiply integrators systems in [17], which contains multi-type nonlinear perturbed terms in a strict feedforward form and uncertain parameters.

Feedforward nonlinear systems come from engineering practice. Many realistic models of mechanical system can be described as feedforward systems with multiple integrators or multiple oscillators as the nominal. It seems that related studies for strict feedforward nonlinear systems are more general and common. In [18]−[21], saturated control has been shown to be an effective tool for the strict feedforward nonlinear systems. In contrast, the results on the stabilization for non-strict feedforward nonlinear systems are very scattered. From [22]−[26], state-dependent saturated control design has been proposed and used to realize the stabilization of non-strict feedforward nonlinear systems.

Modeling uncertainties are inherent in practical mechatronic systems and cannot be easily disregarded, as they can significantly compromise the robustness, stability and tracking performance of such systems [27]−[34]. Uncertain parameters contained in control system equations will not only affect the control performance, but also will bring problems to the analysis and computational. Related research results in [12], [16] and [17] show that the nested saturated controller design is suitable for dealing with the uncertain parameters system.

Through a review of the above works, we have found that the nested saturated control design can be used to solve a variety of stabilization problems of strict feedforward nonlinear systems whose normal dynamics are multiple integrators. However, there are fewer works dealing with the stabilization problems of non-strict feedforward nonlinear systems with uncertain parameters by using nested saturated control designs. More importantly, little or no research has considered the global saturation stabilization problems of uncertain

Manuscript received July 15, 2023; revised October 24, 2023; accepted December 16, 2023. This work was supported in part by the National Natural Science Foundation of China (62203178, U1913602, 61936004), the National Key Rsearch and Development Program of China (2021ZD0201300), the China Postdoctoral Science Foundation (2021TQ0116), the Innovation Group Project of the National Natural Science Foundation of China (61821003), the Technology Innovation Project of Hubei Province of China (2019AE A171), and the 111 Project on Computational Intelligence and Intelligent Control (B18024). Recommended by Associate Editor Qing-Long Han. *(Corresponding authors: Zhigang Zeng and Meng Li.)*

Citation: M. Li and Z. Zeng, "Nested saturated control of uncertain complex cascade systems using mixed saturation levels," *IEEE/CAA J. Autom. Sinica*, vol. 11, no. 5, pp. 1163–1174, May 2024.

M. Li is with the Wuhan National Laboratory for Optoelectronics, Wuhan 430074, and also with the School of Artificial Intelligence and Automation, Huazhong University of Science and Technology, Wuhan 430074, China (e-mail: m_li@hust.edu.cn).

Z. Zeng is with the School of Artificial Intelligence and Automation, Huazhong University of Science and Technology, Wuhan 430074, and also with the Key Laboratory of Image Processing and Intelligent Control of Education Ministry of China, Wuhan 430074, China (e-mail: zgzeng@hust.edu.cn).

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Digital Object Identifier 10.1109/JAS.2023.124176

complex cascade systems that are cascaded by multiple integrator systems and non-strict feedforward nonlinear systems. However, some simplified models of practical mechanical systems can be transformed into uncertain complex cascade systems with such structures, so it is necessary to design robust controls for such uncertain complex cascade systems.

In this paper, we endeavor to solve the challenging problem of designing a class of saturated controllers that can guarantee global asymptotic stability for uncertain complex cascade systems, which are composed of multiple integrator systems and non-strict feedforward nonlinear systems.

For the global stabilization problem of the uncertain complex cascade systems, a class of nested saturated control design is proposed in this paper, which contains both constant saturation levels and state-dependent saturation levels. Inspired by the relevant literature [26] and [35]−[39], our paper incorporates a modified differentiable saturation function, facilitating the saturation reduction analysis of uncertain complex cascade systems. Specifically, by utilizing the differentiable saturation function, the saturation reduction analysis of some subsystem can be carried out smoothly through the usual saturated control analysis method. Additionally, combined with the method in [25], by directly calculating inequalities, the non-integrability and slowly-varying property of the state-dependent saturation levels can be ensured and utilized to complete saturation reduction analysis of the remanent system. Subsequently, the convergence analysis can be carried out by calculating the time derivative of the boundary surface and simple Lyapunov function.

In comparison with existing works, the main contributions of this paper are summarized as follows:

1) The proposed uncertain complex cascade systems consist of uncertain parameter, multiple integrator systems and non-strict feedforward nonlinear systems. Compared with existing results [13], [16], [17], and [25], the considered uncertain complex cascade systems have a more intricate and comprehensive structure that better reflects the actual mechanical system.

2) A novel nested saturated control design is proposed, which incorporates both constant saturation levels and state-dependent saturation levels. The proposed control design is different from the nested saturated control design in [18]−[26], which only contains saturation levels or state-dependent saturation levels. Such a proposed control design helps to achieve stabilization of uncertain complex structural systems.

3) Inspired by [26] and [35]−[39], modified differentiable saturation function is proposed to facilitate the reduction analysis of such uncertain complex cascade systems. Based on the modified differentiable saturation function, the saturation reduction analysis will be divided in two steps. Then the state of systems converges globally to small domains by layering. This design has strong robustness, and there are no similar features in the mentioned literature.

The rest of this article is organized as follows. In Section II, problem formulation and preliminaries are given. In Section III, the main results are presented. In Section IV, an application of the proposed control design to a mechatronic model is presented, and Section V draws conclusions to this paper.

II. Problem Formulation and Preliminaries

Consider the following uncertain complex cascade systems:

$$\begin{cases} \dot{x}_1 = c_1 x_2 + q_1(x,\tilde{x}) \\ \dot{x}_2 = c_2 x_3 + q_2(x,\tilde{x}) \\ \qquad \vdots \\ \dot{x}_n = c_n x_{n+1} + q_n(x,\tilde{x}) \\ \dot{x}_{n+1} = c_{n+1} x_{n+2} \\ \qquad \vdots \\ \dot{x}_{n+m} = c_{n+m} u \end{cases} \tag{1}$$

where $x = (x_1,\ldots,x_n)^T$, $\tilde{x} = (x_{n+1},\ldots,x_{n+m})^T$ and u are the system states and control input, respectively. For $i = 1,\ldots,n$, $n+1,\ldots,n+m$, the bounded uncertain parameters satisfy $0 < c_i^- \le c_i \le c_i^+$. The structure of nonlinear disturbance terms $q_1(x,\tilde{x}),\ldots,q_n(x,\tilde{x})$ are in the form of $|q_i(x,\tilde{x})| = a_i|x||\tilde{x}|^i$ and satisfy the following assumption. Without loss of generality, along the relevant conditions in [24], [25], the disturbance vector field $(q_1(x,\tilde{x}),\ldots,q_n(x,\tilde{x}))^T$ is homogeneous of at least order one with respect to the dilation (1) in [25].

Assumption 1: For $i = 1,2,\ldots,n$, there exist know constants a_i such that

$$|q_i(x,\tilde{x})| \le a_i(|x_1|+\cdots+|x_n|)(|x_{n+1}|+\cdots+|x_{n+m}|)^n. \tag{2}$$

Under Assumption 1, we know that subsystem $(x_1,\ldots,x_n)$ of systems (1) is in a non-strict upper-triangular form. The homogeneous property of $(q_1(x,\tilde{x}),\ldots,q_n(x,\tilde{x}))^T$ can also be converted to (2).

The structure of systems similar to system (1) can be observed in practical mechatronic systems, such as the ball and beam system. Therefore, it is of potential practical significance to study the global saturation stabilization of systems (1).

Remark 1: Some studies consider strict feedforward nonlinear systems whose normal dynamics are multiple integrators, similar to the subsystem $(x_{n+1},\ldots,x_{n+m})$ of uncertain complex cascade systems (1) with nonlinear terms in a strict upper-triangular form or the subsystem $(x_1,\ldots,x_n)$ of systems (1), such as [17]−[21]. Fewer works deal with the stabilization problems of non-strict feedforward nonlinear systems, similar to the subsystem $(x_{n+1},\ldots,x_{n+m})$ of uncertain complex cascade systems (1) with nonlinear terms in a non-strict upper-triangular form, such as [22]−[25]. However, little or no research has considered the global saturation stabilization problems of complex cascade systems that are cascaded by multiple integrator systems and non-strict feedforward nonlinear systems [25] and [26]. Furthermore, taking into account the uncertainty in the control system and the application in mechanical systems, the uncertain complex cascade systems (1) subject to non-strict upper-triangular form of Assumption 1 have a more general structure.

The objective of this paper is to design a class of saturated controllers that can be used to achieve the global asymptotic stability of uncertain complex cascade systems (1), which are cascaded by multiple integrator systems and non-strict feedforward nonlinear systems.

III. Main Results

For system (1), the following nested saturated controller with mixed saturation levels is presented:

$$\begin{cases} u = u_{n+m} = -k_{n+m}\text{sat}_{2^m\varepsilon}(x_{n+m} - u_{n+(m-1)}) \\ \quad\vdots \\ u_{n+1} = -k_{n+1}\text{sat}_{2\varepsilon}(x_{n+1} - u_n) \\ u_n = -k_n\text{sat}_{l_n\varepsilon\rho(x)}(x_n - u_{n-1}) \\ \quad\vdots \\ u_2 = -k_2\text{sat}_{l_2\varepsilon\rho(x)}(x_2 - u_1) \\ u_1 = -k_1\text{sat}_{l_1\varepsilon\rho(x)}(x_1) \end{cases} \tag{3}$$

with

$$\rho(x) = (M_{n+1} + M_n x_n^2 + \cdots + M_j x_j^2 + \cdots + M_1 x_1^2)^{\frac{-1}{2(n-1)}}$$
$$M_{n+1} = M_n\Lambda^2,\ M_1 = \Lambda^2,\ M_{j+1} = M_j^2\Lambda^2\ \ (j = 1, 2, \ldots, n-1) \tag{4}$$

where $0 < \varepsilon \le 2^{-1}$, $\Lambda \ge 3$, $k_i > 0$, $l_i > 0$ are the constants to be determined for $i = 1, 2, \ldots, n, \ldots, n+m$. In particular, nested saturated controller (3) contains mixed saturation levels, which are the constant saturation levels $2\varepsilon, \ldots, 2^m\varepsilon$ and the state-dependent saturation levels $l_1\varepsilon\rho(x), \ldots, l_n\varepsilon\rho(x)$.

