Recursive Filtering for Stochastic Systems With Filter-and-Forward Successive Relays

Hailong Tan, Bo Shen, Qi Li, and Hongjian Liu

***Abstract*—In this paper, the recursive filtering problem is considered for stochastic systems over filter-and-forward successive relay (FFSR) networks. An FFSR is located between the sensor and the remote filter to forward the measurement. In the successive relay, two cooperative relay nodes are adopted to forward the signals alternatively, thereby existing switching characteristics and inter-relay interferences (IRI). Since the filter-and-forward scheme is employed, the signal received by the relay is retransmitted after it passes through a linear filter. The objective of the paper is to concurrently design optimal recursive filters for FFSR and stochastic systems against switching characteristics and IRI of relays. First, a uniform measurement model is proposed by analyzing the transmission mechanism of FFSR. Then, novel filter structures with switching parameters are constructed for both FFSR and stochastic systems. With the help of the inductive method, filtering error covariances are presented in the form of coupled difference equations. Next, the desired filter gain matrices are further obtained by minimizing the trace of filtering error covariances. Moreover, the stability performance of the filtering algorithm is analyzed where the uniform bound is guaranteed on the filtering error covariance. Finally, the effectiveness of the proposed filtering method over FFSR is verified by a three-order resistance-inductance-capacitance circuit system.**

***Index Terms*—Filter-and-forward successive relay (FFSR), recursive filtering, relay network, stochastic system, time-varying system.**

I. Introduction

IN control communities, the filtering/state estimation has been one of the fundamental research topics due to its enormous application potentials in many fields [1]–[4]. Till now, a rich body of enlightening filtering strategies have been developed with various performance indexes, such as the H_∞ filtering [5]–[7], the set-membership filtering [8]–[10], the moving horizon estimation [11], [12], the finite impulse response (FIR) filtering [13]–[15], the Kalman filtering [16]–[18] and the robust recursive filtering [19]–[21]. Among them, the Kalman filtering has been viewed as the most efficient state estimation approach for linear stochastic systems with Gaussian noises. Notably, the traditional Kalman filtering method provides an optimal state estimate (in the minimum error variance sense) where the filtering error covariance is recursively given by Riccati equations. In view of their advantages in online computation, the Kalman-type filtering as well as its variants has attracted considerable research interest, see, e.g., [22]–[25].

With the development of wireless communication techniques, the signal transmission between the sensor node and the filter is frequently implemented through wireless networks. In practice, it is always the case that the coverage of the wireless network is essentially limited because of the non-negligible fading phenomenon. As a result, signals transmitted by the sensor may not be successfully received by the filter, especially in the long-distance transmission [26]. In order to broaden the coverage of the network, a typical solution is to arrange a relay to forward the signal from the sensor to the filter. On account of its largely potentiality in the long-distance wireless communication, the relay network has gained particular research attention from communication communities [27]–[29]. Accordingly, many effective relay techniques have been proposed to cater for real engineering requirements, such as the half-duplex relay, the virtual full-duplex relay and the full-duplex relay.

As one of the most common virtual full-duplex relay techniques, the successive relay subtly embeds two synergistic relay nodes with switching modes to alternately forward signals. Specifically, in each time slot, one of the two relay nodes operating in the receiving mode receives signals from the signal source. At the same time, the other relay node is certainly in the transmitting mode that broadcasts signals to the destination. In the next time slot, the modes of two relay nodes are swapped compulsively, which guarantees successive signal transmissions from the signal source to the destination. By utilizing two collaborative relay nodes, the successive relay has shown its great advantages in improving the bandwidth efficiency, thereby becoming an intriguing topic [29]. However, because of the switching modes of relay nodes, there evi-

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dently appear complex switching characteristics in the successive relay, which unavoidably affects the performance of the networks. On the other hand, the signals transmitted by the relay node in the transmitting mode will be also received by another relay node since it is certainly operating in the receiving mode. Therefore, the collaborative relay nodes are vulnerable to the inter-relay interference (IRI). Recently, many related results concerning the switching characteristics and IRI have been available in the existing literature [30], [31].

Apart from relay techniques, relaying schemes have been another counterpart that should be considered seriously in relay systems. The majority of the relaying schemes reported in the literature include, but are not limited to, the amplify-and-forward (AF), the decode-and-forward (DF) and the filter-and-forward (FF). Differently from the AF relaying scheme amplifying undesired noises in signals, the FF relaying retransmits the signal after it passes through a linear filter, which performs higher accuracy. Comparing with the DF relaying where complex decoding and re-encoding are necessary, such a relaying scheme is easy to implement. Therefore, the FF relaying can realize a trade-off between the complexity and the performance improvement [32]. Consequently, the FF relaying has aroused a lot of research interests and considerable efforts have been devoted to the filter design problem for the FF relaying. For example, in [33], the FF relaying has been first proposed where a finite impulse response filter has been employed to reconstruct actual states from noisy measurements. In [34], FF relays with an optimal filter in the minimum mean-square error sense have been designed by solving a set of convex optimization problems. In [35], a jointly source and relay filter design problem has been considered by converting it into a constrained optimization problem in a finite dimensional space.

Recently, some primary research results investigating the filtering problem under relay networks have been available in the existing literature. Typically, in [36], the optimal and sub-optimal relay configuration methods have been given by minimizing the filtering error covariance of the Kalman filter. In [37], the robust filtering problem has been considered for a class of uncertain systems over AF relay network with random transmission power. In [38], a recursive filter has been designed where the measurement has been forwarded by a full-duplex relay. However, for the filtering problem simultaneously considering relaying techniques and schemes, the corresponding results are extremely deficient despite the fact that relaying techniques and schemes are essential indivisible in practical relay systems. To shorten such a gap, we make the first attempt to investigate the filtering problem under typical filter-and-forward successive relay (FFSR) networks.

To handle the filter design problem over FFSR networks, we are confronted with the following challenges: 1) How to eliminate the effects of switching characteristics and IRI of the successive relay? 2) How to cooperatively design the classical Kalman-type filters for both the FFSR and underlying systems; and 3) How to analyze the stability of the proposed filtering algorithms in the presence of FFSR. Hence, we are endeavoring to overcome the identified challenges by providing the filter design method for the stochastic system with FFSR. Accordingly, the main contributions of this paper are summarized as follows: 1) The FFSR network is, for the first time, considered in the filtering problem of stochastic systems; 2) New optimal recursive filters with switching parameters are jointly designed for the FFSR and the stochastic system when there appear switching characteristics and IRI of FFSR; and 3) The boundedness stability is analyzed for the proposed filtering algorithm under FFSR networks.

II. Problem Formulation

A. System Description

Consider a stochastic system as follows:

$$\begin{cases} x_{k+1} = \Gamma_k x_k + \Sigma_k w_k \\ y_k = \Psi_k x_k + \Theta_k v_k \end{cases} \tag{1}$$

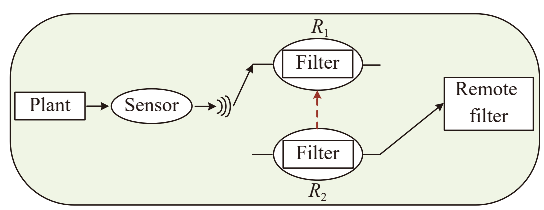
where $x_k \in \mathbb{R}^{n_x}$ denotes the state vector at sampling instant k, $y_k \in \mathbb{R}^{n_y}$ is the measurement output of the sensor node, $w_k \in \mathbb{R}^{n_w}$ and $v_k \in \mathbb{R}^{n_v}$ stand for the zero-mean Gaussian distributed noises with

$$\mathbb{E}\left\{ [w_k \quad v_k][w_l \quad v_l]^T \right\} = \delta(k,l) \begin{bmatrix} W_k & 0 \\ 0 & V_k \end{bmatrix} \tag{2}$$

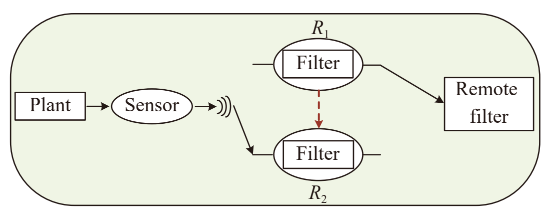
where $\Gamma_k \in \mathbb{R}^{n_x \times n_x}$, $\Psi_k \in \mathbb{R}^{n_y \times n_x}$, $\Sigma_k \in \mathbb{R}^{n_x \times n_w}$, $\Theta_k \in \mathbb{R}^{n_y \times n_v}$, $W_k \in \mathbb{R}^{n_w \times n_w} > 0$ and $V_k \in \mathbb{R}^{n_v \times n_v} > 0$ are pre-setting matrices, $\delta(k,l) = 1$ for $k = l$, otherwise, $\delta(k,l) = 0$. Moreover, the initial value of the discrete-time system follows the Gaussian distribution with mean $\bar{x}_0$ and covariance P_0.

B. Relay Scheme

The considered filtering problem over FFSR network is visually shown in Fig. 1. In detail, two relays (R_1 and R_2) with embedded filters switch between the receiving mode and the transmission mode. In the even time instant, relay R_1 in the receiving mode receives from the sensor and further generates an estimate by its filter. Meanwhile, the relay R_2 in the transmitting mode forwards signals to the remote filter. It should be mentioned that, since both relays can receive the



(a) The even time instant



(b) The odd time instant

Fig. 1. Diagram for the filtering problem over FFSR network.

signals transmitted by each other, there inevitably exists IRI indicated by red dotted lines. In the next odd time instant, R_1 and R_2 exchange their operation modes to assist the signal transmission.

According to [39], the signals arriving at each relay are denoted as

$$\begin{cases} y_{1,k} = \gamma_{1,k} y_k + \gamma_{3,k} \hat{y}_{2,k} + \xi_{1,k}, & k \text{ is even} \\ y_{2,k} = \gamma_{2,k} y_k + \gamma_{3,k} \hat{y}_{1,k} + \xi_{2,k}, & k \text{ is odd} \end{cases} \tag{3}$$

where $y_{i,k}$ $(i = 1,2)$ is the received signal of the relay R_i, $\gamma_{1,k}$ and $\gamma_{2,k}$ stand for the stochastic channel coefficients from sensor to R_i $(i = 1,2)$, $\gamma_{3,k}$ represents the coefficient of relay-relay channel, $\xi_{i,k}$ $(i = 1,2)$ is the channel noise, $\hat{y}_{i,k} = \Psi_k \vartheta_{i,k-1}$ $(i = 1,2)$ is the signal transmitted by relay R_i and $\vartheta_{i,k-1}$ is the estimate signal of the relay which will be generated later.

