Construction of Slope-Consistent Trapezoidal Interval Type-2 Fuzzy Sets for Simplifying the Perceptual Reasoning Method

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Abstract— Computing with words (CWW) proposed by Zadeh is an useful paradigm to mimic the human decision-making ability in a wide variety of physical and mental tasks. To realize CWW, Mendel proposed a specific architecture called perceptual computer, in which interval type-2 (IT2) fuzzy sets (FSs) and perceptual reasoning (PR) method are adopted. The PR method has been proved to have good properties (e.g. it can output intuitive IT2 FSs) and has found several applications in decision making. In this study, we focus on simplifying this method by avoiding its α-cuts based inference process. We first present a novel property for the inference of the PR method. We observe from the property that, if the IT2 FSs in the consequents of the IF-THEN rules are trapezoidal and have consistent slopes, then the output IT2 FS will be strictly trapezoidal and can be determined easily. In this case, the computation of the PR method can be simplified. To achieve such simplification, the trapezoidal IT2 FSs without consistent slopes should be approximated by the slope-consistent trapezoidal IT2 FSs. This issue is also studied in this paper by solving the constrained linear-quadratic optimization problem. At last, examples are given. The simplified PR method will be useful when the CWW models are utilized in the modeling and/or control problems of complex systems or multivariable dynamic systems.

I. INTRODUCTION

IN complex systems, e.g. the social systems and the management systems, problems are usually described and analyzed using natural language. Humans are used to reasoning and calculating on the basis of the premises expressed by words or perceptions [1], [2]. To solve the modeling, analysis, decision-making, evaluation and/or management problems in complex systems, Zadeh [1], [2] proposed the theory of Computing with Words (CWW). In CWW, the operation objects are linguistic variables, i.e., the value of the variable is the word or perception which can be modeled by fuzzy sets (FSs). It is a necessary tool when the available information is perception-based or not precise enough to use numbers.

Since the appearance of CWW, lots of papers and books have studied the theories and applications of CWW using type-1 fuzzy sets (T1 FSs) [3]–[19]. Recently, Mendel [20] claimed that the linguistic words should be modeled at least

by interval type-2 fuzzy sets (IT2 FSs), as "*words mean different things to different people, and so are uncertain*". Also, to realize CWW using IT2 FSs, in [21]–[24], Mendel proposed a specific architecture called perceptual computer (PC) and explored the functions of the encoder, CWW engine, and decoder of the PC. The encoder transforms linguistic perceptions into IT2 FSs according to the codebook [21]–[24]. The CWW engine maps the input IT2 FSs from the encoder into the output IT2 FS [21]–[24]. And then, the decoder maps the output IT2 FS of the CWW engine into a word in the codebook [21]–[25].

In the PC, the CWW engine is the core and can be realized by different approximate reasoning method, e.g. the perceptual reasoning (PR) method [26]–[28], the similarity-based PR method [29] and the linguistic summarizations [30]. The PR method was proved to have good property that the output IT2 FS can resemble the shapes of the IT2 FSs in the codebook. That is to say, it can output intuitive IT2 FSs. Because of this property, the PR method has found lots of applications, such as investment judgment [21], social judgment [31], weapon evaluation [32] and journal publication judgment [33].

To realize the fuzzy reasoning, the PR method should compute different levels of α-cuts of the output IT2 FS using the iterative Karnik-Mendel algorithms. By avoiding the α-cut computations of the output IT2 FS, we can simplify the computation process of the PR method. In this study, this is realized by setting the consequent IT2 FSs in the fuzzy IF-THEN rules to be trapezoidal and slope-consistent. If such conditions can be satisfied, the output IT2 FS of the CWW engine will also be trapezoidal, and its mathematical expressions can be derived easily. In order to achieve this simplification, the trapezoidal IT2 FSs without consistent slopes need to be approximated by the slope-consistent trapezoidal IT2 FSs. This issue is also studied in this paper by transforming the approximation to a constrained linear-quadratic optimization problem. To show the effectiveness of the IT2 FSs' approximation method, examples are finally given. The simplified PR method will be useful when the CWW paradigm is adopted as the modeling and/or control tool to deal with some complex systems or multivariable dynamic systems in which the number of fuzzy IF-THEN rules is usually large and the computation always needs to be done online.

The rest of this paper is organized as follows. Section II introduces type-2 fuzzy sets and the perceptual reasoning method. Section III studies one novel property of the perceptual reasoning method. Section IV presents how to

construct the slope-consistent trapezoidal IT2 FSs for the trapezoidal IT2 FSs without consistent slopes. Section V gives an example to show the effectiveness of the IT2 FSs' approximation method. Finally, conclusions are drawn in Section VI.

II. TYPE-2 FUZZY SETS AND PERCEPTUAL REASONING METHOD

This section briefly introduces the definition of type-2 fuzzy sets and the perceptual reasoning method.