Throughout this paper, the standard saturation function is defined as $\text{sat}_\varepsilon(s) = \text{sign}(s)\min\{|s|, \varepsilon\}, \varepsilon > 0, s \in \mathbb{R}$. Then the state-dependent saturation levels can be formulated as $\text{sat}_{l_i\varepsilon\rho(x)}(s) = l_i\varepsilon\rho(x)\text{sat}_1(\frac{s}{l_i\varepsilon\rho(x)})$, $x \in \mathbb{R}^n$, $s \in \mathbb{R}$.

Remark 2: In order to globally asymptotically stabilize system (1), the nested saturated controller (3) with mixed saturation levels has been proposed. In fact, the constant saturation levels $2\varepsilon, \ldots, 2^m\varepsilon$ are designed to deal with multiple integrator systems of systems (1), and the state-dependent saturation levels $l_1\varepsilon\rho(x), \ldots, l_n\varepsilon\rho(x)$ are used to handle the non-strict feedforward nonlinear systems of systems (1). In addition, the nested structure of the controller helps to deal with uncertain complex cascade systems (1) subject to uncertain parameters. In engineering practice, the saturation nonlinear constraint of the controller widely exists, so the saturated control method that considers the saturation factor in advance is created. This saturated control design can avoid the parameter configuration problems in the process of using the controller to a certain extent. The nested saturated control design (3) appears to be complex due to the form of construction, but it is not inherently complex. If the relevant parameters of the proposed controller satisfy the explicit parameter conditions for the global stabilization of the considered closed-loop system, then the controller can be put directly into the system to achieve the global stabilization of the system, thus avoiding the readjustment of the control parameters. For practical applications, this control design is characterized by high efficiency.

In the following, we will show that the nested saturated controller (3) with mixed saturation levels can globally asymptotically stabilize uncertain complex cascade systems (1).

Based on the proposed nested saturated controller with mixed saturation levels, the saturation reduction analysis will be conducted in a bottom-up recursive manner, once the system states reach a boundary surface, by calculating the time derivative of the boundary surface, a set of explicit parameter conditions are obtained to ensure that the states eventually tend to a small domain.

Due to the presence of both constant saturation levels and state-dependent saturation levels, the bottom-up saturation reduction analysis will face the problem that the differential calculation of saturation function is subject to state-dependent saturation levels, which will affect the whole reduction analysis process. Inspired with [26] and [35]–[39], a modified differentiable saturation function is proposed in our paper, which helps to solve the differential calculation problems related to saturation function with state-dependent saturation levels, and further contributes to the whole saturation reduction analysis of uncertain complex cascade systems in the presence of mixed saturation levels.

A simple Lyapunov function is then used to demonstrate the asymptotic stability of the reduced system.

The algorithm in this paper is divided into three parts.

A. Reduction Analysis of Subsystem $(x_{n+1}, \ldots, x_{n+m})$

In this step, we will address the saturation reduction analysis of saturation terms $x_{n+m} - u_{n+(m-1)}, \ldots, x_{n+1} - u_n$.

We will compute the time derivatives of saturation functions of subsystem $(x_{n+1}, \ldots, x_{n+m})$ of system (1) with controller (3) in small domains, and prove that the non-positive property of time derivatives which are ensured by the parameter conditions.

In the process of computing the time derivatives of saturation functions of subsystem $(x_{n+1}, \ldots, x_{n+m})$, the difficulty of this part is that each step will utilize the information of $\dot{u}_1, \ldots, \dot{u}_n$. From controller (3), we know that

$$u_n = -k_n\text{sat}_{l_n\varepsilon\rho(x)}(x_n - u_{n-1})$$
$$\vdots$$
$$u_2 = -k_2\text{sat}_{l_2\varepsilon\rho(x)}(x_2 - u_1)$$
$$u_1 = -k_1\text{sat}_{l_1\varepsilon\rho(x)}(x_1)$$

where the saturated control design contains state-dependent saturation levels $l_1\varepsilon\rho(x), \ldots, l_n\varepsilon\rho(x)$. Therefore, the differential problem of saturated controllers with state-dependent saturation levels needs to be treated cautiously. In this regard, the works in [26] and [35]–[39] have proposed some methods. Inspired by the literature, the following definition is introduced in this paper.

Definition 1: For $s \in \mathbb{R}$ and $i = 1, \ldots, n$, there hold

$$s \cdot \text{sat}_{l_i\varepsilon\rho(x)}(s) \ge 0 \tag{5a}$$
$$\text{sat}_{l_i\varepsilon\rho(x)}(s) = s,\ \ \forall |s| \le h_i \le l_i\varepsilon\rho(x) \le H_i \tag{5b}$$
$$\text{sat}_{l_i\varepsilon\rho(x)}(s) = f(s),\ \ \forall h_i \le |s| \le H_i \tag{5c}$$
$$\text{sat}_{l_i\varepsilon\rho(x)}(s) = N_i(\text{sign}(s)),\ \ \forall |s| \ge H_i \tag{5d}$$
$$0 \le \frac{d[\text{sat}_{l_i\varepsilon\rho(x)}(s)]}{ds} \le 1 \tag{5e}$$

where h_i, H_i, N_i are positive design parameters that satisfy $h_i = \beta_i H_i$ with $\beta_i \le 1$ and $N_i = h_i(1+\gamma_i)$ with $\gamma_i \ge 0$. $f(s)$ is a non-decreasing function of s. Here, $\mathrm{sat}_{l_i\varepsilon\rho(x)}(s)$ on (h_i, H_i) and $(-H_i, -h_i)$ can be any non-decreasing curves that smoothly connect $\mathrm{sat}_{l_i\varepsilon\rho(x)}(s) = s$ and N_i and $-N_i$, respectively. In particular, let $\gamma_i = 0$, $\beta_i = 1$, which yield $h_i = H_i$ and $N_i = h_i$, respectively, then the saturation function in Definition 1 can be transferred into the standard saturation function. The sate-dependent saturation function $\mathrm{sat}_{l_i\varepsilon\rho(x)}(s) = l_i\varepsilon\rho(x) \times \mathrm{sat}_1(\frac{s}{l_i\varepsilon\rho(x)})$ can also be included into (5). $l_i\varepsilon\rho(x)$ represents related function of system states.

Remark 3: Those usual saturation functions often inherit the standard saturation function in Teel's approach [15]. In order to deal with situations of feedforward nonlinear systems, disturbances, convergence performance, uncertain parameters, or global stability, the relevant works which are based on standard saturation function and its simple deformation form can be found in [12], [13] and [16], [17]. To improve the convergence performance of systems to some extent, a class of modified saturation function with positive tunable parameters has been introduced in [13], [17], and [35], [36]. For some non-strict feedforward nonlinear systems, state-dependent saturated control design has been proposed in [22]–[26]. In order to overcome the related problem under uncertainties and disturbances, modified differentiable saturation functions in a general form have been applied in [35]–[38]. Some differentiable saturation functions in numerical forms have been presented in [26] and [39]. Inspired by the aforementioned literature, another modified differentiable saturation functions (5) in a general form is proposed to deal with the problem that the differential calculation of saturation function is subject to state-dependent saturation levels, such that the saturation reduction analysis of consider system under mixed saturation levels can be carried out smoothly. As $h_i = \beta_i H_i$ with $\beta_i \le 1$ and $N_i = h_i(1+\gamma_i)$ with $\gamma_i \ge 0$, by using $N_i = \beta_i H_i(1+\gamma_i)$, it is easy to design the value of h_i and γ_i, β_i to get the value of H_i, N_i. Then we can get the boundary's values of modified differentiable saturation functions (5). From the perspective of theoretical calculation, it is reasonable to utilize h_i, H_i, N_i as positive design parameters of the modified differentiable saturation functions (5). At the same time, the boundary values of the saturated nonlinear constraints present in the controlled object are generally known or can be obtained by calculating the relevant parameters provided by the device. By designing the values of h_i and γ_i, β_i, the positive design parameters of H_i, N_i can be obtained. Furthermore, all of the positive design parameters of (5) will be utilized to obtain the values of control parameters. Here, we have completed the configuration of the parameters of (5).

