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Stability Robustness of Nonlinear Dynamic Inversion Based Controller

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***Abstract*—This paper presents an analysis on stability robustness of the widely used nonlinear dynamic inversion algorithm. By using nonlinear dynamic inversion algorithm, the multiple-inputs-multiple-outputs affine nonlinear system which has bounded uncertainty and disturbances is transformed into a linear time varying system with special structure. Thus the stability robustness issues of nonlinear dynamic inversion based controller are formulated as the stability problem of linear time varying differential equation and solved by linear time varying system theory. Sufficient conditions of stability and tracking error bound for the closed loop system are given and an illustrative example is also presented.**

Keywords—stability; robustness; nonlinear dynamic inversion; linear time varying system;

I. INTRODUCTION

Nonlinear dynamic inversion (NDI) is a well-known controller design technique for nonlinear systems, and has been researched for many years. By assuming that the nonlinearities of the plant dynamic is perfectly known, NDI uses the inverse dynamics to cancel the plant nonlinearities directly to get a linear system from input to output, so as to generate an integral response. This is often called the linearization process. After linearization, a lots of linear system control law design techniques such as PID, pole assignment, LQR etc, can be used to obtain the desired input-output response. Hence it is simple, intuitionistic, and convenient for control synthesis and has been widely used, especially for flight control design.

As early as 1988, Lane first proposed the flight control system design method based on nonlinear inverse dynamics[1]. Since then, NDI has been widely used in flight control. Snell discussed the use of NDI in the design of a flight control system for a supermaneuverable aircraft, where the dynamics of the aircraft were separated into fast and slow variables[2]. In the end of [2], the authors analyzed the robustness properties of NDI control laws. The assumptions that the aircraft dynamics could be modeled exactly and the states were measurable may be not easy to satisfy. The main disadvantage of dynamic inversion is that it does not provide a guarantee for the closed loop system stability. Usually nonlinear dynamic inversion is used in an inner-loop of control system to obtain a linearized system and an outer-loop is designed to guarantee the stability robustness of the closed loop system. Many advanced techniques can be used in the outer-loop to strengthen the stability robustness.

By taking the advantage of structured singular value synthesis, Adams used NDI and structured singular value synthesis to get robust performance across the flight envelope[3]. In 1998, instead of using conventional static state feedback, Snell et.al. used a dynamic compensator combined dynamic inversion as a general tool to design decoupling laws for flight control system, which was less sensitive to the unmodeled dynamics[4]. Bajodah proposed generalized dynamic inversion control design methodologies for linear spacecraft attitude control[5]. NDI control techniques have also been successfully used in aviation industry to design flight control law for fixed-wing aircraft. For example, NDI has been applied to the United Kingdom VAAC (Vector thrust Aircraft Advanced Control) Harrier, the Robust and Efficient Autoland control Law design project (REAL), the Joint Strike Fighter F35, F-18 HARV, and X-38[6].

Papageorgiod gave a robustness analysis by using linear matrix inequality tools for NDI control laws when quasi-LPV models in presence of time-varying, parametric uncertainty were used to approximate the nonlinear aircraft dynamics[7][8]. Ducard investigated the effects of parameter and measurement uncertainties on the inversion process when NDI was used in pitch rate controller for an unmanned aircraft. A systematic procedure for the selection of parameters in dynamic inversion algorithm was given to guarantee the stability of closed loop system[9]. Based on μ-synthesis, S. Brinker gave a stability robustness of a dynamic inversion aircraft control law[10]. By using Lyapunov analysis, Schumacher examined the stability of dynamic inversion controller with an inner-loop dynamic inversion and an out-loop dynamic inversion, and showed that if the design frequency of inner-loop was large enough and the inversion of inner-loop was exact, then an exponential stability of the outer-loop was achieved[11].

The above stability robustness analysis for dynamic inversion based controller are performanced on linear plants. In this paper, we'll give stability robustness analysis for nonlinear dynamic inversion based controller on nonlinear plants by linear time varying systems theory. We'll show that the closed loop system resulting from nonlinear dynamic inversion algorithm can be formed as a linear time

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varying system with disturbance. Based on the theory of linear time varying systems, sufficient condition and bound of tracking error for the stability of dynamic inversion based controller are given.

The rest of this paper is organized as follows. In section Ⅱ, a brief introduction of nonlinear dynamic is given and the stability robustness issue is presented. In section Ⅲ, the sufficient condition for the stability of linear time varying systems is given, and the boundary analysis of linear time varying differential equation with disturbance is also given. Then the sufficient condition and bound of tracking error for dynamic inversion based controller are presented. In section Ⅳ, we give an illustrative example. Section Ⅴ concludes this paper.

