Letter

Output Feedback Stabilization of High-Order Nonlinear Time-Delay Systems With Low-Order and High-Order Nonlinearities

Meng-Meng Jiang, Kemei Zhang, and Xue-Jun Xie

Dear Editor,

This letter deals with the output feedback stabilization of a class of high-order nonlinear time-delay systems with more general low-order and high-order nonlinearities. By constructing reduced-order observer, based on homogeneous domination theory together with the adding a power integrator method, an output feedback controller is developed to guarantee the equilibrium of the closed system globally uniformly asymptotically stable.

We consider high-order nonlinear time-delay systems as follows:

$$\begin{aligned}&\dot{\eta}_i(t)=\eta_{i+1}^{p_i}(t)+\phi_i(t,\eta(t),\eta_1(t-\tau_1(t)),\ldots,\eta_i(t-\tau_i(t)))\\&\qquad i=1,\ldots,n-1\\&\dot{\eta}_n(t)=u^{p_n}(t)+\phi_n(t,\eta(t),\eta_1(t-\tau_1(t)),\ldots,\eta_n(t-\tau_n(t)))\\&y(t)=\eta_1(t)\end{aligned}\tag{1}$$

where $\eta(t)=[\eta_1(t),\ldots,\eta_n(t)]^T$, $u(t)$ and $y(t)$ are the system state, control input and output respectively. For $i=1,\ldots,n$, $\tau_i(t)$ is time-varying delay with $0\le\tau_i(t)\le\varepsilon_i$, where ε_i is a positive constant; $p_i\in R_{odd}^{\ge 1}\triangleq\{\frac{p}{q}\in R^+$: p and q are odd integers, $p\ge q\}$; ϕ_i is an unknown C function with $\phi_i(t,0,0)=0$. The system's initial condition is $\eta(\theta)=\zeta_0(\theta)$, $\forall\theta\in[-\varepsilon_M,0]$ with $\varepsilon_M=\max\{\varepsilon_1,\ldots,\varepsilon_n\}$ and $\zeta_0(\theta)$ being a specified C function. System (1) is called as high-order system if there exists at least one $i\in\{1,\ldots,n\}$ such that $p_i>1$.

Particularly, when $\tau_i(t)=0$, most of these results require that the nonlinearity ϕ_i satisfies certain restrictive conditions, that is, ϕ_i depends on the output y, or the states in the bounding functions are of an order equal to $\frac{1}{p_j\cdots p_{i-1}}$, or greater than $\frac{1}{p_j\cdots p_{i-1}}$, or less than $\frac{1}{p_j\cdots p_{i-1}}$, e.g., see [1]–[4] and the reference therein.

Recently, the restrictive condition was relaxed by [5]–[7], in which the assumptions can be summarized as the following form:

$$|\phi_i|\le c\sum_{j=1}^{i}(|\eta_j(t)|^{\nu_{lj}}+|\eta_j(t)|^{\nu_{uj}})\tag{2}$$

where low-order $\nu_{lj}=\frac{1}{p_j\cdots p_{i-1}}$ and high-order $\nu_{uj}=\frac{\bar{r}_i+\bar{\omega}_2}{\bar{r}_j}$ are some ratios of odd integers in $[\frac{1}{p_j\cdots p_{i-1}},+\infty)$ with $\bar{r}_1=1,\bar{r}_{i+1}=\frac{\bar{r}_i+\bar{\omega}_2}{p_i}$ and $\bar{\omega}_2\ge 0$.

For high-order nonlinear systems (1) with $\tau_i(t)\ne 0$, since time-delay is always encountered in many practical control systems and its emergence often causes instability and serious deterioration in the systems performance, many attention has been paid on the control design of time-delay system (1) and there have been some results achieved, see [8]–[10] and the reference therein. However, [8] only considered high-order nonlinearities, [9] only had an order in $(0,+\infty)$, [10] allowed low-order to be $\frac{1}{p_j\cdots p_{i-1}}$ and high-order to take any value in $[\frac{1}{p_j\cdots p_{i-1}},+\infty)$.