Fact 1: With the relevant parameters satisfying

$$\begin{aligned}
&c^-_{n+m}k_{n+m}2^m\varepsilon > c^+_{n+(m-1)}k_{n+(m-1)}(2^m + k_{n+(m-1)}2^{m-1})\varepsilon\\
&\quad + c^+_{n+(m-2)}k_{n+(m-1)}k_{n+(m-2)}(2^{m-1} + k_{n+(m-2)}2^{m-2})\varepsilon\\
&\quad + c^+_{n+(m-3)}k_{n+(m-1)}k_{n+(m-2)}k_{n+(m-3)}(2^{m-2} + k_{n+(m-3)}2^{m-3})\varepsilon\\
&\quad + \cdots + c^+_{n+2}k_{n+(m-1)}k_{n+(m-2)}\times\cdots\times k_{n+3}k_{n+2}(2^3 + k_{n+2}2^2)\varepsilon\\
&\quad + c^+_{n+1}k_{n+(m-1)}k_{n+(m-2)}\times\cdots\times k_{n+3}k_{n+2}k_{n+1}(2^2 + k_{n+1}2^1)\varepsilon\\
&\quad + k_{n+(m-1)}k_{n+(m-2)}\times\cdots\times k_{n+3}k_{n+2}k_{n+1}k_n\\
&\quad \times[(c^+_n(k_nN_n + 2^1\varepsilon) + Q_n(x,\tilde{x}))\\
&\quad + k_{n-1}(c^+_{n-1}(k_{n-1}N_{n-1} + H_n) + Q_{n-1}(x,\tilde{x}))\\
&\quad + \cdots + k_{n-1}\cdots k_2(c^+_2(k_2N_2 + H_3) + Q_2(x,\tilde{x}))\\
&\quad + k_{n-1}\cdots k_1(x_2c^+_1(k_1N_1 + H_2) + Q_1(x,\tilde{x}))]\\
&\qquad\vdots\\
&c^-_{n+1}k_{n+1}2^1\varepsilon > c^+_{n+1}2^2\varepsilon + k_n[(c^+_n(k_nN_n + 2^1\varepsilon) + Q_n(x,\tilde{x}))\\
&\quad + k_{n-1}(c^+_{n-1}(k_{n-1}N_{n-1} + H_n) + Q_{n-1}(x,\tilde{x}))\\
&\quad + \cdots + k_{n-1}\cdots k_2(c^+_2(k_2N_2 + H_3) + Q_2(x,\tilde{x}))\\
&\quad + k_{n-1}\cdots k_1(x_2c^+_1(k_1N_1 + H_2) + Q_1(x,\tilde{x}))]
\end{aligned}\tag{6}$$

where $Q_i(x,\tilde{x})$ $(i = 1,\ldots,n)$ is denoted as follows:

$$\begin{aligned}
Q_i(x,\tilde{x}) = {}& a_i(H_1 + \cdots + (k_{n-1}N_{n-1} + H_n))((k_nN_n\\
&+ 2^1\varepsilon) + \cdots + (k_{n+m-1}\cdot 2^{m-1}\varepsilon + 2^m\varepsilon))^n
\end{aligned}$$

then the saturation functions of $(x_{n+1},\ldots,x_{n+m})$ subsystem of system (1) with controller (3) is unsaturated in a finite time.

Proof: The proof is following the analysis method of the nested saturated control schemes in [16] and [17].

Consider the x_{n+m} subsystem. If $x_{n+m}(t) - u_{n+(m-1)}(t) > 2^m\varepsilon$ holds for all $t \in [0,\infty)$, we have

$$\begin{aligned}
\dot{x}_{n+m} &= -c_{n+m}k_{n+m}\mathrm{sat}_{2^m\varepsilon}(x_{n+m} - u_{n+(m-1)})\\
&\le -c^-_{n+m}k_{n+m}2^m\varepsilon.
\end{aligned}\tag{7}$$

Together with $|u_{n+(m-1)}| \le k_{n+(m-1)}2^{m-1}\varepsilon$, for all $t \in [0,\infty)$ there holds $2^m\varepsilon < x_{n+m}(t) - u_{n+(m-1)}(t) \le x_{n+m}(0) - c^-_{n+m}\times k_{n+m}2^m\varepsilon t + k_{n+(m-1)}2^{m-1}\varepsilon$. Then there yield the following contradiction $2^m\varepsilon < x_{n+m}(t) - u_{n+(m-1)}(t) \le 0$, as $t \to \infty$.

Hence, a finite time t_{n+m} exists such that

$$x_{n+m}(t_{n+m}) - u_{n+(m-1)}(t_{n+m}) = 2^m\varepsilon.$$

In particular, as the constant saturation levels $2\epsilon\cdots 2^m\epsilon$ and state-dependent saturation levels $l_1\varepsilon\rho(x),\ldots,l_n\varepsilon\rho(x)$ exist, based on the notion of standard saturation function, it is easy to verify that there hold

$$\frac{d}{dt}u_j = 0,\ \forall|x_j - u_{j-1}| > 2^{j-n}\varepsilon,\ j = n+(m-1),\ldots,n+1.$$

Furthermore, based on (7), we obtain $\frac{d}{dt}(x_{n+m} - u_{n+(m-1)}) \le 0$. According to the Definition 1, we derive that

$$\frac{d}{dt}u_i = 0,\ \forall|x_i - u_{i-1}| > H_i,\ i = n,\ldots,1.$$

Do not lose generality, and then we only need to compute $\frac{d}{dt}(x_{n+m} - u_{n+(m-1)})$ at the time instant t_{n+m} in the domain

$$\forall|x_j - u_{j-1}| \le 2^{j-n}\varepsilon,\ j = n+(m-1),\ldots,n+1\tag{8}$$

and

$$\forall|x_i - u_{i-1}| < H_i,\ i = n,\ldots,1.\tag{9}$$

In view of the Definition 1, one has

$$|u_n| \le k_nN_n,\ldots,|u_2| \le k_2N_2, |u_1| \le k_1N_1.$$

From (8) and (9), together with (2) and Definition 1, it yields

$$\begin{aligned}|q_i(x,\tilde{x})| &\le a_i(|x_1|+\cdots+|x_n|)(|x_{n+1}|+\cdots+|x_{n+m}|)^n\\ &\le a_i(H_1+\cdots+(k_{n-1}N_{n-1}+H_n))((k_nN_n\\ &\quad+2^1\varepsilon)+\cdots+(k_{n+m-1}\cdot 2^{m-1}\varepsilon+2^m\varepsilon))^n. \qquad (10)\end{aligned}$$

Keeping (7)–(10) in mind, and according to (5e) of the Definition 1, we derive that

$$\begin{aligned}&\frac{d}{dt}(x_{n+m}(t)-u_{n+(m-1)}(t))\Big|_{t=t_{n+m}}\\
&\le \frac{d}{dt}[x_{n+m}(t_{n+m})+k_{n+(m-1)}x_{n+(m-1)}(t_{n+m})\\
&\quad+k_{n+(m-1)}k_{n+(m-2)}x_{n+(m-2)}(t_{n+m})\\
&\quad+k_{n+(m-1)}k_{n+(m-2)}k_{n+(m-3)}x_{n+(m-3)}(t_{n+m})\\
&\quad+\cdots+k_{n+(m-1)}k_{n+(m-2)}\times\cdots\times k_{n+3}k_{n+2}x_{n+2}(t_{n+m})\\
&\quad+k_{n+(m-1)}k_{n+(m-2)}\times\cdots\times k_{n+3}k_{n+2}k_{n+1}x_{n+1}(t_{n+m})\\
&\quad+k_{n+(m-1)}k_{n+(m-2)}\times\cdots\times k_{n+3}k_{n+2}k_{n+1}\\
&\quad\times k_n sat_{l_n\varepsilon\rho(x)}(x_n-u_{n-1})]\\
&\le -c^-_{n+m}k_{n+m}2^m\varepsilon+c^+_{n+(m-1)}k_{n+(m-1)}(2^m+k_{n+(m-1)}2^{m-1})\varepsilon\\
&\quad+c^+_{n+(m-2)}k_{n+(m-1)}k_{n+(m-2)}(2^{m-1}+k_{n+(m-2)}2^{m-2})\varepsilon\\
&\quad+c^+_{n+(m-3)}k_{n+(m-1)}k_{n+(m-2)}k_{n+(m-3)}(2^{m-2}+k_{n+(m-3)}2^{m-3})\varepsilon\\
&\quad+\cdots+c^+_{n+2}k_{n+(m-1)}k_{n+(m-2)}\times\cdots\times k_{n+3}k_{n+2}(2^3+k_{n+2}2^2)\varepsilon\\
&\quad+c^+_{n+1}k_{n+(m-1)}k_{n+(m-2)}\times\cdots\times k_{n+3}k_{n+2}k_{n+1}(2^2+k_{n+1}2^1)\varepsilon\\
&\quad+k_{n+(m-1)}k_{n+(m-2)}\times\cdots\times k_{n+3}k_{n+2}k_{n+1}k_n\\
&\quad\times[(c^+_n(k_nN_n+2^1\varepsilon)+q_n(x,\tilde{x}))\\
&\quad+k_{n-1}(c^+_{n-1}(k_{n-1}N_{n-1}+H_n)+q_{n-1}(x,\tilde{x}))\\
&\quad+\cdots+k_{n-1}\cdots k_2(c^+_2(k_2N_2+H_3)+q_2(x,\tilde{x}))\\
&\quad+k_{n-1}\cdots k_1(x_2c^+_1(k_1N_1+H_2)+q_1(x,\tilde{x}))]. \qquad (11)\end{aligned}$$

Under (10) and the parameters conditions (6), one has $\frac{d}{dt}(x_{n+m}(t)-u_{n+(m-1)}(t))\big|_{t=t_{n+m}}<0$. Hence, there holds $x_{n+m}(t)-u_{n+(m-1)}(t)\le 2^m\varepsilon,\ \forall t\ge t_{n+m}$.