II. NONLINEAR DYNAMIC INVERSION ALGORITHM

Consider the following Multiple Input Multiple Output (MIMO) affine nonlinear system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u} \tag{1}$$

where $\mathbf{x} \in \Omega$ is the system state. Ω, which lies in a compact subset of $\mathbf{R}^n$, i.e., $\Omega \subset \mathbf{R}^n$, is the state space of the system (1). $\mathbf{u} \in \mathbf{R}^n$ is the control input, $\mathbf{f} : \Omega \to \mathbf{R}^n$ is a continuous vector function, $\mathbf{g} : \Omega \to \mathbf{R}^n \times \mathbf{R}^n$ is a matrix, whose elements are continuous functions on set Ω. And it is assuming that $\mathbf{g}$ is invertible. (1) describes the dynamic of the control input $\mathbf{u}$ to the system state $\mathbf{x}$. If reference trajectory $\mathbf{x}_r$ is given, then by using nonlinear dynamic inversion algorithm, we can set the control $\mathbf{u}$ as follows

$$\begin{aligned} \mathbf{u} &= \mathbf{g}^{-1}[\mathbf{v} - \mathbf{f}(\mathbf{x})] \\ \mathbf{v} &= -\mathbf{K}(\mathbf{x} - \mathbf{x}_r) + \dot{\mathbf{x}}_r \end{aligned} \tag{2}$$

which uses the inversion dynamics form $\mathbf{u}$ to $\mathbf{x}$. $\mathbf{K}$ is a state feedback matrix which can be chosen to make the closed loop system has the desired dynamics. And we can obtain the closed loop system dynamics as follows

$$\dot{\mathbf{x}} = \mathbf{v} \tag{3}$$

where $\mathbf{v}$ is a virtual control, and can be chosen appropriately to let $\mathbf{x}$ track its reference trajectory with favorable performance. It is clear that if we know all dynamics of the system, then nonlinear dynamic inversion algorithm can generate a good performance, but we always can't get the precise dynamics of nonlinear systems. There always exist external and internal disturbances, parameter uncertainties, which can result in unmolded dynamics. More precisely, the actual system dynamics should be described as follows

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \Delta\mathbf{f}(\mathbf{x}) + [\mathbf{g}(\mathbf{x}) + \Delta\mathbf{g}(\mathbf{x})]\mathbf{u} \tag{4}$$

where $\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$ represents the nominal part of the system dynamics which can be modeled exactly, and $\Delta\mathbf{f}(\mathbf{x})$ and $\Delta\mathbf{g}(\mathbf{x})\mathbf{u}$ represent the unmodeled part of the system dynamics. The exact expression of $\Delta\mathbf{f}(\mathbf{x})$ and $\Delta\mathbf{g}(\mathbf{x})$ are unknown. Only some bound information can be acquired. Therefore the following assumptions are made in our analysis.

Assumption 1 : There exist nonnegative constants $\varepsilon_i, i = 1, \cdots, 6$, *such that*

$$\sup_{\mathbf{x}\in\Omega}\left\|\Delta\mathbf{g}(\mathbf{x})\mathbf{g}^{-1}(\mathbf{x})\right\| \le \varepsilon_1, \quad \sup_{\mathbf{x}\in\Omega}\left\|\frac{\partial \Delta\mathbf{g}(\mathbf{x})\mathbf{g}^{-1}(\mathbf{x})}{\partial \mathbf{x}}\right\| \le \varepsilon_2$$

$$\sup_{\mathbf{x}\in\Omega}\left\|\mathbf{f}(\mathbf{x})\right\| \le \varepsilon_3, \quad \sup_{\mathbf{x}\in\Omega}\left\|\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}\right\| \le \varepsilon_4,$$

$$\sup_{\mathbf{x}\in\Omega}\left\|\Delta\mathbf{f}(\mathbf{x})\right\| \le \varepsilon_5, \sup_{\mathbf{x}\in\Omega}\left\|\frac{\partial \Delta\mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}\right\| \le \varepsilon_6.$$

Because the presence of unmodeled dynamics, when dynamic inversion algorithm is used, perfect performance can't be guaranteed any more, or even worse, the closed loop system becomes unstable. So we look forward to find out the conditions for maintaining the stability of dynamic inversion in presence of unmodeled dynamics, and the way to choose dynamic inversion algorithm parameters to get a much more robust closed loop system.

For the system dynamics described by (4), if we still choose $\mathbf{u}$ as (2), then