Furthermore, the relays alternately forward estimate signals to the remote filter. Then, the signal z_k forwarded to the remote filter is depicted as follows:

$$z_k = \begin{cases} \gamma_{4,k} \hat{y}_{1,k} + \tau_{1,k}, & k \text{ is odd} \\ \gamma_{5,k} \hat{y}_{2,k} + \tau_{2,k}, & k \text{ is even} \end{cases} \tag{4}$$

where $\gamma_{i,k}$ $(i = 4,5)$ is the stochastic channel coefficient of relay-filter channel, $\tau_{m,k}$ $(m = 1,2)$ is the channel noise.

Remark 1: In this paper, the measurements are transmitted through FFSR with switching characteristics and IRI. Based on measurement models (3) and (4), the transmission characteristics of FFSR are described explicitly. In what follows, we will cooperatively design easy-to-implement filters for the relay and the stochastic system.

The stochastic variables $\gamma_{i,k}$ $(i = 1,2,3,4,5)$, $\xi_{j,k}$ $(j = 1,2)$ and $\tau_{m,k}$ $(m = 1,2)$ are mutually independent and auto-uncorrelated and satisfy

$$\begin{aligned} &\mathbb{E}\xi_{j,k} = 0,\ \mathbb{E}\tau_{m,k} = 0,\ \mathbb{E}\gamma_{i,k} = \bar{\gamma}_{i,k} \\ &\mathbb{E}\{\xi_{j,k}\xi_{j,k}^T\} = G_{j,k},\ \mathbb{E}\{\tau_{m,k}\tau_{m,k}^T\} = O_{m,k} \\ &\mathbb{E}\{(\gamma_{i,k} - \bar{\gamma}_{i,k})^2\} = \vec{\gamma}_{i,k} \end{aligned} \tag{5}$$

where $\bar{\gamma}_{i,k}$ and $\vec{\gamma}_{i,k} > 0$ $(i = 1,2,3,4,5)$ are known parameters, $G_{j,k} > 0$ $(j = 1,2)$ and $O_{m,k} > 0$ $(m = 1,2)$ are given positive-definite matrices. Moreover, the statistical characteristics of channel coefficients and noises are available for both relays and the remote filter.

C. Recursive Filter

To obtain the estimate signal $\vartheta_{i,k}$ $(i = 1,2)$ in the relay, we construct the following filter for relay R_i:

$$\vartheta_{i,k} = \Gamma_{k-1}\vartheta_{i,k-1} + \delta_{i,k} K_{i,k}(y_{i,k} - \bar{\gamma}_{i,k}\Psi_k\Gamma_{k-1}\vartheta_{i,k-1} - \bar{\gamma}_{3,k}\hat{y}_{i,k}) \tag{6}$$

where $K_{i,k}$ denotes the gain matrix which should be designed later and $\delta_{i,k}$ $(i = 1,2)$ is an auxiliary variable satisfying

$$\begin{aligned} &\delta_{1,k} = \theta_k,\ \delta_{2,k} = 1 - \theta_k \\ &\theta_k = \begin{cases} 1, & k \text{ is even} \\ 0, & k \text{ is odd.} \end{cases} \end{aligned} \tag{7}$$

In addition, the initial value is selected as $\vartheta_{i,0} = \bar{x}_0$ and $\hat{y}_{i,0} = \Psi_0 \vartheta_{i,0}$.

Remark 2: In this paper, a novel filter (6) is constructed for the FF relaying schemes in FFSR. The distinctive features of the filter are summarized as the following two aspects: 1) An auxiliary variable $\delta_{i,k}$ is introduced to accommodate the switching characteristic of successive relay; and 2) The term $\bar{\gamma}_{3,k}\hat{y}_{i,k}$ is included to eliminate the effects of IRI. Notably, if the filter is designed as

$$\begin{aligned} \vartheta_{i,k} &= \Gamma_{k-1}\vartheta_{i,k-1} + \delta_{i,k} K_{i,k}(y_{i,k} - \bar{\gamma}_{i,k}\Psi_k\Gamma_{k-1}\vartheta_{i,k-1} \\ &\quad - \bar{\gamma}_{3,k}\hat{y}_{j,k}) \\ &= \Gamma_{k-1}\vartheta_{i,k-1} + \delta_{i,k} K_{i,k}(\gamma_{i,k} y_k + \xi_{i,k} \\ &\quad - \bar{\gamma}_{i,k}\Psi_k\Gamma_{k-1}\vartheta_{i,k-1} + (\gamma_{3,k} - \bar{\gamma}_{3,k})\hat{y}_{j,k}). \end{aligned}$$

We obtain that $\mathbb{E}\{(\gamma_{3,k} - \bar{\gamma}_{3,k})\hat{y}_{j,k}\} = 0$, which means that the effects of IRI are eliminated in the mean sense. However, due mainly to the stochastic channel coefficient $\gamma_{3,k}$, it is technically impossible for R_i to capture the exact output $\hat{y}_{j,k}$ of R_j. Therefore, for relay R_i, the term $\hat{y}_{j,k}$ in IRI is replaced by $\hat{y}_{i,k}$ in the constructed filter (6).

With the help of the switched parameter θ_k, z_k is rewritten as follows:

$$z_k = \delta_{2,k}(\gamma_{4,k}\hat{y}_{1,k} + \tau_{1,k}) + \delta_{1,k}(\gamma_{5,k}\hat{y}_{2,k} + \tau_{2,k}). \tag{8}$$

Utilizing signal z_k, we construct the following remote filter for the stochastic system over FFSR network:

$$\hat{x}_k = \Gamma_{k-1}\hat{x}_{k-1} + L_k(z_k - \bar{\delta}_{2,k}\Psi_k\hat{x}_{k-1} - \bar{\delta}_{1,k}\Psi_k\hat{x}_{k-1}) \tag{9}$$

where $\hat{x}_k$ denote the estimate for x_k, L_k is the filter gain to be meticulously designed, $\bar{\delta}_{1,k} = \delta_{1,k}\bar{\gamma}_{5,k}$ and $\bar{\delta}_{2,k} = \delta_{2,k}\bar{\gamma}_{4,k}$. The initial value of the filter (9) is set as $\hat{x}_0 = \bar{x}_0$.

Define $\tilde{\vartheta}_{i,k} = x_k - \vartheta_{i,k}$. Then, it is easily obtained from (1), (3) and (6) that

$$\begin{aligned} \tilde{\vartheta}_{i,k} &= (\Gamma_{k-1} - \delta_{i,k}K_{i,k}\bar{\Gamma}_{i,k-1})\tilde{\vartheta}_{i,k-1} - \delta_{i,k}K_{i,k}\xi_{i,k} \\ &\quad - \delta_{i,k}(\gamma_{i,k} - \bar{\gamma}_{i,k})K_{i,k}\Psi_k\Gamma_{k-1}x_{k-1} + \Sigma_{k-1}w_{k-1} \\ &\quad - \delta_{i,k}(\gamma_{3,k} - \bar{\gamma}_{3,k})K_{i,k}\Psi_k x_{k-1} - \delta_{i,k}\gamma_{i,k}K_{i,k}\Theta_k v_k \\ &\quad + \delta_{i,k}\gamma_{3,k}K_{i,k}\Psi_k\tilde{\vartheta}_{j,k-1} - \delta_{i,k}\gamma_{i,k}K_{i,k}\Psi_k\Sigma_{k-1}w_{k-1} \end{aligned} \tag{10}$$

where $i, j \in \{1,2\}$ and $i \neq j$, $\bar{\Gamma}_{i,k-1} = \bar{\gamma}_{i,k}\Psi_k\Gamma_{k-1} + \bar{\gamma}_{3,k}\Psi_k$.

Letting the filtering error of the remote filter be $\tilde{x}_k = x_k - \hat{x}_k$, it then follows from (1), (8) and (9) that:

$$\begin{aligned} \tilde{x}_k &= [\Gamma_{k-1} - (\bar{\delta}_{1,k} + \bar{\delta}_{2,k})L_k\Psi_k]\tilde{x}_{k-1} + \Sigma_{k-1}w_{k-1} \\ &\quad - [\delta_{2,k}(\gamma_{4,k} - \bar{\gamma}_{4,k}) + \delta_{1,k}(\gamma_{5,k} - \bar{\gamma}_{5,k})]L_k\Psi_k x_{k-1} \\ &\quad + \delta_{2,k}\gamma_{4,k}L_k\Psi_k\tilde{\vartheta}_{1,k-1} + \delta_{1,k}\gamma_{5,k}L_k\Psi_k\tilde{\vartheta}_{2,k-1} \\ &\quad - \delta_{2,k}L_k\tau_{1,k} - \delta_{1,k}L_k\tau_{2,k}. \end{aligned} \tag{11}$$

Based on the filtering errors $\tilde{\vartheta}_{i,k}$ $(i = 1,2)$ and $\tilde{x}_k$, we define $\Xi_{i,k} = \mathbb{E}\{\tilde{\vartheta}_{i,k}\tilde{\vartheta}_{i,k}^T\}$ and $P_k = \mathbb{E}\{\tilde{x}_k\tilde{x}_k^T\}$. In this paper, our main objective is to synergistically design the filter gain matrices $K_{i,k}$ $(i = 1,2)$ and L_k such that the filtering error covariances $\Xi_{i,k}$ and P_k are minimized in the trace sense, i.e.,

$$K_{i,k} = \arg\min_{K_{i,k}} \mathrm{tr}(\Xi_{i,k}),\ L_k = \arg\min_{L_k} \mathrm{tr}(P_k). \tag{12}$$

Remark 3: It should be mentioned that, the performance of the remote filter is inevitably affected by the FF relaying scheme. So, the filtering scheme of relays and the remote filter should be coordinately designed to achieve more accurate state estimates. In this case, the filtering methods with the minimum mean square error will be simultaneously developed for both FFSR and stochastic systems in this paper.

III. Main Results

A. Filter Design

In this section, the desired filter gain matrices are provided for the FFSR and the remote filter.

Lemma 1: Let $\Pi_k = \mathbb{E}\{x_k x_k^T\}$. Π_k can be deduced from the following recursions:

$$\begin{aligned}\Pi_k &= \Gamma_{k-1}\Pi_{k-1}\Gamma_{k-1}^T + \Sigma_{k-1}V_{k-1}\Sigma_{k-1}^T\\ \Pi_0 &= P_0 + \bar{x}_0\bar{x}_0^T. \end{aligned}\tag{13}$$

Proof: The proof is obtained immediately from (1) and the details are omitted here. ∎

Let $\Xi_{ij,k} = \mathbb{E}\{\tilde{\vartheta}_{i,k}\tilde{\vartheta}_{j,k}^T\}$ and $\Xi_{xi,k} = \mathbb{E}\{x_k\tilde{\vartheta}_{i,k}^T\}$ $(i,j = 1,2, i \neq j)$. Then, the covariance $\Xi_{i,k}$ $(i = 1,2)$ is obtained in the following lemma.