Similarly, it can be proved: there exists a finite time $t_{(n+m)*}$ such that $x_{n+m}(t)-u_{n+(m-1)}(t)\ge -2^m\varepsilon$ holds for all $t\ge t_{(n+m)*}$. So, we have

$$\big|x_{n+m}(t)-u_{n+(m-1)}(t)\big|\le 2^m\varepsilon,\quad \forall t\ge T_{n+m}=\max\{t_{n+m},t_{(n+m)*}\}. \qquad (12)$$

Using the above method, we next analyze subsystems $x_{n+m-1},\dots,x_{n+1}$. Under the parameter conditions (6), we obtain

$$\begin{cases}\big|x_{n+(m-1)}(t)-u_{n+(m-2)}(t)\big|\le 2^{m-1}\varepsilon, & \forall t\ge T_{n+(m-1)}\\ \big|x_{n+(m-2)}(t)-u_{n+(m-3)}(t)\big|\le 2^{m-2}\varepsilon, & \forall t\ge T_{n+(m-2)}\\ \quad\vdots & \\ |x_{n+2}(t)-u_{n+1}(t)|\le 2^2\varepsilon, & \forall t\ge T_{n+2}\\ |x_{n+1}(t)-u_n(t)|\le 2^1\varepsilon, & \forall t\ge T_{n+1}\end{cases} \qquad (13)$$

where

$$T_{n+(m-2)},\dots,T_{n+1}\ (T_{n+m}\le T_{n+(m-1)}\le T_{n+(m-2)}\le\cdots\le T_{n+1})$$

are finite time instants. ■

After a finite time T_{n+1}, under the state constants (12) and (13), the nested saturated controller (3) with mixed saturation level reduces to

$$\begin{cases}u=u_{n+m}=-k_{n+m}(x_{n+m}-u_{n+(m-1)})\\ \quad\vdots\\ u_{n+1}=-k_{n+1}(x_{n+1}-u_n)\\ u_n=-k_n\mathrm{sat}_{l_n\varepsilon\rho(x)}(x_n-u_{n-1})\\ \quad\vdots\\ u_2=-k_2\mathrm{sat}_{l_2\varepsilon\rho(x)}(x_2-u_1)\\ u_1=-k_1\mathrm{sat}_{l_1\varepsilon\rho(x)}(x_1).\end{cases} \qquad (14)$$

Remark 4: Since the considered systems (1) is cascaded by multiple integrator systems and non-strict feedforward nonlinear systems, the subsystems $x_{n+m},\dots,x_{n+1}$ need to be analyzed at first in bottom-up saturation reduction analysis, then the subsystems $x_n,\dots,x_1$ are analyzed far behind, respectively. In the process of computing the time derivatives of saturation terms $x_{n+m}-u_{n+(m-1)},\dots,x_{n+1}-u_n,\ \dot{u}_1,\dots,\dot{u}_n$'s information needs to be utilized at each step. Because of the sequential of bottom-up saturation reduction analysis, the presence of complex cascade structures and state-dependent saturation levels $l_1\varepsilon\rho(x),\dots,l_n\varepsilon\rho(x)$ for $u_1,\dots,u_n$, the differential calculation of saturation function subject to state-dependent saturation levels cannot be directly obtained to advance the saturation reduction analysis. Although it is not possible to solve the above problem directly, one possible solution can easily be considered. Based on the property of state-dependent saturation function and the relevant design of state-dependent saturation levels in (4), one can obtain

$$\begin{aligned}|u_n| &= \big|-k_n\mathrm{sat}_{l_n\varepsilon\rho(x)}(x_n-u_{n-1})\big|\le k_nl_n\varepsilon\rho(x)\\ &=\frac{k_nl_n\varepsilon}{(M_{n+1}+M_nx_n^2+\cdots+M_ix_i^2+\cdots+M_1x_1^2)^{1/(2(n-1))}}\\ &\le\frac{k_nl_n\varepsilon}{(M_{n+1})^{1/(2(n-1))}}=k_nl_n\varepsilon M_{n+1}^{-1/(2(n-1))}.\end{aligned}$$

Furthermore, we get $\frac{d}{dt}|u_n|\le\frac{d}{dt}[k_nl_n\varepsilon M_{n+1}^{-1/(2(n-1))}]=0$. Under this situation, the differential calculation of saturation function subject to state-dependent saturation levels can be obtained and the calculating for the time derivative of the boundary surface can be carried out. However, it is not difficult to find that there are certain loopholes in the mentioned solution. By utilizing $\frac{d}{dt}|u_n|\le 0$, the conservatism of computation will increase and some state information of systems will be lost in the reduction analysis calculation. To some extent, this solution is not appropriate. In order to overcome these difficulties, the modified differentiable saturation functions (5) in Definition 1 are presented. By using Definition 1, the time derivatives calculation of saturation function that contains state-dependent saturation levels can be dealt with as follows:

$$\begin{aligned}\frac{d}{dt}|u_n| &= \frac{d}{dt}[k_n sat_{l_n\varepsilon\rho(x)}(x_n-u_{n-1})]\\ &= k_n\frac{d}{dt}[sat_{l_n\varepsilon\rho(x)}(x_n-u_{n-1})]\cdot\frac{d}{dt}[x_n-u_{n-1}]\\ &\le k_n\cdot 1\cdot\frac{d}{dt}[x_n-u_{n-1}]\end{aligned}$$

and then, the calculating for the time derivative of the boundary surface can be carried out as (11) in the proof of Fact 1. By introducing the differential saturation function Definition 1, the differential calculation of saturation function subject to state-dependent saturation levels can be directly obtained. By using the computation form $\frac{d}{dt}|u_n|\le k_n\cdot 1\cdot\frac{d}{dt}[x_n-u_{n-1}]$, all the state information will be utilized in the computation of saturation reduction analysis. In the meantime, this analysis scheme will decrease the conservatism of computation.

From Remark 4, it can be known that the preliminary calculation information of $x_n,\dots,x_1$ cannot be obtained directly in bottom-up saturation reduction analysis. This situation will cause problems in the analysis of saturation degradation at the outset. With the introduction of differential saturation function, this problem has been addressed in some way. By constructing hierarchical and known boundaries with the help of differential saturation functions, we avoid directly dealing with the saturation reduction analysis of state-dependent saturation function first, which in turn facilitates the smooth development of saturated reduction analysis.

Remark 5: Since the mixed saturation levels exist, the reduction analysis is divided into two parts for both constant saturation levels and state-dependent saturation levels, namely, reduction analysis of subsystem $(x_{n+1},\dots,x_{n+m})$ and reduction analysis of subsystem $(x_1,\dots,x_n)$. In the process of reduction analysis of subsystem $(x_{n+1},\dots,x_{n+m})$, combined with Definition 1, the time derivatives calculation of saturation functions are carried out in the following small domains:

$$\begin{cases}\left|x_j-u_{j-1}\right|\le 2^{j-n}\epsilon, & j=n+m,\dots,n+1\\ |x_i-u_{i-1}|<H_i, & i=n,\dots,1.\end{cases}$$

Because of the proposed of Definition 1, the reduction analysis can be carried out in the above way, and it also paves the way for completing the reduction analysis of subsystem $(x_1,\dots,x_n)$.

B. Reduction Analysis of Subsystem $(x_1,\dots,x_n)$

Then, we deal with the saturation reduction analysis of saturation terms x_n-u_{n-1}, $x_{n-1}-u_{n-2},\dots,$ x_2-u_1,x_1 for system (1) with controller (14). Considering the subsystem $(x_1,\dots,$ $x_n)$, we need to handle the saturation reduction analysis of saturation function with state-dependent saturation levels. The non-integrable property of the state-dependent saturation function here is a sufficient condition to prove that the system state reaches the boundary surface in a finite time, and the slowly-varying properties of the state-dependent saturation function is a sufficient condition to ensure that the system state enters a small domain.

Fact 2: With the relevant parameters satisfying

$$\begin{aligned}c_n^-k_nl_n\varepsilon M_{n+1}^{-1/(2(n-1))} &> c_n^+2^1\varepsilon+2^{-n}M_{n+1}^{-1/(2(n-1))}+k_{n-1}\\ &\times\max\{l_{n-1}\varepsilon 2^{-n}M_{n+1}^{-1/(2(n-1))},\\ &[(c_{n-1}^+(k_{n-1}N_{n-1}+l_n\varepsilon M_{n+1}^{-1/(2(n-1))})\\ &+2^{-n}M_{n+1}^{-1/(2(n-1))})+\cdots\\ &+k_{n-2}\cdots k_2(c_2^+(k_2N_2+l_3\varepsilon M_{n+1}^{-1/(2(n-1))})\\ &+2^{-n}M_{n+1}^{-1/(2(n-1))})\\ &+k_{n-2}\cdots k_1(c_1^+(k_1N_1+l_2\varepsilon M_{n+1}^{-1/(2(n-1))})\\ &+2^{-n}M_{n+1}^{-1/(2(n-1))})]\}\\ &+l_n\varepsilon 2^{-n}M_{n+1}^{-1/(2(n-1))}\\ &\vdots\\ c_1^-k_1l_1\varepsilon M_{n+1}^{-1/(2(n-1))} &> c_1l_2\varepsilon M_{n+1}^{-1/(2(n-1))}+2^{-n}M_{n+1}^{-1/(2(n-1))}\\ &+l_1\varepsilon 2^{-n}M_{n+1}^{-1/(2(n-1))}\end{aligned}\tag{15}$$

where $M_{n+1}=M_n\Lambda^2$, $M_{i+1}=M_i^2\Lambda^2$ $(i=1,2,\dots,n-1)$, $M_1=\Lambda^2$ holds, then the saturation functions of $(x_1,\dots,x_n)$ subsystem of system (1) with controller (14) is unsaturated in a finite time.

Proof: The following four Steps 1−4 are given to complete the saturation reduction analysis of saturation terms $x_n-u_{n-1},\dots,x_2-u_1,\ x_1$.

Step 1: Some estimates need to be obtained

From the state-dependent saturation function $\rho(x)$ in (4) of the nested saturated controller (3) with mixed saturation levels, one has

$$\begin{aligned}\rho(x) &= (M_{n+1}+M_nx_n^2+\cdots+M_ix_i^2+\cdots+M_1x_1^2)^{-1/(2(n-1))}\\ &\le\frac{1}{(M_{n+1})^{1/(2(n-1))}}=M_{n+1}^{-1/(2(n-1))}\end{aligned}\tag{16}$$

where M_{n+1},M_{i+1} $(i=1,2,\dots,n-1),M_1$ and Λ satisfy the design in (4).