Based on the above discussion, an interesting problem is immediately proposed: For high-order nonlinear time-delay system (1), under the condition

$$\begin{aligned}|\phi_i|\le c\sum_{j=1}^{i}(&|\eta_j(t)|^{\nu_{lj}}+|\eta_j(t)|^{\nu_{uj}}\\&+|\eta_j(t-\tau_j(t))|^{\nu_{lj}}+|\eta_j(t-\tau_j(t))|^{\nu_{uj}}).\end{aligned}\tag{3}$$

Is it possible to relax condition (3) by allowing low-order ν_{lj} and high-order ν_{uj} to take any value in $(0,\frac{1}{p_j\cdots p_{i-1}}]$ and $[\frac{1}{p_j\cdots p_{i-1}},+\infty)$, respectively? Under the weaker condition, can an output feedback controller be designed for system (1)?

This letter will substantially solve this problem. By constructing reduced-order observer, a global output feedback controller based on the homogeneous domination theory and the adding a power integrator method is developed to guarantee the equilibrium of the closed-loop system globally uniformly asymptotically stable.

Notations: $\mathbb{R}^+$ stands for the set of all the nonnegative real numbers. For any vector $x=[x_1,\ldots,x_n]^T$, denote $x_t=x(t+\theta)$, $\|x\|=(\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$, $\|x_t\|_C=\sup_{-\varepsilon_M\le\theta\le 0}\|x(t+\theta)\|$. For $i=1,\ldots,n$, $\bar{x}_i\triangleq[x_1,\ldots,x_i]^T\in\mathbb{R}^i$, $\bar{x}_{i,t}\triangleq\bar{x}_i(t+\theta)$. A function $f:\mathbb{R}^n\to\mathbb{R}$ is C if it is continuous and is C^1 if it is continuously differential. $\mathcal{K}$ denotes the set of all functions: $\mathbb{R}^+\to\mathbb{R}^+$ that are continuous, strictly increasing and vanishing at zero, $\mathcal{K}_\infty$ denotes the set of all functions that are of class $\mathcal{K}$ and unbounded.

Problem statement: The purpose is to design an output feedback controller for system (1) such that the closed-loop system is globally uniformly asymptotically stable.

To achieve the purpose, the following assumptions are needed.

Assumption 1: For each $i=1,\ldots,n$, there is a positive constant γ_i such that $\tau_i:\mathbb{R}^+\to\mathbb{R}$ satisfies $\dot{\tau}_i(t)\le\gamma_i<1$.

Assumption 2: For each $i=1,\ldots,n$, there are constants $c>0$, $-\frac{1}{\sum_{l=1}^n p_1\cdots p_{l-1}}<\bar{\omega}_1\le 0$ with $p_0=1$ and $\bar{\omega}_2\ge 0$ such that

$$\begin{aligned}|\phi_i|\le c\sum_{j=1}^{i}\Big(&|\eta_j(t)|^{\frac{\bar{m}_i+\bar{\omega}_1}{\bar{m}_j}}+|\eta_j(t)|^{\frac{\bar{r}_i+\bar{\omega}_2}{\bar{r}_j}}\\&+|\eta_j(t-\tau_j(t))|^{\frac{\bar{m}_i+\bar{\omega}_1}{\bar{m}_j}}+|\eta_j(t-\tau_j(t))|^{\frac{\bar{r}_i+\bar{\omega}_2}{\bar{r}_j}}\Big)\end{aligned}\tag{4}$$

where $\bar{m}_i$ and $\bar{r}_i$ are defined as

$$\bar{m}_1=\bar{r}_1=1,\ \bar{m}_{i+1}=\frac{\bar{m}_i+\bar{\omega}_1}{p_i},\ \bar{r}_{i+1}=\frac{\bar{r}_i+\bar{\omega}_2}{p_i}.\tag{5}$$

The following lemma is indispensable in deriving the main result.

Lemma 1 [11]: Consider system

$$\dot{x}=f(x_t,t)\tag{6}$$

where $x(t)\in\mathbb{R}^n$ and $f:R\times C\to\mathbb{R}^n$.