Lemma 2: The covariance $\Xi_{i,k}$ is the solution to the following matrix equation:

$$\begin{aligned}\Xi_{i,k} &= (\Gamma_{k-1} - \delta_{i,k}K_{i,k}\bar{\Gamma}_{i,k-1})\Xi_{i,k-1}(\Gamma_{k-1} - \delta_{i,k}K_{i,k}\\ &\quad\times \bar{\Gamma}_{i,k-1})^T + \delta_{i,k}\bar{\gamma}_{3,k}\Gamma_{k-1}\Xi_{ij,k-1}\Psi_k^T K_{i,k}^T\\ &\quad + \Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T + \delta_{i,k}K_{i,k}\aleph_{i,k}K_{i,k}^T\\ &\quad + \delta_{i,k}\bar{\gamma}_{3,k}K_{i,k}\Psi_k\Xi_{ij,k-1}^T\Gamma_{k-1}^T\\ &\quad - \delta_{i,k}\bar{\gamma}_{i,k}K_{i,k}\Psi_k\Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T\\ &\quad - \delta_{i,k}\bar{\gamma}_{i,k}\Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T\Psi_k^T K_{i,k}^T \end{aligned}\tag{14}$$

where

$$\begin{aligned}\aleph_{i,k} &= (\vec{\gamma}_{i,k} + \bar{\gamma}_{i,k}^2)\Psi_k\Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T\Psi_k^T + G_{i,k}\\ &\quad + \vec{\gamma}_{3,k}\Psi_k\Pi_{k-1}\Psi_k^T + \vec{\gamma}_{i,k}\Psi_k\Gamma_{k-1}\Pi_{k-1}\Gamma_{k-1}^T\Psi_k^T\\ &\quad + (\vec{\gamma}_{3,k} + \bar{\gamma}_{3,k}^2)\Psi_k\Xi_{j,k-1}\Psi_k^T + (\vec{\gamma}_{i,k} + \bar{\gamma}_{i,k}^2)\Theta_k\\ &\quad \times V_k\Theta_k^T - \bar{\gamma}_{3,k}\bar{\Gamma}_{i,k-1}\Xi_{ij,k-1}\Psi_k^T - \bar{\gamma}_{3,k}\Psi_k\Xi_{ij,k-1}^T\\ &\quad \times \bar{\Gamma}_{i,k-1}^T - \bar{\gamma}_{3,k}\Psi_k(\Xi_{xj,k-1} + \Xi_{xj,k-1}^T)\Psi_k^T, \end{aligned}$$

and $j = 1,2$, $i \neq j$. Moreover, the initial value of the recursion is set as $\Xi_{1,0} = \Xi_{2,0} = P_0$.

Proof: From (10), one has

$$\begin{aligned}\Xi_{i,k} &= \mathbb{E}\{[(\Gamma_{k-1} - \delta_{i,k}K_{i,k}\bar{\Gamma}_{i,k-1})\tilde{\vartheta}_{i,k-1} - \delta_{i,k}K_{i,k}\xi_{i,k}\\ &\quad - \delta_{i,k}(\gamma_{i,k} - \bar{\gamma}_{i,k})K_{i,k}\Psi_k\Gamma_{k-1}x_{k-1} + \Sigma_{k-1}w_{k-1}\\ &\quad - \delta_{i,k}(\gamma_{3,k} - \bar{\gamma}_{3,k})K_{i,k}\Psi_k x_{k-1} - \delta_{i,k}\gamma_{i,k}K_{i,k}\Theta_k v_k\\ &\quad + \delta_{i,k}\gamma_{3,k}K_{i,k}\Psi_k\tilde{\vartheta}_{j,k-1} - \delta_{i,k}\gamma_{i,k}K_{i,k}\Psi_k\Sigma_{k-1}w_{k-1}]\\ &\quad \times [(\Gamma_{k-1} - \delta_{i,k}K_{i,k}\bar{\Gamma}_{i,k-1})\tilde{\vartheta}_{i,k-1} - \delta_{i,k}K_{i,k}\xi_{i,k}\\ &\quad - \delta_{i,k}(\gamma_{i,k} - \bar{\gamma}_{i,k})K_{i,k}\Psi_k\Gamma_{k-1}x_{k-1} + \Sigma_{k-1}w_{k-1}\\ &\quad - \delta_{i,k}(\gamma_{3,k} - \bar{\gamma}_{3,k})K_{i,k}\Psi_k x_{k-1} - \delta_{i,k}\gamma_{i,k}K_{i,k}\Theta_k v_k\\ &\quad + \delta_{i,k}\gamma_{3,k}K_{i,k}\Psi_k\tilde{\vartheta}_{j,k-1} - \delta_{i,k}\gamma_{i,k}K_{i,k}\Psi_k\Sigma_{k-1}w_{k-1}]^T\}. \end{aligned}\tag{15}$$

From (7), it is obtained that $\delta_{i,k}^2 = \delta_{i,k}$. Noting that the channel coefficients and the noises are mutually independent and auto-uncorrelated, one has

$$\begin{aligned}&\mathbb{E}\{w_k\xi_{m,k}^T\} = 0,\ \mathbb{E}\{w_k v_k^T\} = 0,\ \mathbb{E}\{w_k\gamma_{n,k}\} = 0\\ &\mathbb{E}\{\xi_{m,k}v_k^T\} = 0,\ \mathbb{E}\{\xi_{m,k}\gamma_{n,k}\} = 0,\ \mathbb{E}\{v_k\gamma_{n,k}\} = 0 \end{aligned}\tag{16}$$

for all $m = 1,2$ and $n = 1,2,3$. Moreover, we have

$$\begin{aligned}&\mathbb{E}\{\tilde{\vartheta}_{i,k-1}\xi_{i,k}^T\} = 0,\ \mathbb{E}\{\tilde{\vartheta}_{i,k-1}w_k^T\} = 0\\ &\mathbb{E}\{\tilde{\vartheta}_{i,k-1}v_k^T\} = 0,\ \mathbb{E}\{x_{k-1}\xi_{i,k}^T\} = 0\\ &\mathbb{E}\{x_{k-1}w_k^T\} = 0,\ \mathbb{E}\{x_{k-1}v_k^T\} = 0. \end{aligned}\tag{17}$$

Substituting (16) and (17) into (15), we directly obtain (14) through some simple algebraic operations.

In addition, noting that $\vartheta_{1,0} = \vartheta_{2,0} = \bar{x}_0$, we easily have $\Xi_{1,0} = \Xi_{2,0} = P_0$. ∎

From Lemma 2, it is obvious that the cross-covariance terms $\Xi_{ij,k}$ and $\Xi_{xi,k}$ $(i,j = 1,2, i \neq j)$ are essential for the covariance $\Xi_{i,k}$. In what follows, the cross-covariance terms $\Xi_{ij,k}$ and $\Xi_{xi,k}$ are given recursively.

Lemma 3: The cross-covariance $\Xi_{ij,k}$ $(i,j = 1,2, i \neq j)$ follows the following recursion:

$$\begin{aligned}\Xi_{ij,k} &= (\Gamma_{k-1} - \delta_{i,k}K_{i,k}\bar{\Gamma}_{i,k-1})\Xi_{ij,k-1}(\Gamma_{k-1} - \delta_{j,k}\\ &\quad \times K_{j,k}\bar{\Gamma}_{j,k-1})^T + \delta_{j,k}\bar{\gamma}_{3,k}(\Gamma_{k-1} - \delta_{i,k}K_{i,k}\\ &\quad \times \bar{\Gamma}_{i,k-1})\Xi_{i,k-1}\Psi_k^T K_{j,k}^T + \delta_{i,k}\bar{\gamma}_{3,k}K_{i,k}\Psi_k\\ &\quad \times \Xi_{j,k-1}(\Gamma_{k-1} - \delta_{j,k}K_{j,k}\bar{\Gamma}_{j,k-1})^T\\ &\quad - \delta_{i,k}\bar{\gamma}_{i,k}K_{i,k}\Psi_k\Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T\\ &\quad - \delta_{j,k}\bar{\gamma}_{j,k}\Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T\Psi_k^T K_{j,k}^T\\ &\quad + \Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T \end{aligned}\tag{18}$$

where $\Xi_{ij,0} = P_0$.

Proof: It follows from (10) that:

$$\begin{aligned}\Xi_{ij,k} &= \mathbb{E}\{[(\Gamma_{k-1} - \delta_{i,k}K_{i,k}\bar{\Gamma}_{i,k-1})\tilde{\vartheta}_{i,k-1} - \delta_{i,k}K_{i,k}\xi_{i,k}\\ &\quad - \delta_{i,k}(\gamma_{i,k} - \bar{\gamma}_{i,k})K_{i,k}\Psi_k\Gamma_{k-1}x_{k-1} + \Sigma_{k-1}w_{k-1}\\ &\quad - \delta_{i,k}(\gamma_{3,k} - \bar{\gamma}_{3,k})K_{i,k}\Psi_k x_{k-1} - \delta_{i,k}\gamma_{i,k}K_{i,k}\Theta_k v_k\\ &\quad + \delta_{i,k}\gamma_{3,k}K_{i,k}\Psi_k\tilde{\vartheta}_{j,k-1} - \delta_{i,k}\gamma_{i,k}K_{i,k}\Psi_k\\ &\quad \times \Sigma_{k-1}w_{k-1}][(\Gamma_{k-1} - \delta_{j,k}K_{j,k}\bar{\Gamma}_{j,k-1})\tilde{\vartheta}_{j,k-1}\\ &\quad - \delta_{j,k}K_{j,k}\xi_{j,k} - \delta_{j,k}(\gamma_{j,k} - \bar{\gamma}_{j,k})K_{j,k}\Psi_k\Gamma_{k-1}x_{k-1}\\ &\quad + \Sigma_{k-1}w_{k-1} - \delta_{j,k}(\gamma_{3,k} - \bar{\gamma}_{3,k})K_{j,k}\Psi_k x_{k-1}\\ &\quad - \delta_{j,k}\gamma_{j,k}K_{j,k}\Theta_k v_k + \delta_{j,k}\gamma_{3,k}K_{j,k}\Psi_k\tilde{\vartheta}_{i,k-1}\\ &\quad - \delta_{j,k}\gamma_{j,k}K_{j,k}\Psi_k\Sigma_{k-1}w_{k-1}]^T\}. \end{aligned}\tag{19}$$