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) + \Delta\mathbf{f}(\mathbf{x}) + [\mathbf{g}(\mathbf{x}) + \Delta\mathbf{g}(\mathbf{x})]\mathbf{g}^{-1}(\mathbf{x})(\mathbf{v} - \mathbf{f}(\mathbf{x})) \\ &= \mathbf{v} + \Delta\mathbf{f}(\mathbf{x}) + \Delta\mathbf{g}(\mathbf{x})\mathbf{g}^{-1}(\mathbf{x})(\mathbf{v} - \mathbf{f}(\mathbf{x})) \\ &= (\mathbf{I} + \Delta\mathbf{g}(\mathbf{x})\mathbf{g}^{-1}(\mathbf{x}))\mathbf{v} - \Delta\mathbf{g}(\mathbf{x})\mathbf{g}^{-1}(\mathbf{x})\mathbf{f}(\mathbf{x}) + \Delta\mathbf{f}(\mathbf{x}) \\ &= -\mathbf{K}(\mathbf{x} - \mathbf{x}_r) + \dot{\mathbf{x}}_r + \Delta\mathbf{g}(\mathbf{x})\mathbf{g}^{-1}(\mathbf{x})(-\mathbf{K}(\mathbf{x} - \mathbf{x}_r) + \dot{\mathbf{x}}_r) \\ &\quad - \Delta\mathbf{g}(\mathbf{x})\mathbf{g}^{-1}(\mathbf{x})\mathbf{f}(\mathbf{x}) + \Delta\mathbf{f}(\mathbf{x}) \end{aligned} \tag{5}$$

Define $\mathbf{e} = \mathbf{x} - \mathbf{x}_r$. then we have

$$\begin{aligned} &\dot{\mathbf{e}} + \mathbf{K}[\mathbf{I} + \Delta\mathbf{g}(\mathbf{x})\mathbf{g}^{-1}(\mathbf{x})]\mathbf{e} - \\ &\Delta\mathbf{g}(\mathbf{x})\mathbf{g}^{-1}(\mathbf{x})[\dot{\mathbf{x}}_r + \mathbf{f}(\mathbf{x})] - \Delta\mathbf{f}(\mathbf{x}) = 0 \end{aligned} \tag{6}$$

i.e.,

$$\begin{aligned} \dot{\mathbf{e}} = &-\mathbf{K}[\mathbf{I} + \Delta\mathbf{g}(\mathbf{x}_r + \mathbf{e})\mathbf{g}^{-1}(\mathbf{x}_r + \mathbf{e})]\mathbf{e} \\ &+ \Delta\mathbf{g}(\mathbf{x}_r + \mathbf{e})\mathbf{g}^{-1}(\mathbf{x}_r + \mathbf{e})[\dot{\mathbf{x}}_r + \mathbf{f}(\mathbf{x}_r + \mathbf{e})] \\ &+ \Delta\mathbf{f}(\mathbf{x}_r + \mathbf{e}) \end{aligned} \tag{7}$$

Define

$$\begin{aligned} \mathbf{\Psi}(\mathbf{e}) = &\Delta\mathbf{g}(\mathbf{x}_r + \mathbf{e})\mathbf{g}^{-1}(\mathbf{x}_r + \mathbf{e})[\dot{\mathbf{x}}_r + \mathbf{f}(\mathbf{x}_r + \mathbf{e})] \\ &+ \Delta\mathbf{f}(\mathbf{x}_r + \mathbf{e}) \end{aligned} \tag{8}$$

$$\mathbf{F} = -\mathbf{K} \tag{9}$$

$$\Delta\mathbf{F}(t) = \left.\frac{\partial[\Delta\mathbf{g}(\mathbf{x}_r + \mathbf{e})\mathbf{g}^{-1}(\mathbf{x}_r + \mathbf{e})]}{\partial \mathbf{e}}\right|_{\mathbf{e}=0} + \left.\frac{\partial \mathbf{\Psi}}{\partial \mathbf{e}}\right|_{\mathbf{e}=0} \tag{10}$$

$$\mathbf{F}(t) = \mathbf{F} + \Delta\mathbf{F}(t) \tag{11}$$

$$\mathbf{G}(t,\mathbf{e}) = \mathbf{\Psi}(\mathbf{e}) - \mathbf{K}\Delta\mathbf{g}(\mathbf{x}_r + \mathbf{e})\mathbf{g}^{-1}(\mathbf{x}_r + \mathbf{e})\mathbf{e} + \left.\frac{\partial[\Delta\mathbf{g}(\mathbf{x}_r + \mathbf{e})\mathbf{g}^{-1}(\mathbf{x}_r + \mathbf{e})]}{\partial\mathbf{e}}\right|_{\mathbf{e}=\mathbf{0}} - \left.\frac{\partial\mathbf{\Psi}}{\partial\mathbf{e}}\right|_{\mathbf{e}=\mathbf{0}} \tag{12}$$

Then (7) can be rewrite as

$$\dot{\mathbf{e}} = \mathbf{F}(t)\mathbf{e} + \mathbf{G}(t,\mathbf{e}) \tag{13}$$