Suppose that $f:R\times C\to\mathbb{R}^n$ given in (6), maps every $R\times$ (bounded set in C) into a bounded set in R^n, and that $u,v,w:\mathbb{R}^+\to\mathbb{R}^+$ are continuous nondecreasing functions, where additionally $u(s)$ and $v(s)$ are positive for $s>0$, and $u(0)=v(0)=0$. If there exists a continuous differentiable functional $V:R\times C\to R$ such that

$$u(\|\phi(0)\|)\le V(t,\phi)\le v(\|\phi\|_C)$$

and

$$\dot{V}(t,\phi)\le -w(\|\phi(0)\|)$$

then, the trivial solution of (6) is uniformly stable. If $w(s)>0$ for $s>0$, then it is uniformly asymptotically stable. In addition, if $\lim_{s\to\infty}u(s)=\infty$, then it is globally uniformly asymptotically stable.

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Controller design: Introduce the following coordinate transformation:

$$x_i(t) = \frac{\eta_i(t)}{L^{\lambda_i}},\ \ i = 1,\dots,n,\ \ v(t) = \frac{u(t)}{L^{\lambda_{n+1}}} \tag{7}$$

then system (1) is transformed into

$$\begin{aligned}\dot{x}_i(t) &= Lx_{i+1}^{p_i}(t) + f_i(t,x(t),x_1(t-\tau_1(t)),\dots,x_i(t-\tau_i(t)))\\ &\quad i = 1,\dots,n-1\\ \dot{x}_n(t) &= Lv^{p_n}(t) + f_n(t,x(t),x_1(t-\tau_1(t)),\dots,x_n(t-\tau_n(t)))\\ y(t) &= x_1(t)\end{aligned} \tag{8}$$

where $L \geq 1$ is a constant to be determined, $\lambda_1 = 0$, $\lambda_i = \frac{\lambda_{i-1}+1}{p_{i-1}}$, $i = 2,\dots,n+1$, $f_i = \frac{\phi_i}{L^{\lambda_i}}$.

Define $\xi_1 = x_1$,

$$m_1 = r_1 = 1,\ \ m_{i+1} = \frac{m_i+\omega_1}{p_i},\ \ r_{i+1} = \frac{r_i+\omega_2}{p_i} \tag{9}$$

where ω_1 and ω_2 are both ratios of an even integer over an odd integer and satisfy $-\frac{1}{\sum_{l=1}^{n} p_1\cdots p_l} < \omega_1 \leq \bar{\omega}_1 \leq 0$, $\omega_2 \geq \bar{\omega}_2 \geq 0$, respectively. Choose $\mu = \max_{i=1,\dots,n+1}\{\frac{r_i}{m_i}\}$ and

$$\begin{aligned}V_1 &= \frac{\xi_1^{2-m_2p_1+1}}{2-m_2p_1+1} + \frac{\xi_1^{2\mu-r_2p_1+1}}{2\mu-r_2p_1+1}\\ &\quad + \frac{(n+1)L}{1-\gamma_1}\int_{t-\tau_1(t)}^{t}\Big(\xi_1^2(s)ds + \xi_1^{2\mu}(s)\Big)ds\\ &\quad + \frac{nL}{1-\gamma_2}\int_{t-\tau_2(t)}^{t}\Big(\xi_1^2(s)ds + \xi_1^{2\mu}(s)\Big)ds.\end{aligned} \tag{10}$$

By choosing the appropriate virtual controller x_2^*, (10) becomes

$$\begin{aligned}\dot{V}_1 &\leq -nL(\xi_1^2 + \xi_1^{2\mu} + \xi_1^2(t-\tau_1(t)) + \xi_1^{2\mu}(t-\tau_1(t))\\ &\quad + \xi_1^2(t-\tau_2(t)) + \xi_1^{2\mu}(t-\tau_2(t)))\\ &\quad + L\Big(\xi_1^{2-m_2p_1} + \xi_1^{2\mu-r_2p_1}\Big)(x_2^{p_1} - x_2^{*p_1}).\end{aligned} \tag{11}$$