Recurring to (16) and (17), we further obtain

$$\begin{aligned}\Xi_{ij,k} &= (\Gamma_{k-1} - \delta_{i,k}K_{i,k}\bar{\Gamma}_{i,k-1})\Xi_{ij,k-1}(\Gamma_{k-1} - \delta_{j,k}K_{j,k}\\ &\quad \times \bar{\Gamma}_{j,k-1})^T + \delta_{j,k}\bar{\gamma}_{3,k}(\Gamma_{k-1} - \delta_{i,k}K_{i,k}\bar{\Gamma}_{i,k-1})\\ &\quad \times \Xi_{i,k-1}\Psi_k^T K_{j,k}^T + \delta_{i,k}\bar{\gamma}_{3,k}K_{i,k}\Psi_k\Xi_{j,k-1}\\ &\quad \times (\Gamma_{k-1} - \delta_{j,k}K_{j,k}\bar{\Gamma}_{j,k-1}) + \Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T \end{aligned}$$

$$
\begin{aligned}
&+\delta_{i,k}\delta_{j,k}\vec{\gamma}_{3,k}K_{i,k}\Psi_k\Pi_{k-1}\Psi_k^T K_{j,k}^T-\delta_{i,k}\delta_{j,k}\\
&\times\vec{\gamma}_{3,k}K_{i,k}\Psi_k(\Xi_{xi,k-1}+\Xi_{xj,k-1}^T)\Psi_k^T K_{j,k}^T\\
&+\delta_{i,k}\delta_{j,k}(\vec{\gamma}_{3,k}+\bar{\gamma}_{3,k}^2)K_{i,k}\Psi_k\Xi_{ij,k}^T\Psi_k^T K_{j,k}^T\\
&+\delta_{i,k}\delta_{j,k}\bar{\gamma}_{i,k}\bar{\gamma}_{j,k}K_{i,k}\Theta_k V_k\Theta_k^T K_{j,k}^T+\delta_{i,k}\delta_{j,k}\\
&\times\bar{\gamma}_{i,k}\bar{\gamma}_{j,k}K_{i,k}\Psi_k\Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T\Psi_k^T K_{j,k}^T\\
&-\delta_{i,k}\bar{\gamma}_{i,k}K_{i,k}\Psi_k\Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T\\
&-\delta_{j,k}\bar{\gamma}_{j,k}\Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T\Psi_k^T K_{j,k}^T. \qquad (20)
\end{aligned}
$$

On the other hand, we know from (7) that $\delta_{i,k}\delta_{j,k}=0$. Then, the recursion (18) is obtained from (20) directly. Moreover, since $\vartheta_{1,0}=\vartheta_{2,0}=\bar{x}_0$, it is easy to deduce that $\Xi_{ij,0}=P_0$. ■

Then, we will show the derivation of the term $\Xi_{xi,k}$ $(i=1,2)$.

Lemma 4: The expression of cross-covariance $\Xi_{xi,k}$ $(i=1,2)$ is given as

$$
\begin{aligned}
\Xi_{xi,k}=&\,\Gamma_{k-1}\Xi_{xi,k-1}(\Gamma_{k-1}-\delta_{i,k}K_{i,k}\bar{\Gamma}_{i,k-1})^T\\
&+\delta_{i,k}\bar{\gamma}_{3,k}\Gamma_{k-1}\Xi_{xj,k-1}\Psi_k^T K_{i,k}^T\\
&-\delta_{i,k}\bar{\gamma}_{i,k}\Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T\Psi_k^T K_{i,k}^T\\
&+\Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T \qquad (21)
\end{aligned}
$$

where $\Xi_{xi,0}=P_0$.

Proof: By the initial value $\vartheta_{i,0}=\bar{x}_0$, we easily have $\Xi_{xi,0}=P_0$. Through some simple mathematical operations, the equation (21) is achieved from (1), (10), (16) and (17) for all $k>0$. ■

So far, the precise expression of $\Xi_{i,k}$ has been presented in (15), (18) and (21). Next, we aim to compute the filtering error covariance P_k.

Letting $P_{x,k}=\mathbb{E}\{\tilde{x}_k x_k^T\}$ and $P_{i,k}=\mathbb{E}\{\tilde{x}_k\tilde{\vartheta}_{i,k}^T\}$ $(i=1,2)$, $P_{x,k}$ and $P_{i,k}$ are given in the following lemmas.

Lemma 5: $P_{x,k}$ is determined by the following recursion:

$$
\begin{aligned}
P_{x,k}=&\,[\Gamma_{k-1}-(\bar{\delta}_{1,k}+\bar{\delta}_{2,k})L_k\Psi_k]P_{x,k-1}\Gamma_{k-1}^T\\
&+\Sigma_{k-1}W_{k-1}\Sigma_{k-1}+\bar{\delta}_{2,k}L_k\Psi_k\Xi_{x1,k-1}^T\Gamma_{k-1}^T\\
&+\bar{\delta}_{1,k}L_k\Psi_k\Xi_{x2,k-1}^T\Gamma_{k-1}^T \qquad (22)
\end{aligned}
$$

where $\Xi_{xi,k}$ $(i=1,2)$ is presented in (21). Moreover, the initial value is $P_{x,0}=P_0$.

Proof: Since the initial value of the filter (9) is set as $\hat{x}_0=\bar{x}_0$, it is obvious that $P_{x,0}=\mathbb{E}\{(x_0-\bar{x}_0)x_0^T\}=P_0$. Then, we will show that (22) holds for all $k>0$.

When $k>0$, it is clear to see from (1) and (11) that

$$
\begin{aligned}
P_{x,k}=&\,\mathbb{E}\{[(\Gamma_{k-1}-(\bar{\delta}_{1,k}+\bar{\delta}_{2,k})L_k\Psi_k)\tilde{x}_{k-1}+\Sigma_{k-1}w_{k-1}\\
&-(\delta_{2,k}(\gamma_{4,k}-\bar{\gamma}_{4,k})+\delta_{1,k}(\gamma_{5,k}-\bar{\gamma}_{5,k}))L_k\Psi_k x_{k-1}\\
&+L_k\Psi_k(\delta_{2,k}\gamma_{4,k}\tilde{\vartheta}_{1,k-1}+\delta_{1,k}\gamma_{5,k}\tilde{\vartheta}_{2,k-1})\\
&-\delta_{2,k}L_k\tau_{1,k}-\delta_{1,k}L_k\tau_{2,k}][\Gamma_{k-1}x_{k-1}\\
&+\Sigma_{k-1}w_{k-1}]\}. \qquad (23)
\end{aligned}
$$

Noting that $\tilde{x}_{k-1}$ is uncorrelated with w_{k-1}, $\tau_{m,k}$ and $\gamma_{n,k}$ $(m=1,2,n=4,5)$, we directly obtain (22) from (23) by utilizing the statistical properties of w_{k-1}, $\tau_{m,k}$ and $\gamma_{n,k}$, which proves the lemma. ■

Lemma 6: The cross-covariance terms $P_{i,k}$ $(i=1,2)$ satisfy the following recursions:

$$
\begin{aligned}
P_{i,k}=&\,(\Gamma_{k-1}-(\bar{\delta}_{i,k}+\bar{\delta}_{j,k})L_k\Psi_k)P_{i,k-1}(\Gamma_{k-1}-\delta_{i,k}K_{i,k}\\
&\times\bar{\Gamma}_{i,k-1})^T+\delta_{i,k}\bar{\gamma}_{3,k}(\Gamma_{k-1}-(\bar{\delta}_{i,k}+\bar{\delta}_{j,k})L_k\Psi_k)\\
&\times P_{j,k-1}\Psi_k^T K_{i,k}^T+\bar{\delta}_{i,k}\bar{\gamma}_{3,k}L_k\Psi_k\Xi_{j,k-1}\Psi_k^T K_{i,k}^T\\
&+\Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T(I-\delta_{i,k}\bar{\gamma}_{i,k}\Psi_k^T K_{i,k}^T)\\
&+\bar{\delta}_{j,k}L_k\Psi_k\Xi_{i,k-1}(\Gamma_{k-1}-\delta_{i,k}K_{i,k}\bar{\Gamma}_{i,k-1})^T\\
&+\bar{\delta}_{i,k}L_k\Psi_k\Xi_{ij,k-1}^T(\Gamma_{k-1}-\delta_{i,k}K_{i,k}\bar{\Gamma}_{i,k-1})^T \qquad (24)
\end{aligned}
$$

where $\Xi_{i,k}$ and $\Xi_{ij,k}$ $(i=1,2,i\neq j)$ are, respectively, given in (15) and (18). Moreover, $P_{1,0}=P_{2,0}=P_0$.

Proof: From (10) and (11), one obtains

$$
\begin{aligned}
P_{1,k}=&\,\mathbb{E}\{[(\Gamma_{k-1}-(\delta_{2,k}\bar{\gamma}_{4,k}+\delta_{1,k}\bar{\gamma}_{5,k})L_k\Psi_k)\tilde{x}_{k-1}\\
&-[\delta_{2,k}(\gamma_{4,k}-\bar{\gamma}_{4,k})+\delta_{1,k}(\gamma_{5,k}-\bar{\gamma}_{5,k})]\\
&\times L_k\Psi_k x_{k-1}+\delta_{2,k}\gamma_{4,k}L_k\Psi_k\tilde{\vartheta}_{1,k-1}\\
&+\delta_{1,k}\gamma_{5,k}L_k\Psi_k\tilde{\vartheta}_{2,k-1}-\delta_{2,k}L_k\tau_{1,k}\\
&-\delta_{1,k}L_k\tau_{2,k}+\Sigma_{k-1}w_{k-1}][(\Gamma_{k-1}-\delta_{1,k}K_{1,k}\\
&\times\bar{\Gamma}_{1,k-1})\tilde{\vartheta}_{1,k-1}-\delta_{1,k}K_{1,k}\xi_{1,k}-\delta_{1,k}(\gamma_{1,k}\\
&-\bar{\gamma}_{1,k})K_{1,k}\Psi_k\Gamma_{k-1}x_{k-1}-\delta_{1,k}(\gamma_{3,k}-\bar{\gamma}_{3,k})\\
&\times K_{1,k}\Psi_k x_{k-1}+\Sigma_{k-1}w_{k-1}-\delta_{1,k}\gamma_{1,k}\\
&\times K_{1,k}\Theta_k v_k+\delta_{1,k}\gamma_{3,k}K_{1,k}\Psi_k\tilde{\vartheta}_{2,k-1}\\
&-\delta_{1,k}\gamma_{1,k}K_{1,k}\Psi_k\Sigma_{k-1}w_{k-1}]\}. \qquad (25)
\end{aligned}
$$

Noting the statistical properties of $\xi_{1,k}$ and v_k again, we easily know that $\tilde{x}_{k-1}$ is also uncorrelated with $\xi_{1,k}$ and v_k. Substituting (16) and (17) into (25), $P_{1,k}$ is obtained immediately. Similarly, the term $P_{2,k}$ can also be computed as (24). ■