This system can be viewed as a linear time varying system with perturbances $\mathbf{G}(t,\mathbf{e})$. If $\lim_{t\to\infty}\mathbf{e}(t) = 0$, then the system (13) is asymptotically stable and the dynamic inversion algorithm based controller has an asymptotically tracking performance. This means that the stability robustness priorities of nonlinear dynamic inversion based controller can be analyzed by linear time varying system theory. Followed by *Assumption 1,* we also can obtain some conditions for the system (13). It is obvious that

$$\left.\frac{\partial[\Delta\mathbf{g}(\mathbf{x}_r + \mathbf{e})\mathbf{g}^{-1}(\mathbf{x}_r + \mathbf{e})]}{\partial\mathbf{e}}\right|_{\mathbf{e}=0} = \left.\frac{\partial[\Delta\mathbf{g}(\mathbf{x})\mathbf{g}^{-1}(\mathbf{x})]}{\partial\mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_r} \left.\frac{\partial\mathbf{x}}{\partial\mathbf{e}}\right|_{\mathbf{e}=0} = \left.\frac{\partial[\Delta\mathbf{g}(\mathbf{x})\mathbf{g}^{-1}(\mathbf{x})]}{\partial\mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_r}$$

and

$$\left\| \left.\frac{\partial[\Delta\mathbf{g}(\mathbf{x})\mathbf{g}^{-1}(\mathbf{x})]}{\partial\mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_r} \right\| \le \varepsilon_2$$

Similarly we have

$$\left\|\mathbf{\Psi}(\mathbf{e})\right\| \le \varepsilon_1(\left\|\mathbf{x}_r\right\| + \varepsilon_3) + \varepsilon_5 \tag{14}$$

$$\left\| \left.\frac{\partial\mathbf{\Psi}}{\partial\mathbf{e}}\right|_{\mathbf{e}=\mathbf{0}} \right\| \le \varepsilon_1(\left\|\mathbf{x}_r\right\| + \varepsilon_3) + \varepsilon_2\varepsilon_4 + \varepsilon_6 \tag{15}$$

$$\left\|\Delta\mathbf{G}(t)\right\| \le \varepsilon_1(\left\|\mathbf{x}_r\right\| + \varepsilon_3) + \varepsilon_2(1 + \varepsilon_4) + \varepsilon_6 \tag{16}$$

$$\left\|\mathbf{G}(t,\mathbf{e})\right\| \le \sigma + \max_{1\le i\le n}\left|\lambda_i(\mathbf{K})\right|\varepsilon_1\left\|\mathbf{e}\right\| \tag{17}$$

where $\sigma = 2\varepsilon_1(\left\|\mathbf{x}_r\right\| + \varepsilon_3) + \varepsilon_2(1 + \varepsilon_4) + \varepsilon_5 + \varepsilon_6$, and $\lambda_{\max}(\mathbf{K})$ is the maximum eigenvalue of matrix $\mathbf{K}$.

III. STABILITY ANALYSIS OF NONLINEAR DYNAMIC INVERSION ALGORITHM

A. Stability of linear time varying system

In this section, we'll give an introduction to the necessary and sufficient condition for linear time varying system proposed by Luo in [12] first, and then give a boundary analysis on linear time varying systems with nonlinear perturbances. Finally, we'll present a sufficient condition for the stability of nonlinear dynamic inversion based controller and give a tracking error bound for the controller.

For linear time varying system, the stability issue has been researched for many years, and lots of sufficient or necessary and sufficient conditions are obtained[13][14][15][16][17][18]. It is different from linear time invariant systems that for linear time varying systems, even all eigenvalues of the system matrix lie on the left-half plane all the time, it still may be unstable. Meanwhile, for some linear time varying system, it will be stable while all of its system matrix eigenvalues distributed on the right half plane. Therefore, the stability of linear time varying systems can't be determined by the eigenvalues of the system matrix. Some other informations about the system were required. Lots of sufficient conditions for stability require the existences of a Lyapunov function, but it has the same deficiency as other theories based on Lyapunov function, i.e., there lacks systematic way to figure out a Lyapunov function. Some conditions rely on the property of the state transition matrix, hence, for linear time systems with uncertainty or complex dynamics, the verification of the state transition matrix, is unpractical. However, for linear time varying systems with special structure, necessary and sufficient conditions for stability can be easily obtained and verified.

Consider a linear time varying system described by

$$\dot{\mathbf{x}}(t) = [\mathbf{A} + \Delta\mathbf{A}(t)]\mathbf{x}(t) \tag{18}$$

where $\mathbf{A}$ is a time invariant Hurwitz matrix with proper dimensions, $\Delta\mathbf{A}(t)$ is a time varying matrix which describes the time varying parametric uncertainties of the system dynamics. Hence the nominal dynamics of the system is described by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \tag{19}$$

Luo gave a stability robustness analysis for the system (18), and derived the conditions for stability robustness[12].

Let $\mathbf{A}^+$ be the matrix obtained by replacing the entries of $\mathbf{A}$ with their absolute values, and $\mathbf{A}^\#$ be the matrix obtained by replacing the entries of $\mathbf{A}$ with $\mathbf{A}^\#_{ij} = \mathbf{A}^+_{ij}$ for all $i \ne j$ and $\mathbf{A}^\#_{ii} = \mathrm{Re}(\mathbf{A}^+_{ii})\ (i,j = 1,\cdots,n)$.