Furthermore, using $\text{sat}_{l_i\varepsilon\rho(x)}(s)=l_i\varepsilon\rho(x)\text{sat}_1(\frac{s}{l_i\varepsilon\rho(x)})$ and Definition 1, one has

$$\begin{aligned}|u_n| &= \left|-k_n\text{sat}_{l_n\varepsilon\rho(x)}(x_n-u_{n-1})\right|\\ &\le k_nl_n\varepsilon\rho(x)\\ &\le k_nl_n\varepsilon M_{n+1}^{-1/(2(n-1))}\end{aligned}\tag{17}$$

where $k_nl_n\varepsilon M_{n+1}^{-1/(2(n-1))}\le k_nN_n$ holds for suitable Λ and ε.

From Assumption 1, together with (13) and Definition 1, one has

$$\begin{aligned}|q_i(x,\tilde{x})| &\le a_i(|x_1|+\cdots+|x_n|)(|x_{n+1}|+\cdots+|x_{n+m}|)^n\\ &\le a_i(|x_1|+\cdots+|x_n|)((k_nN_n+2^1\varepsilon)\\ &\quad+6\cdots+(k_{n+m-1}\cdot 2^{m-1}\varepsilon+2^m\varepsilon))^n.\end{aligned}\tag{18}$$

Combined with Definition 1, it yields

$$\begin{aligned}|q_i(x,\tilde{x})| &\le a_i(H_1+\cdots+(k_{n-1}N_{n-1}+H_n))((k_nN_n\\ &\quad+2^1\varepsilon)+\cdots+(k_{n+m-1}\cdot 2^{m-1}\varepsilon+2^m\varepsilon))^n.\end{aligned}\tag{19}$$

Step 2: Verifying the non-integrable property of $\rho(x)$

Here, we will verify the non-integrable property of $\rho(x)$. Consider the subsystem $\dot{x}_n = c_n x_{n+1} + q_n(x,\tilde{x})$.

We define a function $V_n = |x_n|$ for x_n subsystem. For $|x_n| \geq |u_{n-1}| + l_n\varepsilon\rho(x)$, using (13), (17) and (19), there holds

$$\begin{aligned}\dot{V}_n &= c_n u_n + c_n(x_{n+1} - u_n) + q_n(x,\tilde{x})\\ &= -c_n k_n \mathrm{sat}_{l_n\varepsilon\rho(x)}(x_n - u_{n-1}) + c_n(x_{n+1} - u_n) + q_n(x,\tilde{x})\\ &\leq -c_n^- k_n N_n + c_n^+ 2^1\varepsilon + q_n(x,\tilde{x}). \end{aligned} \tag{20}$$

Under the parameter conditions (15), it implies that $\dot{V}_n < 0$. Then there exists a finite time such that $|x_n| \leq |u_{n-1}| + l_n\varepsilon\rho(x)$.

From (16) and (17), using Definition 1, one has

$$|u_{n-1}| = \left|-k_{n-1}\mathrm{sat}_{l_{n-1}\varepsilon\rho(x)}(x_{n-1} - u_{n-2})\right| \leq k_{n-1}N_{n-1}$$

and $l_n\varepsilon\rho(x) \leq l_n\varepsilon M_{n+1}^{-1/(2(n-1))}$.

Hence x_n is bounded, and one can obtain $|x_n| \leq b_{n1}$, where b_{n1} is a positive constant. In the meantime, the disturbance vector field $(q_1(x,\tilde{x}),\ldots,q_n(x,\tilde{x}))^T$ is homogeneous of at least order one and satisfies the Assumption 1. Combing with $\dot{x}_{n-1} = c_{n-1}x_n + q_{n-1}(x,\tilde{x})$, and using integral calculation, we can obtain $|x_{n-1}| \leq b_{(n-1)1} + b_{(n-1)2}t$ where $b_{(n-1)1}, b_{(n-1)2}$ are positive constants. By analogy and using the proof of Theorem 1 in [25], we can infer that $\rho(x)$ is non-integrable.

Step 3: Obtaining the slowly-varying estimate of $\rho(x)$

By the state-dependent saturation function $\rho(x)$ in (4), one can obtain

$$\dot{\rho}(x) = -\frac{1}{2(n-1)} \times \frac{(\frac{d}{dt})(M_{n+1} + M_n x_n^2 + \cdots + M_i x_i^2 + \cdots + M_1 x_1^2)}{M_{n+1} + M_n x_n^2 + \cdots + M_i x_i^2 + \cdots + M_1 x_1^2}\rho(x).$$

According to (18), and using the Proposition 2 in [12], we can deduce that

$$\frac{(\frac{d}{dt})(M_{n+1} + M_n x_n^2 + \cdots + M_i x_i^2 + \cdots + M_1 x_1^2)}{M_{n+1} + M_n x_n^2 + \cdots + M_i x_i^2 + \cdots + M_1 x_1^2}$$

can be rendered arbitrarily small by choosing large Λ and small ε. In turn, we have the following slowing-varying estimate

$$\begin{aligned}&|\dot{\rho}(x)|\\ &= \left|-\frac{1}{2(n-1)}\frac{(\frac{d}{dt})(M_{n+1} + M_n x_n^2 + \cdots + M_i x_i^2 + \cdots + M_1 x_1^2)}{M_{n+1} + M_n x_n^2 + \cdots + M_i x_i^2 + \cdots + M_1 x_1^2}\rho(x)\right|\\ &\leq \frac{1}{2(n-1)}\cdot\left|\frac{(\frac{d}{dt})(M_{n+1} + M_n x_n^2 + \cdots + M_i x_i^2 + \cdots + M_1 x_1^2)}{M_{n+1} + M_n x_n^2 + \cdots + M_i x_i^2 + \cdots + M_1 x_1^2}\right|\rho(x)\\ &\leq 2^{-n}\rho(x) \end{aligned} \tag{21}$$

with suitable Λ and ε.

Utilizing Lemma 1 in [25] and combining with (16), Assumption 1 and (21), there holds

$$\begin{aligned}|q_i(x,\tilde{x})| &\leq a_i(|x_1| + \cdots + |x_n|)(|x_{n+1}| + \cdots + |x_{n+m}|)^n\\ &\leq 2^{-n}\rho(x) \leq 2^{-n}M_{n+1}^{-1/(2(n-1))}\end{aligned} \tag{22}$$

with suitable Λ and ε.

Step 4: Verifying the reduction of saturated terms $x_n - u_{n-1},\ldots,x_2 - u_1$, x_1

Consider the subsystem x_n. If $x_n(t) - u_{n-1}(t) > l_n\varepsilon\rho(x)$ holds for all $t \in [T_{n+1},\infty)$, we have

$$\begin{aligned}\dot{x}_n &= c_n u_n + c_n(x_{n+1} - u_n) + q_n(x,\tilde{x})\\ &= -c_n k_n \mathrm{sat}_{l_n\varepsilon\rho(x)}(x_n - u_{n-1}) + c_n(x_{n+1} - u_n) + q_n(x,\tilde{x})\\ &\leq -c_n^- k_n l_n\varepsilon\rho(x) + c_n 2^1\varepsilon + q_n(x,\tilde{x}).\end{aligned} \tag{23}$$

According to (15), one has $\dot{x}_n < 0$. Together with $|u_{n-1}| \leq k_{n-1}N_{n-1}$, for all $t \in [T_{n+1},\infty)$, there holds

$$\begin{aligned}l_n\varepsilon\rho(x) &< x_n(t) - u_{n-1}(t)\\ &\leq x_n(0) + \int_0^t[-c_n^- k_n l_n\varepsilon\rho(x) + c_n 2^1\varepsilon + q_n(x,\tilde{x})]ds\\ &\quad + k_{n-1}N_{n-1}.\end{aligned}$$

From Step 2, we obtain that $\rho(x)$ is non-integrable, then there yields the following contradiction:

$$l_n\varepsilon\rho(x) < x_n(t) - u_{n-1}(t) \leq 0, \text{ as } t \to \infty.$$

Hence, a finite time t_{n+1} exists such that $x_n(t_n) - u_{n-1}(t_n) = l_n\varepsilon\rho(x(t_n))$. It implies that states of the subsystem x_n arrive at the boundary surface $x_n(t_n) - u_{n-1}(t_n) = l_n\varepsilon\rho(x(t_n))$.

Next, we need to prove the non-positive property of the time derivative of the boundary surface $x_n(t_n) - u_{n-1}(t_n) = l_n\varepsilon\rho(x(t_n))$, which guarantees states to enter into small domains.