Through the recursive design method, the nth function V_n and a series of virtual controllers $x_1^*,\dots,x_{n+1}^*$ defined by

$$\begin{aligned}x_1^* &= 0,\ \ x_i^* = -\beta_{i-1}\left(\xi_{i-1} + \xi_{i-1}^{\frac{m_{i-1}r_i}{r_{i-1}m_i}}\right)^{m_i}\\ \xi_{i-1} &= x_{i-1}^{\frac{1}{m_{i-1}}} - x_{i-1}^{*\frac{1}{m_{i-1}}},\ \ i = 2,\dots,n+1\end{aligned} \tag{12}$$

such that

$$\begin{aligned}\dot{V}_n &\leq -L\sum_{j=1}^{n}\Big(\xi_j^2 + \xi_j^{\frac{2\mu m_j}{r_j}} + \xi_j^2(t-\tau_j(t)) + \xi_j^{\frac{2\mu m_j}{r_j}}(t-\tau_j(t))\\ &\quad + \xi_j^2(t-\tau_{j+1}(t)) + \xi_j^{\frac{2\mu m_j}{r_j}}(t-\tau_{j+1}(t))\Big)\\ &\quad + L\Big(\xi_n^{2-m_{n+1}p_n} + \xi_n^{\frac{(2\mu-r_{n+1}p_n)m_n}{r_n}}\Big)\Big(v^{p_n} - x_{n+1}^{*p_n}\Big)\end{aligned}$$

where $\tau_{n+1}(t) = 0$, β_{i-1} is a positive constant.

Introduce variables $z_2,\dots,z_n$ as

$$x_i^{p_{i-1}} = (z_i + l_{i-1}x_{i-1})^{\frac{r_ip_{i-1}}{r_{i-1}}} + (z_i + l_{i-1}x_{i-1})^{\frac{m_ip_{i-1}}{m_{i-1}}} \tag{13}$$

where the gains $l_1 \geq 1,\dots,l_{n-1} \geq 1$ are constants to be determined. By (13), one deduces that

$$\begin{aligned}\dot{z}_i &= -l_{i-1}Lx_i^{p_{i-1}} - l_{i-1}f_{i-1}\\ &\quad + \mathcal{I}_{1,i}^{-1}p_{i-1}x_i^{p_{i-1}-1}(Lx_{i+1}^{p_i} + f_i)(z_i + l_{i-1}x_{i-1})^{-\frac{\omega_1}{m_{i-1}}}\end{aligned} \tag{14}$$

where $x_{n+1} = v, \mathcal{I}_{1,i} = \frac{r_ip_{i-1}}{r_{i-1}}(z_i + l_{i-1}x_{i-1})^{\frac{\omega_2}{r_{i-1}} - \frac{\omega_1}{m_{i-1}}} + \frac{m_ip_{i-1}}{m_{i-1}}$. Based on (13) and (14), the reduced-order observer is constructed

$$\begin{aligned}\dot{\hat{z}}_i &= -l_{i-1}L\hat{x}_i^{p_{i-1}}\\ \hat{x}_i^{p_{i-1}} &= (\hat{z}_i + l_{i-1}\hat{x}_{i-1})^{\frac{r_ip_{i-1}}{r_{i-1}}} + (\hat{z}_i + l_{i-1}\hat{x}_{i-1})^{\frac{m_ip_{i-1}}{m_{i-1}}}\end{aligned} \tag{15}$$

where $\hat{x}_i$ is the estimate of x_i, $i = 2,\dots,n$, $\hat{x}_1 = x_1$. Using the certainty equivalence principle and (12), we obtain output feedback controller of system (8)

$$\begin{aligned}v(t) &= \hat{x}_{n+1}^* = -\beta_n\left(\hat{\xi}_n + \hat{\xi}_n^{\frac{m_nr_{n+1}}{r_nm_{n+1}}}\right)^{m_{n+1}}\\ \hat{\xi}_i &= \hat{x}_i^{\frac{1}{m_i}} - \hat{x}_i^{*\frac{1}{m_i}},\ \ \hat{x}_1^* = 0\\ \hat{x}_{i+1}^* &= -\beta_i\left(\hat{\xi}_i + \hat{\xi}_i^{\frac{m_ir_{i+1}}{r_im_{i+1}}}\right)^{m_{i+1}},\ \ i = 1,\dots,n.\end{aligned} \tag{16}$$