Define $\Delta\mathbb{A}$ as a matrix whose entries are the maximum of the corresponding entries of matrix $\Delta\mathbf{A}^+_{ij}$ and define $\varepsilon = \max_{i=1,\cdots,n;j=1,\cdots,n}(\Delta\mathbb{A}_{ij})$.

Define matrix $\mathbf{E}$ as $\mathbf{E}_{ij} = \frac{\Delta\mathbb{A}_{ij}}{\varepsilon}$. Then the structured uncertainty can be described as

$$\{\Delta\mathbf{A}(t) : \Delta\mathbf{A}^+(t)[\le]\varepsilon\mathbf{E}\ \forall t \ge 0\} \tag{20}$$

where '$[\le]$' represents that the '$\le$' is applied to two matrices element-by-element.

Then for the system (18), the following theorem is given[12].

Theorem 1. *Suppose*

(1) The nominal closed-loop system described by (19)

with $\text{Max Re}\lambda_i(\mathbf{A}) < 0, i = 1,\cdots,n$

(2) $\text{Max Re}\lambda_i(\mathbf{A}^{\#}) < 0, i = 1,\cdots,n$

Then the uncertain system given by (18) is asymptoti- cally stable for all $\Delta\mathbf{A}(t)$ described by (20), if

$$\varepsilon < \frac{1}{\Pi[-\mathbf{A}^{-1}\mathbf{E}]} \qquad (21)$$

where $\Pi[-\mathbf{A}^{-1}\mathbf{E}]$ represents the maximum eigenvalue of the matrix $-\mathbf{A}^{-1}\mathbf{E}$. And we have

$$\mathbf{x}(t)[\leq]e^{(\mathbf{A}^{\#}+\varepsilon\mathbf{E})}\mathbf{x}^{+}(0) \qquad (22)$$

More ever, if $\mathbf{A}^{\#} = \mathbf{A}$,then (21) is the necessary and sufficient condition of asymptotic stability for the uncertain closed loop system given by (18).

Proof: see reference [12].

Now we consider the following linear time varying system with perturbances

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t,\mathbf{x}) \qquad (23)$$

where $\mathbf{f}(t,\mathbf{x}) \in C[\mathrm{I}\times \mathrm{R}^n, \mathrm{R}^n]$ represent perturbances. $\mathbf{f}(t,\mathbf{x}) \equiv 0$ if and only if $\mathbf{x} = 0$. For the stability of the system (23), we have the following theorem.

Theorem 2*. For system (23)*

(a) If $\forall\varepsilon > 0, \exists\pi(\varepsilon)$, and $\mathbf{x} \in \mathbf{D} = \{\mathbf{x} \mid \|\mathbf{x}\| < \pi\}$,

$$\|\mathbf{f}(t,\mathbf{x})\| < \varepsilon\|\mathbf{x}\| \qquad (24)$$

holds, then the trivial solution of (23) is exponentially stable if the trivial solution of homogeneous part $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$ is exponentially stable.

(b) There exists a constant c that If

$$\sup_{t\in[0,+\infty),\mathbf{x}\in\Omega} \|\mathbf{f}(t,\mathbf{x})\| < c \qquad (25)$$

then the trivial solution of (23) is bounded if the trivial solution of $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$ is exponentially stable.

Proof.

(a) Suppose that the trivial solution of $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$ is exponential stable, then there exist a constant m and a constant α with $0 < \alpha < -\max\limits_{1\leq j\leq n} \mathrm{Re}\,\lambda_j(\mathbf{A})$, that

$$\|\Phi(t,t_0)\| \leq me^{-\alpha(t-t_0)} \qquad (26)$$

where $\Phi(t,t_0)$ represents the state transition matrix of $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$.

According to linear system theory

$$\begin{aligned}\|\mathbf{x}(t)\| &= \left\|\Phi(t,t_0)\,\mathrm{x}(t_0) + \int_{t_0}^{t}\Phi(t,\tau)\mathbf{f}(\mathbf{x}(\tau),\tau)d\tau\right\| \\ &\leq \|\Phi(t,t_0)\|\|\mathrm{x}(t_0)\| + \left\|\int_{t_0}^{t}\Phi(t,\tau)\mathbf{f}(\mathbf{x}(\tau),\tau)d\tau\right\| \\ &\leq me^{-\alpha(t-t_0)}\|\mathrm{x}(t_0)\| + \left\|\int_{t_0}^{t}\varepsilon m e^{\alpha(t-\tau)}\mathbf{x}(\tau)d\tau\right\|\end{aligned} \qquad (27)$$

According to Gronwall inequality, we have

$$\|\mathbf{x}(t)\|e^{\alpha t} \leq me^{\alpha t_0+\varepsilon m(t-t_0)}\|\mathbf{x}(t_0)\| \qquad (28)$$

i.e.,

$$\|\mathbf{x}(t)\| \leq me^{\alpha(t_0-t)+\varepsilon m(t-t_0)}\|\mathbf{x}(t_0)\| \qquad (29)$$

If $\varepsilon < \dfrac{\alpha}{m}$, then $\alpha(t_0 - t) + \varepsilon m(t - t_0) < 0$, the system is exponentially stable.