Lemma 7: The filtering error covariance P_k is determined by the following recursion:

$$
\begin{aligned}
P_k=&\,(\Gamma_{k-1}-(\bar{\delta}_{1,k}+\bar{\delta}_{2,k})L_k\Psi_k)P_{k-1}(\Gamma_{k-1}\\
&-(\bar{\delta}_{1,k}+\bar{\delta}_{2,k})L_k\Psi_k)^T+L_k\Psi_k\mathfrak{R}_k\Psi_k^T L_k^T\\
&+\bar{\delta}_{2,k}(\Gamma_{k-1}P_{1,k-1}\Psi_k^T L_k^T+L_k\Psi_k P_{1,k-1}^T\Gamma_{k-1}^T)\\
&+\bar{\delta}_{1,k}(\Gamma_{k-1}P_{2,k-1}\Psi_k^T L_k^T+L_k\Psi_k P_{2,k-1}^T\Gamma_{k-1}^T)\\
&+\Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T+L_k(\delta_{2,k}O_{1,k}+\delta_{1,k}O_{2,k})L_k^T \qquad (26)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathfrak{R}_k=&\,(\delta_{2,k}\vec{\gamma}_{4,k}+\delta_{1,k}\vec{\gamma}_{5,k})\Pi_{k-1}-\delta_{2,k}\bar{\gamma}_{4,k}^2(P_{1,k-1}\\
&+P_{1,k-1}^T)-\delta_{1,k}\bar{\gamma}_{5,k}^2(P_{2,k-1}+P_{2,k-1}^T)\\
&+\delta_{2,k}(\vec{\gamma}_{4,k}+\bar{\gamma}_{4,k}^2)\Xi_{1,k-1}+\delta_{1,k}(\vec{\gamma}_{5,k}+\bar{\gamma}_{5,k}^2)\\
&\times\Xi_{2,k-1}-\delta_{2,k}\vec{\gamma}_{4,k}(\Xi_{x1,k-1}+\Xi_{x1,k-1}^T)\\
&-\delta_{1,k}\vec{\gamma}_{5,k}(\Xi_{x2,k-1}+\Xi_{x2,k-1}^T).
\end{aligned}
$$

Proof: Noting (11), (16) and (17) again, we have

$$\begin{aligned}
P_k = {} & (\Gamma_{k-1}-(\bar{\delta}_{1,k}+\bar{\delta}_{2,k})L_k\Psi_k)P_{k-1}(\Gamma_{k-1} \\
& -(\bar{\delta}_{1,k}+\bar{\delta}_{2,k})L_k\Psi_k)^T+\Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T \\
& +\bar{\delta}_{2,k}(\Gamma_{k-1}-(\bar{\delta}_{1,k}+\bar{\delta}_{2,k})L_k\Psi_k)^T P_{1,k-1} \\
& \times\Psi_k^T L_k^T+\bar{\delta}_{2,k}L_k\Psi_k P_{1,k-1}^T(\Gamma_{k-1}-(\bar{\delta}_{1,k} \\
& +\bar{\delta}_{2,k})L_k\Psi_k)^T+\bar{\delta}_{1,k}L_k\Psi_k P_{2,k-1}^T(\Gamma_{k-1} \\
& -(\bar{\delta}_{1,k}+\bar{\delta}_{2,k})L_k\Psi_k)^T+\bar{\delta}_{1,k}(\Gamma_{k-1} \\
& -(\bar{\delta}_{1,k}+\bar{\delta}_{2,k})L_k\Psi_k)P_{2,k-1}\Psi_k^T L_k^T \\
& +(\delta_{2,k}\vec{\gamma}_{4,k}+\delta_{1,k}\vec{\gamma}_{5,k})L_k\Psi_k\Pi_{k-1}\Psi_k^T L_k^T \\
& -\delta_{2,k}\vec{\gamma}_{4,k}L_k\Psi_k(\Xi_{x1,k-1}+\Xi_{x1,k-1}^T)\Psi_k^T L_k^T \\
& -\delta_{1,k}\vec{\gamma}_{5,k}L_k\Psi_k(\Xi_{x2,k-1}+\Xi_{x2,k-1}^T)\Psi_k^T L_k^T \\
& +\delta_{2,k}(\vec{\gamma}_{4,k}+\bar{\gamma}_{4,k}^2)L_k\Psi_k\Xi_{1,k-1}\Psi_k^T L_k^T \\
& +\delta_{1,k}(\vec{\gamma}_{5,k}+\bar{\gamma}_{5,k}^2)L_k\Psi_k\Xi_{2,k-1}\Psi_k^T L_k^T \\
& +L_k(\delta_{2,k}O_{1,k}+\delta_{1,k}O_{2,k})L_k^T. \qquad (27)
\end{aligned}$$

Since $\delta_{1,k}\delta_{2,k}=0$, one easily has

$$\begin{aligned}
\bar{\delta}_{2,k}(\bar{\delta}_{1,k}+\bar{\delta}_{2,k}) &= \delta_{2,k}\bar{\gamma}_{4,k}^2 \\
\bar{\delta}_{1,k}(\bar{\delta}_{1,k}+\bar{\delta}_{2,k}) &= \delta_{1,k}\bar{\gamma}_{5,k}^2. \qquad (28)
\end{aligned}$$

Furthermore, (26) is immediately obtained from (27) and (28). ■

So far, the filtering error covariances of the FFSR and the remote filter are recursively calculated through (14) and (26), respectively. In the following theorems, the desired filter gain matrices are simultaneously designed to minimize the filtering error covariances $\Xi_{i,k}$ $(i=1,2)$ and P_k.

Theorem 1: If the filter gain $K_{i,k}$ $(i=1,2)$ is selected as

$$K_{i,k}=\begin{cases}(\Gamma_{k-1}(\Xi_{i,k-1}\bar{\Gamma}_{i,k-1}^T-\bar{\gamma}_{3,k}\Xi_{ij,k-1}\Psi_k^T) \\ \quad +\bar{\gamma}_{i,k}\Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T\Psi_k^T) \\ \quad \times(\bar{\Gamma}_{i,k-1}\Xi_{i,k-1}\bar{\Gamma}_{i,k-1}^T+\aleph_{i,k})^{-1}, & \delta_{i,k}=1 \\ 0, & \delta_{i,k}=0\end{cases} \qquad (29)$$

then the filtering error covariance $\Xi_{i,k}$ is minimized at each time instant.

Proof: According to [40], the partial derivative of the trace of $\Xi_{i,k}$ satisfies

$$\begin{aligned}
\frac{\partial \mathrm{tr}\{\Xi_{i,k}\}}{\partial K_{i,k}} = {} & -2\delta_{i,k}(\Gamma_{k-1}-\delta_{i,k}K_{i,k}\bar{\Gamma}_{i,k-1})\Xi_{i,k-1}\bar{\Gamma}_{i,k-1}^T \\
& +2\delta_{i,k}\bar{\gamma}_{3,k}\Gamma_{k-1}\Xi_{ij,k-1}\Psi_k^T+2\delta_{i,k}K_{i,k}\aleph_{i,k} \\
& -2\delta_{i,k}\bar{\gamma}_{i,k}\Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T\Psi_k^T. \qquad (30)
\end{aligned}$$

When $\delta_{i,k}=0$, it is seen from (30) that $\frac{\partial \mathrm{tr}\{\Xi_{i,k}\}}{\partial K_{i,k}}$ is identically equal to zero. So, the filter gain $K_{i,k}$ can be chosen arbitrarily. To reduce computational burden of the filtering scheme, we set $K_{i,k}=0$ for all $\delta_{i,k}=0$.

If $\delta_{i,k}=1$, (30) can be written as

$$\begin{aligned}
\frac{\partial \mathrm{tr}\{\Xi_{i,k}\}}{\partial K_{i,k}} = {} & -2(\Gamma_{k-1}-K_{i,k}\bar{\Gamma}_{i,k-1})\Xi_{i,k-1}\bar{\Gamma}_{i,k-1}^T \\
& +2\bar{\gamma}_{3,k}\Gamma_{k-1}\Xi_{ij,k-1}\Psi_k^T+2K_{i,k}\aleph_{i,k} \\
& -2\bar{\gamma}_{i,k}\Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T\Psi_k^T.
\end{aligned}$$

Letting $\frac{\partial \mathrm{tr}\{\Xi_{i,k}\}}{\partial K_{i,k}}=0$, we find that

$$\begin{aligned}
K_{i,k} = {} & (\bar{\gamma}_{i,k}\Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T\Psi_k^T-\bar{\gamma}_{3,k}\Gamma_{k-1}\Xi_{ij,k-1}\Psi_k^T \\
& +\Gamma_{k-1}\Xi_{i,k-1}\bar{\Gamma}_{i,k-1}^T)(\bar{\Gamma}_{i,k-1}\Xi_{i,k-1}\bar{\Gamma}_{i,k-1}^T+\aleph_{i,k})^{-1}.
\end{aligned}$$ ■

Theorem 2: If L_k is chosen as

$$L_k=\mathcal{P}_kO_k^{-1} \qquad (31)$$

where

$$\begin{aligned}
O_k = {} & (\bar{\delta}_{1,k}+\bar{\delta}_{2,k})^2\Psi_kP_{k-1}\Psi_k^T+\Psi_k\Re_k\Psi_k^T \\
& +\delta_{2,k}O_{1,k}+\delta_{1,k}O_{2,k} \\
\mathcal{P}_k = {} & (\bar{\delta}_{1,k}+\bar{\delta}_{2,k})\Gamma_{k-1}P_{k-1}\Psi_k^T-\Gamma_{k-1}(\bar{\delta}_{2,k}P_{1,k-1} \\
& +\bar{\delta}_{1,k}P_{2,k-1})\Psi_k^T
\end{aligned}$$

then the filtering error covariance P_k is locally minimized for all $k\in\mathbf{N}^+$.

Proof: Similarly to the proof of Theorem 1, taking the partial derivative of the trace of P_k with respect to L_k and further letting the partial derivative be zero, we have

$$\begin{aligned}
\frac{\partial \mathrm{tr}\{P_k\}}{\partial L_k} = {} & -2(\Gamma_{k-1}-(\bar{\delta}_{1,k}+\bar{\delta}_{2,k})L_k\Psi_k)P_{k-1} \\
& \times(\bar{\delta}_{1,k}+\bar{\delta}_{2,k})\Psi_k^T+2L_k\Psi_k\Re_k\Psi_k^T \\
& +2\Gamma_{k-1}(\bar{\delta}_{2,k}P_{1,k-1}+\bar{\delta}_{1,k}P_{2,k-1})\Psi_k^T \\
& +2L_k(\delta_{2,k}O_{1,k}+\delta_{1,k}O_{2,k}) \\
= {} & 2L_kO_k-2\mathcal{P}_k=0. \qquad (32)
\end{aligned}$$

Obviously, when $L_k=\mathcal{P}_kO_k^{-1}$, the filtering error covariance achieves its minimal value. ■

By now, the filter error covariances of FFSR and the remote filter have been derived and further minimized by quantitatively devising the filter gain matrices. The proposed filtering method is summarized in Algorithm 1.