For the subsystem x_n, we need to compute the derivative $\frac{d}{dt}(x_n(t_n) - u_{n-1}(t_n) - l_n\varepsilon\rho(x(t_n)))|_{t=t_n}$ under two conditions

$$|x_i - u_{i-1}| > l_i\varepsilon\rho(x),\ \ i = n-1,\ldots,1$$

and

$$|x_i - u_{i-1}| \leq l_i\varepsilon\rho(x),\ \ i = n-1,\ldots,1.$$

Under $|x_i - u_{i-1}| > l_i\varepsilon\rho(x)$, $i = n-1,\ldots,1$, combined with (16)–(18), (21), (22), one obtains

$$\begin{aligned}&\frac{d}{dt}(x_n(t_n) - u_{n-1}(t_n) - l_n\varepsilon\rho(x(t_n)))|_{t=t_n}\\ &\leq \dot{x}_n + \dot{u}_{n-1} + l_n\varepsilon\dot{\rho}(x(t_n))\\ &\leq (-c_n^- k_n l_n\varepsilon\rho(x) + c_n 2^1\varepsilon + q_n(x,\tilde{x}))\\ &\quad + k_{n-1}l_{n-1}\varepsilon\dot{\rho}(x) + l_n\varepsilon\dot{\rho}(x(t_n))\\ &\leq (-c_n^- k_n l_n\varepsilon\rho(x) + c_n 2^1\varepsilon + 2^{-n}\rho(x)) + k_{n-1}l_{n-1}\varepsilon 2^{-n}\rho(x)\\ &\quad + l_n\varepsilon 2^{-n}\rho(x)\\ &\leq (-c_n^- k_n l_n\varepsilon M_{n+1}^{-1/(2(n-1))} + c_n 2^1\varepsilon + 2^{-n}M_{n+1}^{-1/(2(n-1))})\\ &\quad + k_{n-1}l_{n-1}\varepsilon 2^{-n}M_{n+1}^{-1/(2(n-1))} + l_n\varepsilon 2^{-n}M_{n+1}^{-1/(2(n-1))}.\end{aligned} \tag{24}$$

Under $|x_i - u_{i-1}| \leq l_i\varepsilon\rho(x)$, $i = n-1,\ldots,1$, it yields

$$\begin{aligned}&\frac{d}{dt}(x_n(t_n) - u_{n-1}(t_n) - l_n\varepsilon\rho(x(t_n)))|_{t=t_n}\\ &\leq (-c_n^- k_n l_n\varepsilon\rho(x) + c_n 2^1\varepsilon + q_n(x,\tilde{x}))\\ &\quad + D^+(k_{n-1}sat_{l_{n-1}\varepsilon\rho(x)}(x_{n-1} - u_{n-2})) + l_n\varepsilon\dot{\rho}(x)\\ &\leq (-c_n^- k_n l_n\varepsilon\rho(x) + c_n 2^1\varepsilon + 2^{-n}\rho(x))\\ &\quad + k_{n-1}[\dot{x}_{n-1} + \cdots + k_{n-2}\cdots k_2\dot{x}_2\\ &\quad + k_{n-2}\cdots k_1\dot{x}_1] + l_n\varepsilon 2^{-n}\rho(x)\end{aligned}$$

$$\begin{aligned}&\le(-c_n^-k_nl_n\varepsilon M_{n+1}^{-1/(2(n-1))}+c_n2^1\varepsilon+2^{-n}M_{n+1}^{-1/(2(n-1))})\\&+k_{n-1}[(c_{n-1}^+(k_{n-1}N_{n-1}+l_n\varepsilon M_{n+1}^{-1/(2(n-1))})+q_{n-1}(x,\tilde{x}))\\&+\cdots+k_{n-2}\cdots k_2(c_2^+(k_2N_2+l_3\varepsilon M_{n+1}^{-1/(2(n-1))})+q_2(x,\tilde{x}))\\&+k_{n-2}\cdots k_1(c_1^+(k_1N_1+l_2\varepsilon M_{n+1}^{-1/(2(n-1))})+q_1(x,\tilde{x}))]\\&+l_n\varepsilon 2^{-n}M_{n+1}^{-1/(2(n-1))}.\end{aligned}\tag{25}$$

By using (22) and the parameters conditions (15), it yields that $\frac{d}{dt}(x_n(t_n)-u_{n-1}(t_n)-l_n\varepsilon\rho(x(t_n)))|_{t=t_n}<0$ holds under two conditions: $|x_i-u_{i-1}|>l_i\varepsilon\rho(x)$, $i=n-1,\dots,1$, and $|x_i-u_{i-1}|\le l_i\varepsilon\rho(x)$, $i=n-1,\dots,1$. Hence, there holds$x_n(t)-u_{n-1}(t)\le l_n\varepsilon\rho(x)$, $\forall t\ge t_n$.

Similarly, it can be proved: there exists a finite time t_{n*} such that $x_n(t)-u_{n-1}(t)\ge -l_n\varepsilon\rho(x)$, $\forall t\ge t_{n*}$ holds for all $t\ge t_{n*}$. So, we have

$$|x_n(t)-u_{n-1}(t)|\le l_n\varepsilon\rho(x),\ \ \forall t\ge T_n=\max\{t_n,t_{n*}\}.\tag{26}$$

We next analyze subsystems $x_{n-1},\dots,x_1$ by using the same method. Under the parameter conditions (15), we obtain

$$\begin{cases}|x_{n-1}(t)-u_{n-2}(t)|\le l_{n-1}\varepsilon\rho(x),\ \ \forall t\ge T_n\\|x_{n-2}(t)-u_{n-3}(t)|\le l_{n-2}\varepsilon\rho(x),\ \forall t\ge T_{n-1}\\\quad\vdots\\|x_2(t)-u_1(t)|\le l_2\varepsilon\rho(x),\ \forall t\ge T_2\\|x_1(t)|\le l_1\varepsilon\rho(x),\ \forall t\ge T_1\end{cases}\tag{27}$$

where $T_n,\dots,T_1$ ($T_n\le T_{n-1}\le\cdots\le T_1$) are finite time instants. ■

After a finite time T_1, under the state constants (12), (13), (26), and (27), the nested saturated controller (3) with mixed saturation level reduces to

$$\begin{cases}u=u_{n+m}=-k_{n+m}(x_{n+m}-u_{n+(m-1)})\\\quad\vdots\\u_{n+1}=-k_{n+1}(x_{n+1}-u_n)\\u_n=-k_n(x_n-u_{n-1})\\\quad\vdots\\u_2=-k_2(x_2-u_1)\\u_1=-k_1(x_1).\end{cases}\tag{28}$$

Remark 6: Inspired with [25], the same method has been used to verify the non-integrable property and slowly-varying estimate of $\rho(x)$, then the reduction of saturated terms $x_n-u_{n-1},\dots,x_2-u_1$, $x_1x_n-u_{n-1},\dots,x_2-u_1$, x_1 can be addressed. But differently from [25], the nested saturated control design has been proposed to deal with the problem of global asymptotic stability of uncertain complex cascade systems.

Remark 7: In the process of reduction analysis of subsystem $(x_1,\dots,x_n)$, the time derivatives calculations of saturation functions are carried out in the following small domains $|x_i-u_{i-1}|\le l_i\varepsilon\rho(x)$, $i=n-1,\dots,1$.

C. Asymptotical Stability Analysis of Reduced System

In the following, we analyze the asymptotic stability of the reduced systems (1) with (28).

Fact 3: Under the following parameters conditions:

$$\begin{aligned}&(k_{n+m-1}2^{m-1}\varepsilon+2^m\varepsilon)c_{n+m}^-[k_{n+m}(k_{n+m-1}2^{m-1}\varepsilon+2^m\varepsilon)\\&+\cdots+k_{n+m}k_{n+m-1}\cdots k_{n+1}(k_nN_n+2^1\varepsilon)\\&+k_{n+m}k_{n+m-1}\cdots k_{n+1}k_n(k_{n-1}N_{n-1}+l_n\varepsilon M_{n+1}^{-1/(2(n-1))})\\&+\cdots+k_{n+m}k_{n+m-1}\cdots k_2k_1l_1\varepsilon M_{n+1}^{-1/(2(n-1))}]\\&>l_1\varepsilon M_{n+1}^{-1/(2(n-1))}[(c_1^+(k_1N_1+l_2\varepsilon M_{n+1}^{-1/(2(n-1))})\\&+2^{-n}M_{n+1}^{-1/(2(n-1))})]+\cdots\\&+(k_{n-1}N_{n-1}+l_n\varepsilon M_{n+1}^{-1/(2(n-1))})[(c_n^+(k_nN_n+2^1\varepsilon)\\&+2^{-n}M_{n+1}^{-1/(2(n-1))})]\\&+(k_nN_n+2^1\varepsilon)c_{n+1}^+(k_{n+1}2\varepsilon+2^2\varepsilon)\\&+\cdots+(k_{n+m-2}2^{m-2}\varepsilon+2^{m-1}\varepsilon)c_{n+m-1}^+(k_{n+m-1}2^{m-1}\varepsilon+2^m\varepsilon)\end{aligned}\tag{29}$$

the reduced systems (1) with controller (24) are asymptotically stable.

Proof: Consider the Lyapunov function

$$U=2^{-1}x_1^2+2^{-1}x_2^2+\cdots+2^{-1}x_n^2+2^{-1}x_{n+1}^2+\cdots+2^{-1}x_{n+m}^2.$$

The derivative of U along solutions of the reduced systems (1) with (28) is

$$\begin{aligned}\dot{U}&=x_1\dot{x}_1+\cdots+x_n\dot{x}_n+x_{n+1}\dot{x}_{n+1}+\cdots+x_{n+m}\dot{x}_{n+m}\\&\le|x_1||c_1x_2+q_1(x,\tilde{x})|+\cdots+|x_n||c_nx_{n+1}+q_n(x,\tilde{x})|\\&\quad+|x_{n+1}|c_{n+1}|x_{n+2}|+\cdots+|x_{n+m}|c_{n+m}\\&\quad\times[-k_{n+m}|x_{n+m}|-\cdots-k_{n+m}k_{n+m-1}\cdots k_{n+1}|x_{n+1}|\\&\quad-k_{n+m}k_{n+m-1}\cdots k_{n+1}k_n|x_n|-\cdots-k_{n+m}k_{n+m-1}\cdots k_2k_1|x_1|].\end{aligned}$$