(b) Suppose $\sup\limits_{t\in[0,+\infty),\mathbf{x}\in\Omega}\|\mathbf{f}(t,\mathbf{x})\| < c$, then

$$\begin{aligned}\|\mathbf{x}(t)\| &\leq me^{-\alpha(t-t_0)}\|\mathrm{x}(t_0)\| + \left\|\int_{t_0}^{t} cme^{-\alpha(t-\tau)}d\tau\right\| \\ &= \|\mathrm{x}(t_0)\|me^{-\alpha(t-t_0)} + \frac{mc}{\alpha}\end{aligned}$$

Because $\alpha > 0$, hence $\lim\limits_{t\to\infty}\|\mathrm{x}(t_0)\|me^{-\alpha(t-t_0)} = 0$.

Then it is easy to verify that (b) is true.

B. Sufficient stability condition for NDI based controller

Now we'll give the sufficient conditions for stability of nonlinear dynamic inversion based controller. For this purpose, considering the system (4) and the controller (2), we choose the state feedback matrix $\mathbf{K}$ as follows

$$\mathbf{K} = \mathrm{diag}(k_1,\cdots,k_n), k_i > 0, i = 1,\cdots,n \qquad (30)$$

and define $\mathbf{E}=[\mathbf{E}_{ij}]_{n\times n}, \mathbf{E}_{ij}=1, i=1,\dots n, j=1,\dots,n$. Then the determinant of $\lambda\mathbf{I} + \mathbf{F}^{-1}\mathbf{E}$ is

$$\mathrm{Det}[\lambda\mathbf{I} + \mathbf{F}^{-1}\mathbf{E}] = \lambda^n - (\textstyle\sum_{i=1}^{n} k_i^{-1})\lambda^{n-1} \qquad (31)$$

And the characteristic values of $-\mathbf{F}^{-1}\mathbf{E}$ are $\sum_{i=1}^{n} k_i^{-1}, \underbrace{0,\cdots,0}_{n-1}$. Hence

$$\Pi(-\mathbf{F}^{-1}\mathbf{E}) = \lambda_{\max}(-\mathbf{F}^{-1}\mathbf{E}) = \textstyle\sum_{i=1}^{n} k_i^{-1} \qquad (32)$$

where $\lambda_{\max}(-\mathbf{F}^{-1}\mathbf{E})$ represents the maximum eigenvalue of $-\mathbf{F}^{-1}\mathbf{E}$.

According to *Theorem 1*, the homogeneous part of the system (13) is asymptotically stable if

$$\varepsilon < \frac{1}{\Pi(-\mathbf{F}^{-1}\mathbf{E})} = \frac{1}{\lambda_{\max}(-\mathbf{F}^{-1}\mathbf{E})} = \frac{1}{\sum_{i=1}^{n} k_i^{-1}} \qquad (33)$$

where

$$\varepsilon \triangleq \varepsilon_1(\left\|\mathbf{x}_r\right\| + \varepsilon_3) + \varepsilon_2(1+\varepsilon_4) + \varepsilon_6 \quad (34)$$

The state transition matrix of $\dot{\mathbf{e}} = \mathbf{F}\mathbf{e}$ is $\Phi(t,t_0) = e^{\mathbf{F}(t-t_0)}$.

Hence

$$\left\|\Phi(t,t_0)\right\| \le e^{-\min\limits_{1\le i\le n}\{\left|\lambda_i(\mathbf{K})\right|\}(t-t_0)} = e^{-\min\limits_{1\le i\le n}\{k_i\}(t-t_0)} \quad (35)$$

According to *Theorem 2 (a) and (b)*, if the condition (33) holds, i.e., the homogeneous part of (13) is stable, and

$$\max_{1\le i\le n}\left|\lambda_i(\mathbf{K})\right|\varepsilon_1 \le \min_{1\le i\le n}\left|\lambda_i(\mathbf{K})\right| \quad (36)$$

then the system (13) is stable and the norm $\left\|\mathbf{e}\right\|$ is bounded by $\sigma \big/ \min\limits_{1\le i\le n}\{k_i\}$.

The sufficient condition for the stability robustness of nonlinear dynamic inversion based controller is summarized in the following theorem.

Theorem 3. *For the nonlinear system (4) and the nonlinear dynamic inversion based controller (2), suppose that Assumption 1 holds, then the closed loop system is stable and has a bounded tracking if the conditions (33) and (36) are hold.*

Through the above analysis, we know that if we choose such nonlinear dynamic inversion parameters $k_i(i=1,\cdots,n)$ that $\sum_{i=1}^{n} k_i^{-1}$ is small, then the closed loop system admits a large unmodeled while the stability still holds.