Algorithm 1 Recursive filtering algorithm

Step 1: Set $k=0$ and the time horizon $[0,N]$. Give the initial values $\bar{x}_0$ and P_0. Then, obtain the matrix Π_0.

Step 2: Compute the cross-covariance matrices $\Xi_{ij,k}$, $\Xi_{xi,k}$, $P_{i,k}$ and $P_{x,k}$ via (18), (21), (22) and (24), recursively.

Step 3: Obtain filter gain matrices $K_{i,k}$ and L_k from (29) and (31). Furthermore, generate the optimal state estimates through filters (6) and (9).

Step 4: Compute matrices Π_{k+1}, $\Xi_{i,k+1}$ and P_{k+1} from recursions (1), (14) and (26), respectively.

Step 5: Set $k=k+1$. If $k<N$, return to Step 2, else go to Step 6.

Step 6: Stop.

Remark 4: In Algorithm 1, the filtering error covariances $\Xi_{i,k}$ and P_k are derived via recursions (14) and (26), which are unrelated to the measurement output y_k. As a result, the filter gain matrices $K_{i,k}$ and L_k can be calculated in advance. In this case, the developed filtering algorithm is capable of online computation and easy-to-implement.

B. Stability Analysis

In this section, the boundedness stability is strictly analyzed for the proposed algorithms.

Assumption 1: There exist positive scalars $\bar{\tau}$, $\bar{w}$, $\bar{f}$, $\bar{\mu}$, $\bar{\varphi}$, $\bar{g}$, $\bar{p}_0$, $\underline{\tau}$, $\underline{w}$, $\underline{f}$, $\underline{\mu}$, $\underline{\varphi}$, $\underline{\varrho}$, $\underline{g}$, $\underline{p}_0$ and $\underline{v}$ such that

$$\underline{\tau}I \le \Gamma_k\Gamma_k^T \le \bar{\tau}I,\ \underline{f}I \le \Psi_k\Psi_k^T \le \bar{f}I,\ \underline{v}I \le \Theta_k V_k \Theta_k^T$$
$$\underline{w}I \le \Sigma_k W_k \Sigma_k^T \le \bar{w}I,\ \underline{p}_0 I \le P_0 \le \bar{p}_0 I,\ \underline{\varrho}I \le O_{i,k}O_{i,k}^T$$
$$\underline{\mu} \le |\bar{\gamma}_{m,k}| \le \bar{\mu}, \underline{g} \le G_{i,k} \le \bar{g},\ \underline{\varphi} \le \vec{\gamma}_{m,k} \le \bar{\varphi}$$

for all $i = 1,2$ and $m = 1,2,3,4,5$.

First, we will analyze the boundedness of the covariance $\Xi_{i,k}$ $(i = 1,2)$.

Theorem 3: Under the Assumption 1, if there exists a positive scalar $\bar{m}_k$ determined as follows:

$$\bar{m}_k = \bar{\tau}\bar{m}_{k-1} + \bar{w} \tag{33}$$

then we have $\Xi_{i,k} \le \bar{m}_k I$. Moreover, if $\tilde{n} - \frac{2\bar{\mu}^2\bar{f}(2\bar{\tau}+3)\bar{\mu}^2}{\underline{\varphi}}\bar{m}_{k-1} > 0$, one obtains that $\Xi_{i,k} \ge \underline{m}_k I$, where

$$\begin{aligned}
&\underline{m}_k = \rho_{k-1}\underline{m}_{k-1} + \tilde{w}\\
&\rho_{k-1} = \frac{\tilde{n} - 2\bar{\mu}^2\bar{f}(2\bar{\tau}+3)\frac{\bar{\mu}^2}{\underline{\varphi}}\bar{m}_{k-1}}{\tilde{n} + 2\bar{\mu}^2\bar{f}(2\bar{\tau}+3)\bar{m}_{k-1}}\underline{\tau}\\
&\tilde{n} = (\underline{\mu}^2 + \underline{\varphi})(\underline{f}\,\underline{w} + \underline{v}) + \underline{g}\\
&\tilde{w} = \varrho\bar{w},\ 0 < \varrho < 1\\
&\bar{m}_0 = \bar{p}_0,\ \underline{m}_0 = \underline{p}_0.
\end{aligned} \tag{34}$$

Proof: When the filter gain matrix $K_{i,k}$ is chosen as (29), the covariance matrix $\Xi_{i,k}$ is written as follows:

$$\begin{aligned}
\Xi_{i,k} = {} & \Gamma_{k-1}\Xi_{i,k-1}\Gamma_{k-1}^T + \Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T\\
& - \delta_{i,k}(\Gamma_{k-1}(\Xi_{i,k-1}\bar{\Gamma}_{i,k-1}^T - \bar{\gamma}_{3,k}\Xi_{ij,k-1}\Psi_k^T)\\
& + \bar{\gamma}_{i,k}\Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T\Psi_k^T)(\bar{\Gamma}_{i,k-1}\Xi_{i,k-1}\bar{\Gamma}_{i,k-1}^T\\
& + \aleph_{i,k})^{-1}(\Gamma_{k-1}(\Xi_{i,k-1}\bar{\Gamma}_{i,k-1}^T - \bar{\gamma}_{3,k}\Xi_{ij,k-1}\Psi_k^T)\\
& + \bar{\gamma}_{i,k}\Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T\Psi_k^T)^T.
\end{aligned} \tag{35}$$

For the term $\bar{\Gamma}_{i,k-1}\Xi_{i,k-1}\bar{\Gamma}_{i,k-1}^T + \aleph_{i,k}$, it is easy to get from the definition of $\aleph_{i,k}$ that

$$\bar{\Gamma}_{i,k-1}\Xi_{i,k-1}\bar{\Gamma}_{i,k-1}^T + \aleph_{i,k} \ge G_{i,k} > 0. \tag{36}$$

Since $\delta_{i,k} \in \{0,1\}$, it is immediately derived from (35) and (36) that

$$\Xi_{i,k} \le \Gamma_{k-1}\Xi_{i,k-1}\Gamma_{k-1}^T + \Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T. \tag{37}$$

From [41], we further conclude that $\Xi_{i,k} \le \bar{m}_k I$.

Next, we utilize the mathematical induction method to show that $\Xi_{i,k} \ge \underline{m}_k I$. First, it is easily obtained from Assumption 1 and $\Xi_{i,0} = P_0$ that $\Xi_{i,0} \ge \underline{m}_0 I$. Assuming that $\Xi_{i,k-1} \ge \underline{m}_{k-1} I$, we will demonstrate that $\Xi_{i,k} \ge \underline{m}_k I$ for all $l > 0$.

Since $\Xi_{i,k-1} \ge \underline{m}_{k-1} I$, it is known that $\Xi_{i,k-1}$ is invertible. Letting $\tilde{\Gamma}_{i,k-1} = \bar{\Gamma}_{i,k-1} - \bar{\gamma}_{3,k}\Psi_k\Xi_{ij,k-1}^T\Xi_{i,k-1}^{-1}$ and $\varsigma < \frac{\underline{\varphi}}{\bar{\mu}^2}$, we deduce from (35) and (36) that

$$\begin{aligned}
\Xi_{i,k} \ge {} & (1+\varsigma^{-1})\Gamma_{k-1}[\Xi_{i,k-1} - \Xi_{i,k-1}\tilde{\Gamma}_{i,k-1}^T(\tilde{\Gamma}_{i,k-1}\\
& \times \Xi_{i,k-1}\tilde{\Gamma}_{i,k-1}^T + \tilde{\aleph}_{i,k})^{-1}\tilde{\Gamma}_{i,k-1}\Xi_{i,k-1}]\Gamma_{k-1}^T\\
& - \Gamma_{k-1}\Xi_{i,k-1}\Gamma_{k-1}^T + \Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T\\
& - (1+\varsigma)\bar{\gamma}_{i,k}^2\Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T\Psi_k^T\\
& \times (\bar{\Gamma}_{i,k-1}\Xi_{i,k-1}\bar{\Gamma}_{i,k-1}^T + \aleph_{i,k})^{-1}\Psi_k\\
& \times \Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T
\end{aligned} \tag{38}$$

where

$$\begin{aligned}
\tilde{\aleph}_{i,k} = {} & (\vec{\gamma}_{i,k} + \bar{\gamma}_{i,k}^2)\Psi_k\Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T\Psi_k^T + G_{i,k}\\
& + \vec{\gamma}_{3,k}\Psi_k\Pi_{k-1}\Psi_k^T + \vec{\gamma}_{i,k}\Psi_k\Gamma_{k-1}\Pi_{k-1}\Gamma_{k-1}^T\Psi_k^T\\
& + (\vec{\gamma}_{3,k} + \bar{\gamma}_{3,k}^2)\Psi_k\Xi_{j,k-1}\Psi_k^T + (\vec{\gamma}_{i,k} + \bar{\gamma}_{i,k}^2)\Theta_k V_k\Theta_k^T\\
& - \vec{\gamma}_{3,k}\Psi_k(\Xi_{xj,k-1} + \Xi_{xj,k-1}^T)\Psi_k^T - \bar{\gamma}_{3,k}^2\Psi_k\Xi_{ij,k-1}^T\\
& \times \Xi_{i,k-1}^{-1}\Xi_{ij,k-1}\Psi_k^T.
\end{aligned} \tag{39}$$

From the definitions of $\Xi_{xi,k-1}$ and $\Xi_{ij,k-1}$, it is obvious that

$$\mathbb{E}\left\{\begin{bmatrix}\tilde{\vartheta}_{i,k}\\ \tilde{\vartheta}_{j,k}\end{bmatrix}\begin{bmatrix}\tilde{\vartheta}_{i,k} & \tilde{\vartheta}_{j,k}\end{bmatrix}\right\} = \begin{bmatrix}\Xi_{i,k-1} & \Xi_{ij,k-1}\\ \Xi_{ij,k-1}^T & \Xi_{j,k-1}\end{bmatrix} \ge 0, \tag{40}$$

which implies that

$$\Xi_{j,k-1} - \Xi_{ij,k-1}^T\Xi_{i,k-1}^{-1}\Xi_{ij,k-1} \ge 0 \tag{41}$$

and

$$\Xi_{i,k-1}^{-1}\Xi_{ij,k-1}\Xi_{ij,k-1}^T\Xi_{i,k-1}^{-1} \le \bar{m}_{k-1}\Xi_{i,k-1}^{-1}. \tag{42}$$

Moreover, it is not difficult to know that

$$\Xi_{xj,k-1} + \Xi_{xj,k-1}^T \le \Pi_{k-1} + \Xi_{j,k-1}. \tag{43}$$

Substituting (40)−(43) into (39), we immediately have $\tilde{\aleph}_{i,k} \ge G_{i,k} > 0$, which implies that $\tilde{\aleph}_{i,k}$ is invertible as well.