Combined with (12), (13), (16), (19), (26), (27), and under parameters conditions (29), it yields

$$\begin{aligned}\dot{U}&\le l_1\varepsilon\rho(x)[(c_1^+(k_1N_1+l_2\varepsilon\rho(x))+q_1(x,\tilde{x}))]+\cdots\\&\quad+(k_{n-1}N_{n-1}+l_n\varepsilon\rho(x))[(c_n^+(k_nN_n+2^1\varepsilon)\\&\quad+q_n(x,\tilde{x}))]\ +(k_nN_n+2^1\varepsilon)c_{n+1}^+(k_{n+1}2\varepsilon+2^2\varepsilon)\\&\quad+\cdots+(k_{n+m-1}2^{m-1}\varepsilon+2^m\varepsilon)c_{n+m}^-\\&\quad\times[-k_{n+m}(k_{n+m-1}2^{m-1}\varepsilon+2^m\varepsilon)\\&\quad-\cdots-k_{n+m}k_{n+m-1}\cdots k_{n+1}(k_nN_n+2^1\varepsilon)\\&\quad-k_{n+m}k_{n+m-1}\cdots k_{n+1}k_n(k_{n-1}N_{n-1}+l_n\varepsilon\rho(x))\\&\quad-\cdots-k_{n+m}k_{n+m-1}\cdots k_2k_1l_1\varepsilon\rho(x)]\\&\le l_1\varepsilon M_{n+1}^{-1/(2(n-1))}[(c_1^+(k_1N_1+l_2\varepsilon M_{n+1}^{-1/(2(n-1))})\\&\quad+q_1(x,\tilde{x}))]+\cdots\\&\quad+(k_{n-1}N_{n-1}+l_n\varepsilon M_{n+1}^{-1/(2(n-1))})[(c_n^+(k_nN_n+2^1\varepsilon)\\&\quad+q_n(x,\tilde{x}))]+(k_nN_n+2^1\varepsilon)c_{n+1}^+(k_{n+1}2\varepsilon+2^2\varepsilon)\\&\quad+\cdots+(k_{n+m-1}2^{m-1}\varepsilon+2^m\varepsilon)c_{n+m}^-\\&\quad\times[-k_{n+m}(k_{n+m-1}2^{m-1}\varepsilon+2^m\varepsilon)\\&\quad-\cdots-k_{n+m}k_{n+m-1}\cdots k_{n+1}(k_nN_n+2^1\varepsilon)\end{aligned}$$

$$- k_{n+m}k_{n+m-1}\cdots k_{n+1}k_n(k_{n-1}N_{n-1} + l_n\varepsilon M_{n+1}^{-1/(2(n-1))})$$
$$- \cdots - k_{n+m}k_{n+m-1}\cdots k_2k_1l_1\varepsilon M_{n+1}^{-1/(2(n-1))}] < 0.$$

Hence, the reduced systems (1) with controller (24) are asymptotically stable. ∎

From Facts 1−3, the main results of this paper can be shown as follows.

Theorem 1: With parameter conditions (6), (15) and (29) holding, let $0 < \varepsilon < 2^{-1}$ and $\Lambda > 3$, the nested saturated controller (3) can globally stabilizes uncertain complex cascade systems (1).

Based on Facts 1−3, systems (1) with controller (3) are globally attractive and locally asymptotically stable, and thus the closed-loop system is globally asymptotically stable at the origin [40]. Then Theorem 1 can be proved.

Remark 8: For the problem of global asymptotic stability of uncertain complex cascade systems, the nested saturated control design has been proposed in this paper. Specifically, a modified differentiable saturation function (5) is proposed to facilitate the saturation reduction analysis. Based on the differentiable saturation function (5), the saturation reduction analysis has been divided into two steps. By constructing hierarchical and known boundaries with the help of differential saturation functions, the time derivatives calculations of the boundary surface will be carried out in the following small domains:

$$\begin{cases} \left|x_j - u_{j-1}\right| \le 2^{j-n}\epsilon, & j = n+m,\ldots,n+1 \\ |x_i - u_{i-1}| < H_i, & i = n,\ldots,1 \end{cases}$$

for subsystems $x_{n+m},\ldots,x_{n+1}$; and then are analyzed in the small domains:

$$|x_i - u_{i-1}| \le l_i\varepsilon\rho(x) \le H_i,\ i = n,\ldots,1$$

for the subsystems $x_n,\ldots,x_1$. Along such hierarchical convergence strategy, all states will first converge to the domains $|x_j| \le |u_{j-1}| + 2^{j-n}\varepsilon$, $j = n+m,\ldots,n+1$, and $|x_i| < |u_{i-1}| + H_i$, $i = n,\ldots,1$, then states x_j, $j = n+m,\ldots,n+1$ will keep staying in the domains $|x_j| \le |u_{j-1}| + 2^{j-n}\varepsilon$. In turn, states x_i, $i = n,\ldots,1$ with respect to $|x_i| < |u_{i-1}| + H_i$ will converge further into the domains $|x_i| \le |u_{i-1}| + l_i\varepsilon\rho(x)$. At last, all states will converge globally to a small region through the above hierarchical convergence strategies. Then the closed-loop system is globally attractive and locally asymptotically stable, and thus is global asymptotic stability. Without using the differentiable saturation function (5), the time derivatives calculations of the boundary surface need to be first expanded with respect to the following small domains:

$$\begin{cases} \left|x_j - u_{j-1}\right| \le 2^{j-n}\epsilon, & j = n+m,\ldots,n+1 \\ |x_i - u_{i-1}| < l_i\epsilon\rho(x), & i = n,\ldots,1. \end{cases}$$

As the differential calculation of saturation function subject to state-dependent saturation levels is hard to be handled, the saturation reduction analysis may be difficult to develop in the above cases without using the differentiable saturation function (5). One-step global convergence to a small domain is certainly ideal. For uncertain complex cascade systems (1) and novel nested saturated control design (3), it is algorithmically impossible to achieve one-step global convergence to small domain now. However, based on the differentiable saturation function (5), the proposed hierarchical convergence strategies can achieve state global convergence to small regions. Compared with one-step global convergence, the hierarchical convergence strategy is somewhat easier to achieve global convergence and has stronger overall robustness to perturbations to some extent.

IV. Simulation Results

In this paper, we present an application to verify the effectiveness of the proposed method.

The ball and beam system can be found in many control laboratories, which is a typical nonlinear control system. It has the characteristics of simple mechanism and easy observation. It is a common experimental equipment in control laboratories and is usually used to test the effect of a control strategy [22]−[26] and [41]−[47]. The ball and beam system shares numerous similarities with robot systems, and many practical systems can be abstracted as models based on the concept of ball and beam. Due to the extensive application background of ball and beam systems in practical robotics and mechanical systems, as well as their simplicity of operation in the laboratory settings, the study of ball and beam systems has garnered significant scholarly attention.

In order to understand the simple principle of actual ball and beam model, we take an embedded ball and beam model (type: GBN2004-E2) of GOOGOL TECH-PARADOX (a company in China) as an example for a simple explanation. This type ball and beam model is shown in Fig. 1 (This figure can be found on website: http://new.paradoxtech.cn/product-detail/NoRjA5GB).

Fig. 1. The actual ball and beam model.

The ball and beam system is a typical single input, double output system. The beam is made to rotate in a vertical plane by applying a torque at the center of rotation and the ball is free to roll along the beam [44]. The sensor on the beam and the encoder on the motor detect the actual position of the ball and the actual position of the motor respectively, and then they feed back the relevant information to the controller to calculate the control quantity. By controlling the rotation of the motor, the beam is made to rotate so that the ball can roll along it.

Consider the simplified ball and beam system in [43] and [44] that is described as follows:

$$\begin{cases} 0 = (\frac{J_b}{R^2} + M)\ddot{r} + MG\sin\theta - Mr\dot{\theta}^2 \\ \tau = (Mx_1^2 + J + J_b)\ddot{\theta} + 2Mr\dot{r}\dot{\theta} + MGr\cos\theta \end{cases} \tag{30}$$

where r and θ represents the ball position and the beam angle, respectively, τ is the torque applied to the beam. The nominal value of the various system parameters are given as follows:

$$M = 0.05\ \mathrm{kg},\ \ R = 0.01\ \mathrm{m},\ \ J = 0.02\ \mathrm{kg}\cdot\mathrm{m}^2$$
$$J_b = 2\times 10^{-6}\ \mathrm{kg}\cdot\mathrm{m}^2,\ \ G = 9.81\ \mathrm{m/s}^2$$

where the meaning represented by the above parameters can be consulted with the ball and beam system in [43] and [44]. The illustrative diagram of ball and beam systems for system (30) is presented in Fig. 2. The principle of the ball and beam system in Fig. 2 is analogous to the principle of the actual ball and beam model in Fig. 1.

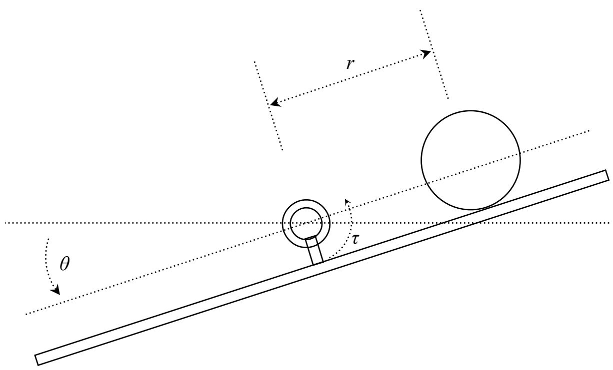

Fig. 2. An illustrative diagram of the ball and beam system.

Following [43], let$(r,\dot{r},\theta,\dot{\theta})^T = (x_1,x_2,x_3,x_4)^T$, the dynamics equation of the ball and beam system is described by:

$$\begin{cases}\dot{x}_1 = x_2\\ \dot{x}_2 = Hx_1x_4^2 - GH\sin x_3\\ \dot{x}_3 = x_4\\ \dot{x}_4 = -\dfrac{2Mx_1x_2x_4 + MGx_1\cos x_3}{Mx_1^2+J+J_b} + \dfrac{\tau}{Mx_1^2+J+J_b}\end{cases} \tag{31}$$

where H represents $H = M/(J_b/R^2+M) = 0.7134$.