IV. Simulation

In this section, we'll give an numerical example to show the efficiency of the sufficients condition (33) and (36). Consider the following MIMO nonlinear system with unmodeled dynamics.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \Delta F(x_1,x_2,t) + \Delta G(x_1,x_2,t)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad (37)$$

where x_1, x_2 are system states, u_1, u_2 are control inputs, $f_1 = -x_1 - x_2^2, f_2 = -x_1^2$, $g_{11} = 10$, $g_{12} = \sin^2 x_2$, $g_{21} = x_1^2$, $g_{22} = 1$. $\Delta F(x_1,x_2,t)$ and $\Delta G(x_1,x_2,t)$ represent the bounded unmodeled dynamics. The initial conditions are $x_1(0) = 1$, $x_2(0) = -1$. The reference trajectories are $x_{1r}(t) = 0$, $x_{2r}(t) = 0$. For nominal dynamics of (37), according to dynamic inversion algorithm, the control inputs can be chosen as

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = -\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}\begin{pmatrix} 10 & \sin^2 x_2 \\ x_1^2 & 1 \end{pmatrix}^{-1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (38)$$

k_1, k_2 are chosen as $k_1 = k_2 = 10$.

Figure 1 gives the time histories of the closed loop system states when $\Delta F(x_1,x_2,t) = \Delta G(x_1,x_2,t) = 0$. Obviously, when there are no unmodeled dynamics, the nonlinear dynamic inversion based controller can generate a perfect response. If there exists unmodeled dynamic

$$\Delta F(x_1,x_2,t) = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \Delta G(x_1,x_2,t) = 0 \quad (39)$$

It is easy to verify that the conditon (36) is holds. Then according to (33), the closed loop system is asymptotically stable if

$$\varepsilon < \frac{1}{\sum_{i=1}^{2} 1/k_i} = 5 \quad (40)$$

where $\varepsilon = \varepsilon_6 = \max\limits_{i=1,2,3,4}\{\left|d_i\right|\}$. Figure 2 and Figure 3 describe the state responses when $d_i(i=1,2,3,4)=2,4,4.9.5.1$. These curves show that the closed loop system is stable when the condition (40) holds, i.e., $\max\limits_{i=1,2,3,4}\{\left|d_i\right|\} < 5$, and the closed loop system is unstable when the condition (40) does not hold, i.e., $\max\limits_{i=1,2,3,4}\{\left|d_i\right|\} > 5$.

Figure 4 shows a comparison of the states responses to different control parameters when $d_i(i=1,2,3,4)=5.1$, i.e., $\varepsilon = 5.1$. If the control parameters $k_i=10, i=1,2$, then the closed loop system is unstable. If $k_i=12, i=1,2$, then the closed loop system is stable.

I. Conclusions

In this paper, we present an analysis on stability robustness of nonlinear dynamic inversion based controller. The closed loop system of multiple-inputs-multiple-outputs affine nonlinear systems which have bounded uncertainty and disturbances and nonlinear dynamics inversion algorithm can be described by a linear time varying system model with special structure. Based on linear time varying system theory, sufficient conditions of stability and tracking error bound for the closed loop system. Example illustrates the proposed conditions.

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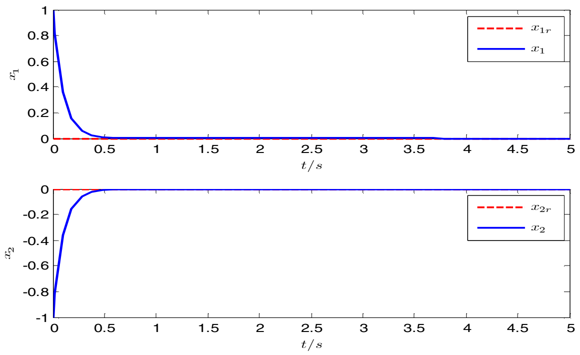

Figure 1. Time histories of system states (without unmodeled dynamics)

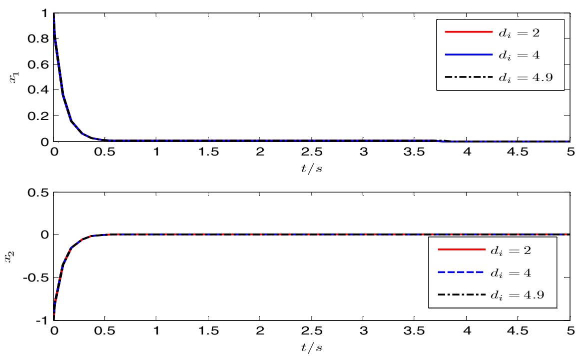

Figure 2. Time histories of system states (with unmodeled dynamics, $k_1=k_2=10$)

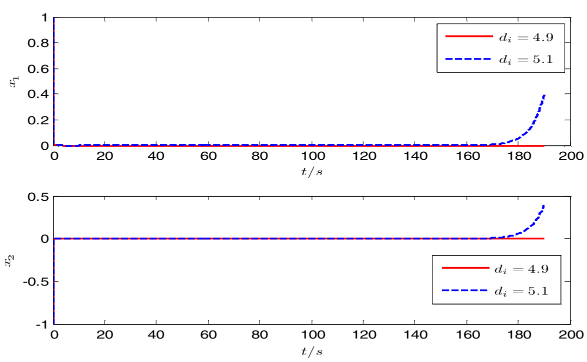

Figure 3. Time histories of system states (with unmodeled dynamics, $k_1=k_2=10$)

REFERENCES

[1] Stephen H. Lane, Robert F. Stengel. Flight control design using non-linear inverse dynamics. *Automatica*, Vol. 24, Iss. 4, pp.471-483, 1988.