Defining the estimate error as $e_i = z_i - \hat{z}_i$, $i = 2,\dots,n$, using (14), (15) yield

$$\begin{aligned}\dot{e}_i &= -l_{i-1}L(x_i^{p_{i-1}} - \hat{x}_i^{p_{i-1}}) - l_{i-1}f_{i-1}\\ &\quad + \mathcal{I}_{1,i}^{-1}p_{i-1}x_i^{p_{i-1}-1}(Lx_{i+1}^{p_i} + f_i)(z_i + l_{i-1}x_{i-1})^{-\frac{\omega_1}{m_{i-1}}}.\end{aligned} \tag{17}$$

Define

$$\begin{aligned}U_i &= \frac{m_{i-1}}{2-\omega_1}e_i^{\frac{2-\omega_1}{m_{i-1}}} + \int_{z_i+l_{i-1}x_{i-1}}^{\hat{z}_i+l_{i-1}x_{i-1}}\Big(s^{\frac{2\mu-r_ip_{i-1}}{r_{i-1}}}\\ &\quad - (z_i + l_{i-1}x_{i-1})^{\frac{2\mu-r_ip_{i-1}}{r_{i-1}}}\Big)ds,\ i = 2,\dots,n.\end{aligned} \tag{18}$$

Choose $T = \sum_{i=1}^{n}U_i + V_n$, then

$$\begin{aligned}\dot{T} &\leq -\frac{L}{4}\sum_{j=1}^{n}\Big(\xi_j^2 + \xi_j^{\frac{2\mu m_j}{r_j}}\Big) - L\Big(2^{2-\frac{2\mu}{r_{n-1}}}\Big(2^{\frac{m_np_{n-1}}{m_{n-1}}} - 1\Big)\\ &\quad \times l_{n-1}^{\frac{2m_np_{n-1}}{m_{n-1}+m_np_{n-1}}} - c_1\Big)\Big(e_n^{\frac{2}{m_{n-1}}} + e_n^{\frac{2\mu}{r_{n-1}}}\Big)\\ &\quad - L\sum_{j=2}^{n-1}\Big(2^{2-\frac{2\mu}{r_{j-1}}}\Big(2^{\frac{m_jp_{j-1}}{m_{j-1}}} - 1\Big)l_{j-1}^{\frac{2m_jp_{j-1}}{m_{j-1}+m_jp_{j-1}}} - c_2\Big)\\ &\quad \times\Big(e_j^{\frac{2}{m_{j-1}}} + e_j^{\frac{2\mu}{r_{j-1}}}\Big) + c_3L^{1-\nu}\Big(\sum_{j=2}^{n}\Big(e_j^{\frac{2}{m_{j-1}}} + e_j^{\frac{2\mu}{r_{j-1}}}\Big)\\ &\quad + \sum_{j=1}^{n}\Big(\xi_j^2 + \xi_j^{\frac{2\mu m_j}{r_j}}\Big)\Big)\end{aligned} \tag{19}$$

where c_1, c_2, c_3 are some appropriate positive constants.

By determining $l_1,\dots,l_{n-1}$ and L as

$$\begin{aligned}l_{n-1} &= \max\Big\{\Big((c_1+1)\Big(2^{\frac{m_np_{n-1}}{m_{n-1}}} - 1\Big)^{-1}\\ &\qquad \times 2^{\frac{2\mu}{r_{n-1}}-2}\Big)^{\frac{m_{n-1}+m_np_{n-1}}{2m_np_{n-1}}}, 1\Big\}\\ l_i &= \max\Big\{\Big((c_2+1)\Big(2^{\frac{m_{i+1}p_i}{m_i}} - 1\Big)^{-1}\\ &\qquad \times 2^{\frac{2\mu}{r_i}-2}\Big)^{\frac{m_i+m_{i+1}p_i}{2m_{i+1}p_i}}, 1\Big\},\ \ i = n-2,\dots,1\\ L &> \max\{(8c_3)^{\frac{1}{\nu}}, 1\}.\end{aligned}$$

Inequality (19) becomes

$$\dot{T} \leq -\frac{L}{8}\sum_{j=1}^{n}\Big(\xi_j^2 + \xi_j^{\frac{2\mu m_j}{r_j}}\Big) - \frac{7L}{8}\sum_{j=2}^{n}\Big(e_j^{\frac{2}{m_{j-1}}} + e_j^{\frac{2\mu}{r_{j-1}}}\Big). \tag{20}$$