The control objective of the ball and beam system is to design a state feedback control law such that the ball can be globally asymptotically positioned at the beam.

By making the following transformation:

$$\tau = 2Mx_1x_2x_4 + MGx_1\cos x_3 + (Mx_1^2+J+J_b)u \tag{32}$$

system (31) is transformed into

$$\begin{cases}\dot{x}_1 = x_2\\ \dot{x}_2 = -GH\sin x_3 + Hx_1x_4^2\\ \dot{x}_3 = x_4\\ \dot{x}_4 = u.\end{cases} \tag{33}$$

Further, by introducing the change of coordinates

$$y_1 = -x_1,\ y_2 = -x_2,\ y_3 = x_3,\ y_4 = x_4 \tag{34}$$

system (33) is rewritten as follows:

$$\begin{cases}\dot{y}_1 = y_2\\ \dot{y}_2 = \left(GH\dfrac{\sin y_3}{y_3}\right)\cdot y_3 + Hy_1y_4^2\\ \dot{y}_3 = y_4\\ \dot{y}_4 = u\end{cases} \tag{35}$$

where u is a new control input; G,H are constant parameters; $\sin y_3/y_3$ can be viewed as bounded uncertain parameter. As $y_3 = x_3 = \theta$ represents the beam angle, the range of the beam angle belongs to $\theta\in[0,\pi/2]$. Hence, one has $\sin y_3/y_3 \in [2/\pi, 1]$.

Based on the above analysis, system (35) is shown in the form of system (1), where the uncertain parameter $GH(\sin y_3/y_3)$ is limited by $GH(\sin y_3/y_3)\in[GH\times 2/\pi, GH] = [4.455354, 6.998454]$.

Following Theorem 1 in Section III, for globally stabilizing systems (35), we design the following control law:

$$\begin{cases}u = u_4 = -k_4\mathrm{sat}_{2^2\varepsilon}(y_4-u_3)\\ u_3 = -k_3\mathrm{sat}_{2^1\varepsilon}(y_3-u_2)\\ u_2 = -k_2\mathrm{sat}_{l_2\varepsilon\rho(y)}(y_2-u_1)\\ u_1 = -k_1\mathrm{sat}_{l_1\varepsilon\rho(y)}(y_1)\end{cases} \tag{36}$$

where $\rho(y)$ satisfies the conditions in the proposed algorithm, namely,

$$\rho(x) = (M_3 + M_2y_2^2 + M_1y_1^2)^{-\frac{1}{2}}$$
$$M_3 = M_2\Lambda^2 = \Lambda^8,\ M_2 = M_1^2\Lambda^2 = \Lambda^6,\ M_1 = \Lambda^2 \tag{37}$$

where $0<\varepsilon\le 2^{-1}, \Lambda\ge 2, k_i>0, l_i>0,\ i = 1,2,3,4$ are the constants to be determined.

By using Theorem 1, we can use controller (36) globally stabilizes system (35). According to the relevant algorithm design in Definition 1, Facts 1−3, and Theorem 1, general parameters of differentiable saturation functions are designed and calculated as follows:

$$\gamma_1 = 0.001,\ \beta_1 = 0.6,\ h_1 = 0.36\times10^{-5}$$
$$\Rightarrow N_1 = 0.36036\times10^{-5},\ H_1 = 0.6\times10^{-5}$$
$$\gamma_2 = 0.001,\ \beta_2 = 0.6,\ h_2 = 0.48\times10^{-4}$$
$$\Rightarrow N_2 = 0.48048\times10^{-4},\ H_2 = 0.8\times10^{-4}. \tag{38}$$

Meanwhile, those control parameters are also determined as follows:

$$k_1 = 1,\ k_2 = 285,\ k_3 = 0.96\times10^4,\ k_4 = 9.1\times10^7$$
$$l_1 = 2,\ l_2 = 1;\ \varepsilon = 0.002,\ \Lambda = 3. \tag{39}$$

As $GH(\sin y_3/y_3)\in[GH\times2/\pi, GH] = [4.455354, 6.998454]$, the bounded uncertain parameter is design as

$$GH\frac{\sin y_3}{y_3} = 4.455354 + 2.5431\,|\sin t|$$

in the simulation. Under parameter design (38) and (39) and the initial state $(x_1(0),x_2(0),x_3(0),x_4(0))^T = (0.0001,0,0,0)$, Figs. 3 and 4 show the effectiveness of the controller (36) for system (35). It can be observed form Figs. 3 and 4 that the states and control design are globally asymptotically convergent to zero.

In fact, the simplified ball and beam systems (30) are originally systems without uncertain parameters. However, after appropriate coordinate transformations, the ball and beam systems (30) can be transformed into systems (35) which can be considered as complex cascade systems with uncertain parameters. Two points should then be noted. On the one hand, the

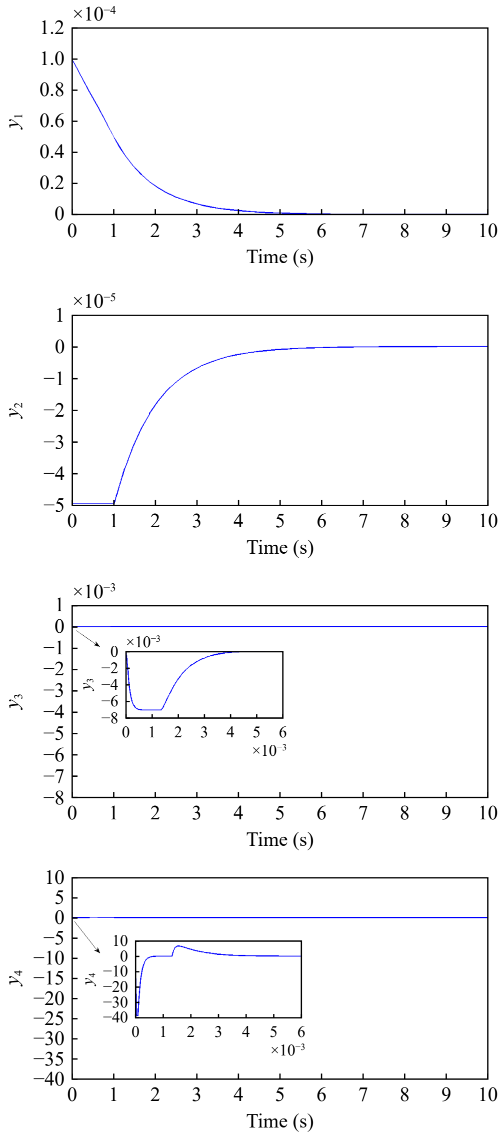

Fig. 3. Trajectories of states of system (35) with controller (36).

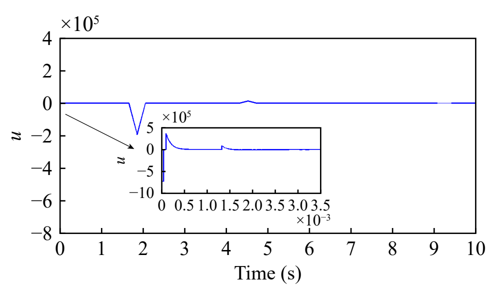

Fig. 4. Trajectories of control input of system (35) with controller (36).

effectiveness of the proposed algorithm for uncertain complex cascade systems can be verified by applying the algorithm proposed in this paper to deal with the equivalent systems (35) of systems (30). On the other hand, this paper has provided different treatments for the global stabilization of a class of simplified mechanical systems. The key is to transform mechanical systems into systems that can be considered as uncertain complex cascade systems.

V. Conclusion

In this paper, the proposed nested saturated control design with mixed saturation levels has been used to achieve the global asymptotic stability of uncertain complex cascade systems. By introducing a modified differentiable saturation function, the saturation reduction analysis of uncertain complex cascade system can be carried out smoothly. Moreover, the modified differentiable saturation function has helped to achieve hierarchical global convergence and to improve the robustness of the control strategy. Combined with the usual saturation schemes, some explicit parameter conditions have been given to guarantee the global asymptotic stability of uncertain complex cascade systems. In this paper, a class of nested saturation control strategies is proposed to deal with the global asymptotic stability problem of a special class of high-order differential equations. Furthermore, the proposed algorithm has been extended to corresponding applications in some practical mechanical models, such as the ball and beam system.

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Meng Li (Member, IEEE) receive the Ph.D. degree in control science and engineering from the School of Automation, Central South University, in 2020. He was a Visiting Ph.D. Student with the Department of Mechanical Engineering, University of Victoria, Canada, from 2019 to 2020. He is currently a Postdoctoral Researcher of control science and engineering with Wuhan National Laboratory for Optoelectronics, and also with the School of Artificial Intelligence and Automation, Huazhong University of Science and Technology. His research interests include uncertain nonlinear control systems, saturated control, and stability analysis of dynamic systems.

Zhigang Zeng (Fellow, IEEE) received the Ph.D. degree in systems analysis and integration from Huazhong University of Science and Technology in 2003. He is currently a Professor with the School of Artificial Intelligence and Automation, Huazhong University of Science and Technology, and also with the Key Laboratory of Image Processing and Intelligent Control of the Education Ministry of China. He has published over 300 international journal articles. His research interests include the theory of functional differential equations and differential equations with discontinuous right-hand sides, and their applications to dynamics of neural networks, memristive systems, and associative memories.

Dr. Zeng has been a Member of the Editorial Board of *Neural Networks* since 2012, *Cognitive Computation* since 2010, and *Applied Soft Computing* since 2013. He was an Associate Editor of the *IEEE Transactions on Neural Networks* from 2010 to 2011, and the *IEEE Transactions on Fuzzy Systems* from 2016 to 2021. He has been an Associate Editor of the *IEEE Transactions on Cybernetics* since 2014.