Meanwhile, we have

$$\begin{aligned}
& \Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T - (1+\varsigma)\bar{\gamma}_{i,k}^2\Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T\Psi_k^T\\
& \quad \times (\bar{\Gamma}_{i,k-1}\Xi_{i,k-1}\bar{\Gamma}_{i,k-1}^T + \aleph_{i,k})^{-1}\Psi_k\Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T\\
& \quad \ge (1 - (1+\varsigma)\frac{\bar{\gamma}_{i,k}^2}{\vec{\gamma}_{i,k} + \bar{\gamma}_{i,k}^2})\Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T.
\end{aligned} \tag{44}$$

Since $\varsigma < \underline{\varphi}/\bar{\mu}^2$, we can find a scalar $0 < \varrho < 1$ such that

$$\begin{aligned}
& (1 - (1+\varsigma)\frac{\bar{\gamma}_{i,k}^2}{\vec{\gamma}_{i,k} + \bar{\gamma}_{i,k}^2})\Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T\\
& \quad \ge \varrho\Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T.
\end{aligned} \tag{45}$$

According to the matrix inversion lemma, it is further obtained from (38), (44) and (45) that

$$\begin{aligned}
\Xi_{i,k} \ge {} & (1 + \frac{\bar{\mu}^2}{\underline{\varphi}})\Gamma_{k-1}(\Xi_{i,k-1}^{-1} + \tilde{\Gamma}_{i,k-1}^T\tilde{\aleph}_{i,k}^{-1}\tilde{\Gamma}_{i,k-1})^{-1}\Gamma_{k-1}^T\\
& - \frac{\bar{\mu}^2}{\underline{\varphi}}\Gamma_{k-1}\Xi_{i,k-1}\Gamma_{k-1}^T + \varrho\Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T.
\end{aligned} \tag{46}$$

Recurring to (39), (41) and (43), we easily get that $\tilde{\aleph}_{i,k}^{-1} \leq \frac{1}{\tilde{n}} I$. Moreover, the term $\tilde{\Gamma}_{i,k-1}^{T}\tilde{\Gamma}_{i,k-1}$ satisfies that

$$\begin{aligned}\tilde{\Gamma}_{i,k-1}^{T}\tilde{\Gamma}_{i,k-1} &\leq 2(\bar{\Gamma}_{i,k-1}^{T}\bar{\Gamma}_{i,k-1} + \bar{\gamma}_{3,k}^{2}\Xi_{i,k-1}^{-1}\Xi_{ij,k-1} \\ &\quad \times \Psi_k^T\Psi_k\Xi_{ij,k-1}^{T}\Xi_{i,k-1}^{-1}) \\ &\leq 4\bar{\mu}^2(\bar{f}\bar{\tau} + \bar{f})I + 2\bar{\mu}^2\bar{f}\bar{m}_{k-1}\Xi_{i,k-1}^{-1}. \end{aligned} \tag{47}$$

Thus, it is obtained from (46) and (47) that

$$\Xi_{i,k} \geq \underline{m}_k I. \tag{48}$$

Consequently, we obtain that $\underline{m}_k I \leq \Xi_{i,k} \leq \bar{m}_k I$. ∎

Furthermore, the uniform upper bound and lower bound are provided for $\Xi_{i,k}$ in the following lemma.

Lemma 8: If $\bar{\tau} < 1$, the filtering error covariance $\Xi_{i,k}$ satisfies $\Xi_{i,k} \leq \bar{m}I$ with $\bar{m} = \bar{\tau}\bar{p}_0 + \frac{\bar{w}}{1-\bar{\tau}}$. Moreover, if $\tilde{n} - \frac{2\bar{\mu}^2\bar{f}(2\bar{\tau}+3)\bar{\mu}^2}{\varphi} \times \bar{m} > 0$, there exists a uniformly lower bound on $\Xi_{i,k}$, i.e.,

$$\Xi_{i,k} \geq \underline{m}I \tag{49}$$

where

$$\underline{m} = \underline{\rho}\underline{p}_0 + \frac{\tilde{w}}{1-\underline{\rho}}$$

$$\underline{\rho} = \frac{\tilde{n} - 2\bar{\mu}^2\bar{f}(2\bar{\tau}+3)\bar{m}\frac{\bar{\mu}^2}{\varphi}}{\tilde{n} + 2\bar{\mu}^2\bar{f}(2\bar{\tau}+3)\bar{m}}\underline{\tau}.$$

Proof: By (33) and (34), one has

$$\begin{aligned}\bar{m}_k &= \bar{\tau}\bar{m}_{k-1} + \bar{w} \\ &= \bar{\tau}^2\bar{m}_{k-2} + \bar{\tau}\bar{v} + \bar{w} \\ &\;\;\vdots \\ &= \bar{\tau}^k\bar{m}_0 + \sum_{i=0}^{k-1}\bar{\tau}^i\bar{w}. \end{aligned} \tag{50}$$

Since $\bar{\tau} < 1$, it is further yielded from (50) that $\Xi_{i,k} \leq \bar{m}I$.

On the other hand, noting that $\bar{m}_k \leq \bar{m}$ and $\underline{\rho} < \bar{\tau} < 1$, we know that $\underline{\rho} \leq \rho_{k-1} < 1$. Hence, it is deduced from (33) that

$$\underline{m}_k \geq \underline{\rho}^k\underline{m}_0 + \sum_{i=0}^{k-1}\underline{\rho}^i\tilde{w} \geq \underline{m}. \tag{51}$$

∎

Next, the upper bound and lower bound are both provided for the filtering error covariance P_k.

Theorem 4: Under Assumption 1, if there is a sequence $\underline{p}_k$ $(k = 0, 1, 2, \ldots)$ satisfying

$$\underline{p}_k = \alpha_{k-1}\underline{p}_{k-1} + \underline{w} \tag{52}$$

where

$$\alpha_{k-1} = \frac{\tau o}{\underline{o} + 4\bar{\mu}^2\bar{f}\bar{m}_{k-1}} \tag{53}$$

then the filtering error covariance P_k is bounded by $\bar{m}_k$ and $\underline{p}_k$, i.e.,

$$\underline{p}_k I \leq P_k \leq \bar{m}_k I. \tag{54}$$

Proof: The mathematical induction method is employed to prove the theorem.

From the initial values, we directly obtain that $\underline{p}_0 I \leq P_0 \leq \bar{m}_0 I$. Then, it is assumed that $\underline{p}_{k-1} I \leq P_{k-1} \leq \bar{m}_{k-1} I$. we aim to show $\underline{p}_k I \leq P_{k-1} \leq \bar{m}_k I$.

When the filter gain matrix L_k is given as (31), the filtering error covariance is further derived as

$$\begin{aligned}P_k &= \Gamma_{k-1}P_{k-1}\Gamma_{k-1}^T + \Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T - \mathcal{P}_k O_k^{-1}\mathcal{P}_k^T \\ &\leq \Gamma_{k-1}P_{k-1}\Gamma_{k-1}^T + \Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T. \end{aligned} \tag{55}$$

Thus, it is directly obtained that

$$P_k \leq \bar{m}_k I. \tag{56}$$

Letting $\tilde{\Psi}_k = (\bar{\delta}_{1,k} + \bar{\delta}_{2,k})\Psi_k - \Psi_k(\bar{\delta}_{2,k}P_{1,k-1} + \bar{\delta}_{1,k}P_{2,k-1})^T \times P_{k-1}^{-1}$, we obtain from (55) that

$$\begin{aligned}P_k &= \Gamma_{k-1}[P_{k-1} - P_{k-1}\tilde{\Psi}_k^T(\tilde{\Psi}_k P_{k-1}\tilde{\Psi}_k^T + \tilde{\mathfrak{R}}_k)^{-1} \\ &\quad \times \tilde{\Psi}_k P_{k-1}]\Gamma_{k-1}^T + \Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T \end{aligned} \tag{57}$$

where

$$\begin{aligned}\tilde{\mathfrak{R}}_k &= (\delta_{2,k}\vec{\gamma}_{4,k} + \delta_{1,k}\vec{\gamma}_{5,k})\Psi_k\Pi_{k-1}\Psi_k^T + \delta_{2,k}(\vec{\gamma}_{4,k} + \bar{\gamma}_{4,k}^2) \\ &\quad \times \Psi_k\Xi_{1,k-1}\Psi_k^T + \delta_{1,k}(\vec{\gamma}_{5,k} + \bar{\gamma}_{5,k}^2)\Psi_k\Xi_{2,k-1}\Psi_k^T \\ &\quad - \delta_{2,k}\vec{\gamma}_{4,k}\Psi_k(\Xi_{x1,k-1} + \Xi_{x1,k-1}^T)\Psi_k^T \\ &\quad - \delta_{1,k}\vec{\gamma}_{5,k}\Psi_k(\Xi_{x2,k-1} + \Xi_{x2,k-1}^T)\Psi_k^T \\ &\quad - \delta_{2,k}\bar{\gamma}_{4,k}^2\Psi_k P_{1,k-1}^T P_{k-1}^{-1}P_{1,k-1}\Psi_k^T \\ &\quad - \delta_{1,k}\bar{\gamma}_{5,k}^2\Psi_k P_{2,k-1}^T P_{k-1}^{-1}P_{2,k-1}\Psi_k^T \\ &\quad + \delta_{2,k}O_{1,k} + \delta_{1,k}O_{2,k}. \end{aligned} \tag{58}$$

Following the same methodology in (41)–(43), we further have:

$$\begin{aligned}\Xi_{x1,k-1} + \Xi_{x1,k-1}^T &\leq \Pi_{k-1} + \Xi_{1,k-1} \\ \Xi_{x2,k-1} + \Xi_{x2,k-1}^T &\leq \Pi_{k-1} + \Xi_{2,k-1} \end{aligned} \tag{59}$$

and

$$\begin{aligned}\Xi_{1,k-1} - P_{1,k-1}^T P_{k-1}^{-1}P_{1,k-1} &\geq 0, \\ \Xi_{2,k-1} - P_{2,k-1}^T P_{k-1}^{-1}P_{2,k-1} &\geq 0. \end{aligned} \tag{60}$$

Thus, it is known from (58)–(60) that $\tilde{\mathfrak{R}}_k > 0$ and therefore is invertible.