Then, we get the output feedback controller of system (1) with the form

$$\begin{aligned}u(t) &= -L^{\lambda_{n+1}}v(t) = -\beta_nL^{\lambda_{n+1}}\left(\hat{\xi}_n + \hat{\xi}_n^{\frac{m_nr_{n+1}}{r_nm_{n+1}}}\right)^{m_{n+1}}\\ \hat{\xi}_i &= \hat{x}_i^{\frac{1}{m_i}} - \hat{x}_i^{*\frac{1}{m_i}},\ x_0^* = 0\\ \hat{x}_i^* &= -\beta_{i-1}\left(\hat{\xi}_{i-1} + \hat{\xi}_{i-1}^{\frac{m_{i-1}r_i}{r_{i-1}m_i}}\right)^{m_i},\ i = 1,\dots,n\end{aligned} \tag{21}$$

where $\hat{x}_2,\dots,\hat{x}_n$ are observed by

$$\dot{\hat{z}}_i = -l_{i-1} L \hat{x}_i^{p_{i-1}}$$
$$\hat{x}_i^{p_{i-1}} = (\hat{z}_i + l_{i-1}\hat{x}_{i-1})^{\frac{r_i p_{i-1}}{r_{i-1}}} + (\hat{z}_i + l_{i-1}\hat{x}_{i-1})^{\frac{m_i p_{i-1}}{m_{i-1}}}. \quad (22)$$

Main result: By the above design, the following stability is obtained.

Theorem 1: If Assumptions 1 and 2 hold for system (1), the output feedback controller (21) and (22) guarantees that

1) All the solutions of the closed-loop system (1), (21) and (22) are well defined on $[-\varepsilon_M, +\infty)$.

2) The equilibrium $\eta = 0$ of the closed-loop system is globally uniformly asymptotically stable.

Proof: 1) Under (21), system (1) can be equivalently transformed into a ξ-system

$$\dot{\xi}_i(t) = \varphi_i(t, \xi(t), \xi_1(t-\tau_1(t)), \ldots, \xi_i(t-\tau_i(t)), \xi_1(t-\tau_2(t)), \ldots, \xi_{i-1}(t-\tau_i(t)), u(t)) \quad (23)$$

where $\varphi_i(\cdot): R^+ \times R^n \times R^i \times R^{i-1} \times R \to R$ is a C^1 function with $\varphi_i(t,0,0,0,0) = 0$. Define $\mathcal{Z} = [\xi_1, \ldots, \xi_n, e_2, \ldots, e_n]^T$, by the existence and continuation of the solution, the solution $\mathcal{Z}(t)$ is defined on $[-\varepsilon_M, t_M)$ with t_M being infinite or not.

It is obvious that $T(\mathcal{Z})$ is C^1, positive definite and radially unbounded.

Then, we show that $\eta(t)$ is well defined on $[-\varepsilon_M, +\infty)$. Noticing that $T(\mathcal{Z})$ and the term $-\frac{L}{8}\sum_{j=1}^n(\xi_j^2 + \xi_j^{\frac{2\mu m_j}{r_j}}) - \frac{7L}{8}\sum_{j=2}^n(e_j^{\frac{2}{m_{j-1}}} + e_j^{\frac{2\mu}{r_{j-1}}})$ on the right-hand side of (20) are positive definite and radially unbounded, one can find $\mathcal{K}_\infty$ functions $\pi_2(\cdot)$ and $\pi_3(\cdot)$ such that

$$T(\mathcal{Z}) \le \pi_2(\|\mathcal{Z}\|), \quad \dot{T}(\mathcal{Z}) \le -\pi_3(\|\mathcal{Z}\|). \quad (24)$$

Since $\pi_1(\|\mathcal{Z}\|)$ is a $\mathcal{K}_\infty$ function, for any $\delta > 0$, one can always find a $\beta = \beta(\delta)$ with $\beta > \delta > 0$ such that $\pi_2(\delta) \le \pi_1(\beta)$. If $\|\mathcal{Z}_0(\theta)\|_C < \delta$, $\theta \in [-\varepsilon_M, 0]$, and (24) yield $\pi_1(\|\mathcal{Z}\|) \le T(\mathcal{Z}_t(\theta)) \le T(\mathcal{Z}_0(\theta)) \le \pi_2(\|\mathcal{Z}_0(\theta)\|_C) \le \pi_1(\beta)$, $\forall t \in [0, t_M)$, which means that $\|\mathcal{Z}(t)\| \le \beta$ for any $t \in [-\varepsilon_M, t_M)$. Hence, t_M is not an escape time, i.e., $\mathcal{Z}(t)$ is well defined on $[-\varepsilon_M, +\infty)$, so is $\eta(t)$.