[2] S. Antony Snell, Dale F. Enns, and Willian L. Garrard. Nonlinear inversion flight control for a supermaneuverable aircraft. *Journal of Guidance, Control, and Dynamics*, 15(4):976–984, 1992.

[3] Richard J. Adams and Siva S. Banda. Robust flight control design using dynamic inversion and structured singular value synthesis. *IEEE Transactions on Control Systems Technology*, 1(2):80–92, 1993.

[4] Antony Snell. Decoupling control design with application to flight. *Journal of Guidance, Control, and Dynamics*, 21(4):647–655, 1998.

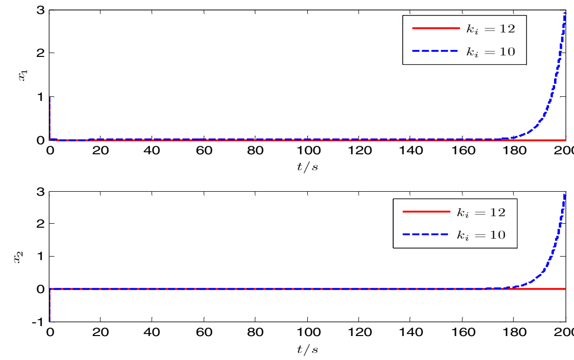

Figure 4. Comparison of different control parameters (k_i=10 and k_i=12)

[5] A.H. Bajodah. Generalised dynamic inversion spacecraft control design methodologies. *IET Control Theory and Applications*, Vol. 3, Iss. 2, pp. 239–251,2009.

[6] Gary J. Balas. Flight control law design: an industry perspective. *European Journal of Control*, 9(2-3):207–226, 2003.

[7] G. Papageorgiou and R. A. Hyde. Analysing the stability of NDI-based flight controllers with LPV methods. *In Guidance, Navigation and Control Conference*, Montreal, Canada, August 2001.

[8] G. Papageorgiod and K. Glover. Robustness analysis of nonlinear dynamic inversion control laws with application to flight control. *In 43rd IEEE Conference on Decision and Control*, pages 3485–3490, Atlantis, Paradies Island, Bahamas, 2004.

[9] Guillaume Ducard and Hans Peter Geering. Stability analysis of a dynamic inversion based pitch rate controller for an unmanned aircraft. *In 2008 IEEE/RSJ International Conference on Intelligent Robots and Systems*, pages 360–366, Nice, France, 2008.

[10] Joseph S. Brinker and Kevin A. Wise. Stability and flying qualities robustness of a dynamic inversion aircraft control law. *Journal of Guidance, Control, and Dynamics*, 19(6):1270-1277, 1996.

[11] Corey Schumacher, Pramod P. Khargonekar. Stability analysis of a missile control system with a dynamic inversion controller. *Journal of Guidance, Control, and Dynamics*, 21(3):508-515, 1998.

[12] J. S. Luo, X. F. Zhang, and Johnson. Stability robustness in linear state-space models with structured time-varying uncertainties. *International journal of Control*, Vol. 60, N0. 3, pp.435-450.

[13] Leonardo A. M., Reinaldo M. P. Stability analysis of linear time-varying systems: Improving conditions by adding more information about parameter variation. *Systems & Control Letters*, Vol. 60, pp.338-343, 2011.

[14] Izchak Lewkowicz. A necessary condition for quantitative exponential stability of linear state-space system. *System & Control Letters*, Vol. 38, pp.1-4, 1999.

[15] Minyen Wu. On stability of linear time varying systems. *Int. J. Systems Sci.*, Vol.15, No.2,pp.137-150, 1984.

[16] Germain Garcia, D. Peres and Sophie Tarbouriech. Necessary and sufficient numerical conditions for asymptotic stability of linear time-varying systems. *Proceedings of the 47th IEEE Conference on Decision and Control*, Cancun, Mexico,2008.

[17] Dirk Aeyels, Joan Peuteman. On exponential of nonlinear time-varying differential equations. *Automatica*, Vol.35,pp.1091-1100,1999.

[18] Leopoldo Jetto and Valentina Orsini. Relaxed sufficient conditions for the stability of continuous and discrete-time linear varying system. *Proceedings of the 46th IEEE Conference on Decision and Control,* New Orleans, USA, 2007.