Utilizing the matrix inversion lemma again, one easily gets that

$$\begin{aligned}P_k &= \Gamma_{k-1}(P_{k-1}^{-1} + \tilde{\Psi}_k^T\tilde{\mathfrak{R}}_k^{-1}\tilde{\Psi}_k)^{-1}\Gamma_{k-1}^T \\ &\quad + \Sigma_{k-1}W_{k-1}\Sigma_{k-1}^T. \end{aligned} \tag{61}$$

Since $\delta_{1,k}\delta_{2,k} = 0$, we have $\tilde{\mathfrak{R}}_k > \delta_{2,k}O_{1,k} + \delta_{1,k}O_{2,k} \geq \underline{o}$. Furthermore, the term $\tilde{\Psi}_k^T\tilde{\Psi}_k$ satisfies

$$\tilde{\Psi}_k^T\tilde{\Psi}_k \leq 2\bar{\mu}^2\bar{f}I + 2\bar{\mu}^2\bar{f}\bar{p}_{k-1}P_{k-1}^{-1}. \tag{62}$$

In this case, it is further derived that

$$P_k \geq \Big(\frac{\tau o}{\underline{o} + 4\bar{\mu}^2\bar{f}\bar{m}_{k-1}}\underline{p}_{k-1} + \underline{w}\Big)I = \underline{p}_k I. \tag{63}$$

Synthesizing (56) and (63), (54) is obtained immediately. ∎

Lemma 9: Give a scalar $\underline{p}$ as follows:

$$\underline{p} = \underline{\alpha}\underline{p}_0 + \frac{w}{1-\underline{\alpha}}$$

with $\underline{\alpha} = \frac{\tau o}{o+4\bar{\mu}^2\bar{f}\bar{m}}$. If $\bar{\tau} < 1$, then we have

$$\underline{p}I \leq P_k \leq \bar{m}I. \quad (64)$$

Proof: The proof of the lemma is directly obtained by following the same methodology in Lemma 8. The details are omitted here. ■

In this part, the uniform bounds have been provided for the filtering error covariances $\Xi_{i,k}$ and P_k. In what follows, a three-order RLC circuit system will be adopted to display the effectiveness of the proposed filtering method.

IV. Illustrative Example

In this section, the effectiveness of the proposed filtering method is demonstrated by a three-order resistance-inductance-capacitance (RLC) circuit system.

The considered RLC circuit system consists of one voltage source ($V(t)$), one inductor (L), two capacitors (C_1 and C_2) and two resistances (T_1 and T_2). According to [42], the dynamics of the three-order RLC circuit system is given as

$$C_1\dot{V}_{C_1}(t) = -\frac{1}{R_2}(V_{C_1}(t) - V_{C_2}(t)) + \frac{1}{R_1}u(t)$$

$$C_2\dot{V}_{C_2}(t) = \frac{1}{R_2}(V_{C_1}(t) - V_{C_2}(t)) + i_L(t)$$

$$L\dot{i}_L(t) = V_{C_2}(t)$$

where $V_{C_1}(t)$ and $V_{C_2}(t)$ are, respectively, the capacitance voltages of resistances T_1 and T_2, and $i_L(t)$ is the inductor's current.

Selecting $V_{C_1}(t)$, $V_{C_2}(t)$ and $i_L(t)$ as the state variable, we further have

$$x(t) = Ax(t) + Bu(t) \quad (65)$$

where

$$x(t) = \begin{bmatrix} V_{C_1}(t) \\ V_{C_2}(t) \\ i_L(t) \end{bmatrix},\ A = \begin{bmatrix} -\frac{1}{R_2C_1} & \frac{1}{R_2C_1} & 0 \\ \frac{1}{R_2C_2} & -\frac{1}{R_2C_2} & -\frac{1}{C_2} \\ 0 & \frac{1}{L} & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{1}{R_1C_1} & 0 & 0 \end{bmatrix}^T.$$

Moreover, the control input $u(t)$ is designed as $u(t) = Kx(t)$.

Discretizing (65) and further considering the effects of stochastic noises, we obtain the following discrete-time stochastic system

$$x_{k+1} = \Gamma_k x_k + \Sigma_k w_k$$

where $\Gamma_k = e^{(A+BK)q}$ and q is the sampling period.

Letting $R_1 = 4\,\Omega$, $R_2 = 2\,\Omega$, $C_1 = C_2 = 1.2$ F, $L = 0.5$ H, $q = 0.6$ s and $K = [-1.5\ 1.638\ 1.810]$, we have

$$\Gamma_k = \begin{bmatrix} 0.6858 & 0.3364 & 0.0935 \\ 0.1790 & 0.7960 & -0.4229 \\ 0.0120 & 0.1069 & 0.9732 \end{bmatrix}.$$

Furthermore, the other parameters are given as follows:

$$\Sigma_k = [0.6\quad 0.5\quad 0.3]^T,\ \Psi_k = [2\quad 1.8\quad 2.5]$$

$$W_k = V_k = G_{i,k} = O_{i,k} = 4\times10^{-6},\ i = 1,2$$

$$\bar{\gamma}_{1,k} = \bar{\gamma}_{3,k} = \bar{\gamma}_{4,k} = 0.9,\ \bar{\gamma}_{2,k} = \bar{\gamma}_{5,k} = 0.92$$

$$\Theta_k = 0.6,\ \vec{\gamma}_{m,k} = 0.01,\ m = 1,2,3,4,5$$

$$\bar{x}_0 = [0.3\quad 0.2\quad -0.2]^T,\ P_0 = \mathrm{diag}\{0.05, 0.02, 0.03\}.$$

By utilizing the aforementioned parameters, the gain matrices can be derived from Theorems 1 and 2. Based on the proposed recursive filtering, the simulation results are presented in Figs. 2−5. Figs. 2−4 show the real state components x_k^i ($i = 1,2,3$) and their estimates in R_1, R_2 and the remote filter respectively. Comparison of the real states and the estimates in Figs. 2−4 demonstrates that the designed filter is competent in estimating the states. The trace of the filtering error covariance is depicted in Fig 5. It is concluded from Figs. 2−5 that, the system state is accurately estimated with a desired performance by employing the developed filtering method.

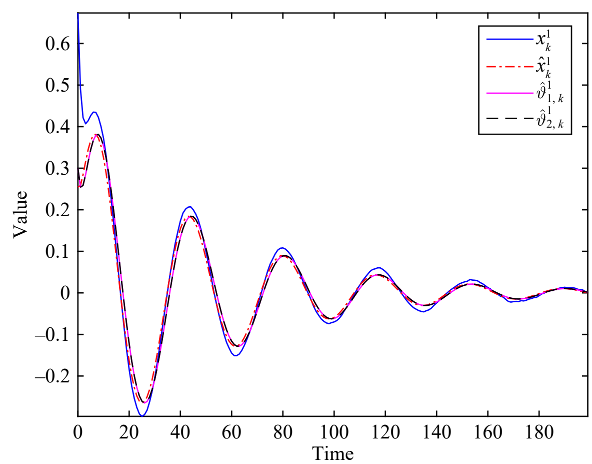


Fig. 2. x_k^1 and the estimates.

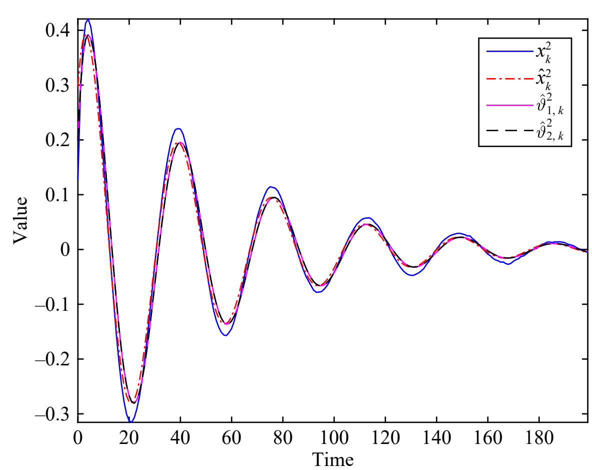


Fig. 3. x_k^2 and the estimates.

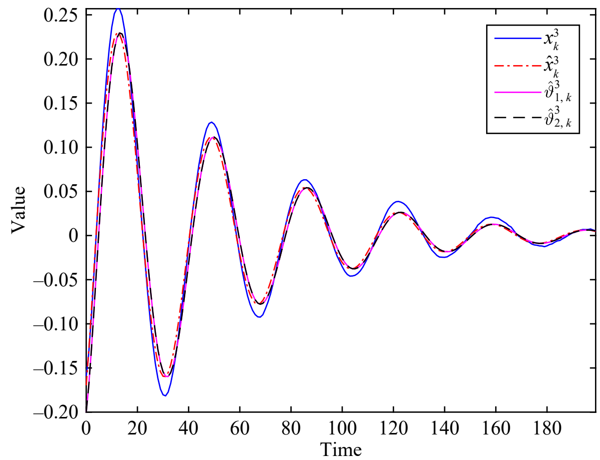


Fig. 4. x_k^3 and the estimates.

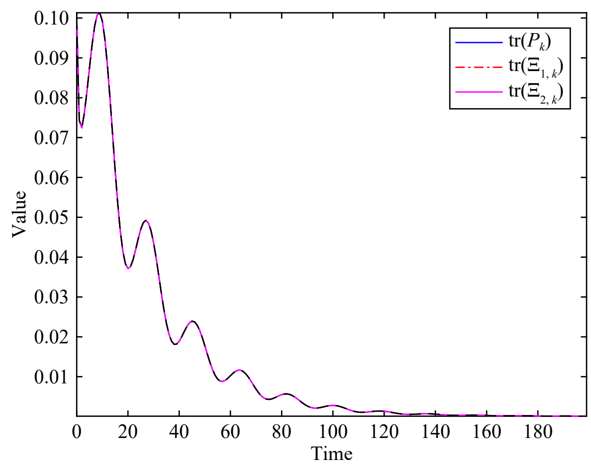


Fig. 5. The trace of filtering error covariances.

V. Conclusions

This paper has investigated the filtering problem for stochastic systems over FFSR networks. The signal has been transmitted to the remote filter through FFSR networks. In the FFSR, two relays have been employed to forward signals alternatively with the FF scheme. First, novel filter structures with switching parameters have been designed for FFSR and stochastic systems to accommodate the switching characteristics and IRI of FFSR networks. By means of the mathematical induction method, the filtering error covariance matrices have been explicitly presented through a class of coupled Riccati-like equations. Then, the desired filter gain matrices have been quantitatively derived by minimizing the trace of filtering error covariances. Moreover, the performance analysis has been conducted to show the boundedness of the filtering error covariances. Finally, a three-order RLC circuit system has been adopted to display the usefulness of the proposed filtering method over FFSR networks.

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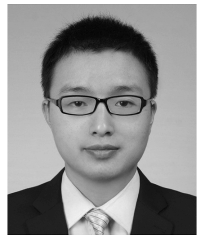

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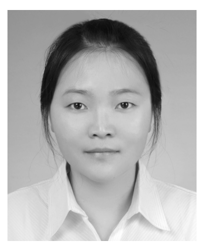

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