2) Since $t_M = +\infty$, according to (24) and Lemma 1, the equilibrium $\mathcal{Z} = 0$ is globally uniformly asymptotically stable. Since $x_i^*(\xi_{i-1})$ is continuous on ξ_{i-1} and $x_i^*(0) = 0$, by (22) and the globally uniformly asymptotic stability of system (17) and (23), it is easy to prove that the equilibrium $\eta = 0$ of the closed-loop system (1), (21) and (22) is globally uniformly asymptotically stable. ■

Numerical example: Consider a simple system

$$\begin{aligned}
\dot{\eta}_1 &= \eta_2^{\frac{21}{19}} \\
\dot{\eta}_2 &= u + \frac{\eta_1^{\frac{17}{23}} \sin\eta_1}{3} + \frac{\ln(1+|\eta_1|^{\frac{21}{19}})}{2(1+\eta_2^2)} + \frac{\eta_1^{\frac{17}{23}}(t - \frac{1}{2}\sin^2 t)}{3} \\
&\quad + \frac{\eta_1^{\frac{21}{19}}(t - \frac{1}{2}\sin^2 t)}{2(1+\eta_2^2(t - \frac{1}{2}\cos^2 t))} \\
y &= \eta_1.
\end{aligned} \quad (25)$$

Following the design procedure, a direct but redundant computation leads to an output feedback controller of system (25) with the form:

$$\begin{aligned}
u &= -L^{\frac{40}{21}}\beta_2\left(\hat{\xi}_2 + \hat{\xi}_2^{\frac{21}{17}}\right)^{\frac{17}{23}} \\
\hat{\xi}_2 &= \hat{x}_2^{\frac{23}{19}} + \beta_1^{\frac{23}{19}}\left(x_1 + x_1^{\frac{23}{19}}\right) \\
\hat{x}_2^{\frac{21}{19}} &= (\hat{z}_2 + l_1 x_1)^{\frac{21}{23}} + (\hat{z}_2 + l_1 x_1)^{\frac{21}{19}} \\
\dot{\hat{z}}_2 &= -l_1 L \hat{x}_2^{\frac{21}{19}}
\end{aligned} \quad (26)$$

where $\beta_1 = 2.68$, $\beta_2 = 600$, $l_1 = 30$, $L = 3$.

By choosing the initial values $\eta_1(\theta) = 0.001$, $\eta_2(\theta) = -0.002$, $\hat{z}_2(\theta) = -0.05$, $\theta \in [-\frac{1}{2}, 0]$, Fig. 1 verifies the effectiveness of the control scheme.

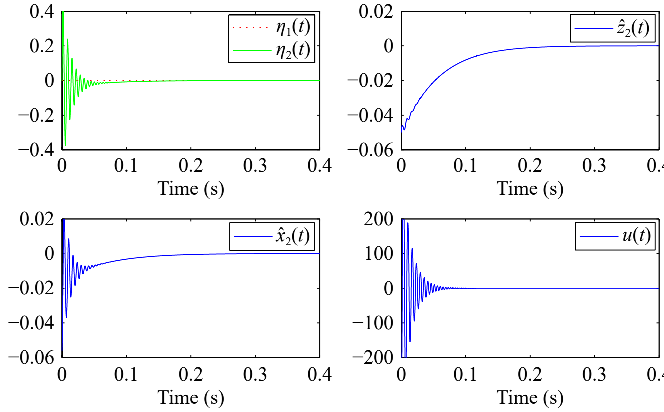

Fig. 1. The response of the closed-loop system (25) and (26).

Conclusion: This letter addresses the global output feedback stabilization of high-order nonlinear time-delay systems with more general low-order and high-order nonlinearities.

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