A. Type-2 Fuzzy Sets

Fuzzy sets were introduced by Zadeh [34] in 1965 as an extension of the classical notion of set. The FS with crisp membership function (MF) is called type-1 (T1) FS, while the FS with fuzzy membership MF is called type-2 FS (T2 FS). The T2 FS $\tilde{Y}$ can be characterized as [34], [35]

$$\tilde{Y} = \int_{x \in X} \mu_{\tilde{Y}}(x)/x = \int_{x \in X} \int_{u \in J_x \subseteq [0,1]} f_x(u)/u \Big/ x, \quad (1)$$

where $\mu_{\tilde{Y}}(x)$ is the fuzzy MF grade of a generic element x, $\int \int$ denotes union over all admissible x and u, $f_x(u)$ is the secondary MF and J_x is the primary membership of x which is the domain of the secondary MF.

When the secondary grades of $\tilde{Y}$ all equal 1, the T2 FS becomes interval type-2 (IT2) FS. An IT2 FS $\tilde{Y}$ can be completely depicted by its lower MF (LMF) $\underline{\mu}_{\tilde{Y}}(x)$ and upper MF (UMF) $\overline{\mu}_{\tilde{Y}}(x)$, i.e. [35]

$$\mu_{\tilde{Y}}(x) = [\underline{\mu}_{\tilde{Y}}(x), \overline{\mu}_{\tilde{Y}}(x)]. \quad (2)$$
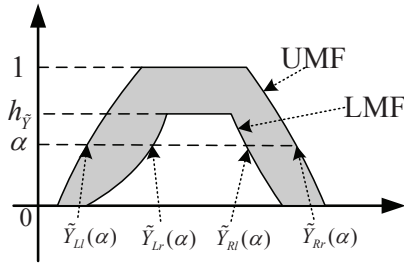

Fig. 1.   An example of interval type-2 fuzzy sets

One example of IT2 FSs is depicted in Fig. 1. The α-cut of the IT2 FS $\tilde{Y}$ is also depicted in this figure. For the IT2 FS $\tilde{Y}$, its α-cut consists of four end-points, which are denoted respectively as $\tilde{Y}_{Ll}(\alpha), \tilde{Y}_{Rr}(\alpha), \tilde{Y}_{Lr}(\alpha)$ and $\tilde{Y}_{Rl}(\alpha)$.

Trapezoidal IT2 FSs shown in Fig. 2 are special cases of general IT2 FSs. We can still depict trapezoidal IT2 FSs using their LMFs and UMFs. A trapezoidal IT2 FS $\tilde{Y}$ can be denoted by its LMF and UMF as [35], [36]

$$\overline{\mu}_{\tilde{Y}}(x) = \begin{cases} \frac{x - \overline{a}}{\overline{b} - \overline{a}} & \overline{a} < x \le \overline{b} \\ 1 & \overline{b} < x \le \overline{c} \\ \frac{\overline{d} - x}{\overline{d} - \overline{c}} & \overline{c} < x \le \overline{d} \\ 0 & else \end{cases} \quad (3)$$
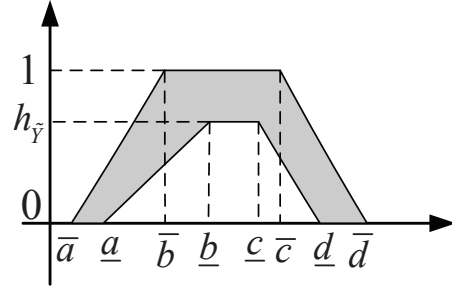

Fig. 2.   Trapezoidal IT2 FS

$$\underline{\mu}_{\tilde{Y}}(x) = \begin{cases} h_{\tilde{Y}} \frac{x - \underline{a}}{\underline{b} - \underline{a}} & \underline{a} < x \le \underline{b} \\ 1 & \underline{b} < x \le \underline{c} \\ h_{\tilde{Y}} \frac{\underline{d} - x}{\underline{d} - \underline{c}} & \underline{c} < x \le \underline{d} \\ 0 & else \end{cases} \quad (4)$$

where $\overline{a}, \overline{b}, \overline{c}, \overline{d}, \underline{a}, \underline{b}, \underline{c}, \underline{d}$ satisfy the orders shown in Fig. 2.

For the trapezoidal IT2 FS $\tilde{Y}$, its UMF and LMF are respectively denoted as

$$\overline{\mu}_{\tilde{Y}}(x) = \overline{\mu}_{\tilde{Y}}(x, \overline{a}, \overline{b}, \overline{c}, \overline{d}) \quad (5)$$

$$\underline{\mu}_{\tilde{Y}}(x) = \underline{\mu}_{\tilde{Y}}(x, \underline{a}, \underline{b}, \underline{c}, \underline{d}, h_{\tilde{Y}}). \quad (6)$$

B. Perceptual Reasoning Method

Suppose that the following type-2 fuzzy rule base is adopted in the CWW engine:

$$\left\{ R^i : \ x_1 = \tilde{X}_1^i, \cdots, \ x_p = \tilde{X}_p^i, \to y = \tilde{Y}^i \right\}_{i=1}^M, \quad (7)$$

where $\tilde{X}_1^i, \cdots, \tilde{X}_p^i, \tilde{Y}^i$ are IT2 FSs.

The PR method [26]–[28] utilizes the following two steps to compute the output word $\tilde{Y}$:

Step 1: Computing the firing strength of each rule

Suppose that the input words are $\tilde{\mathbf{X}} = (\tilde{X}_1, \tilde{X}_2, ..., \tilde{X}_p)$, then, the firing strength of Rule i is an interval $F^i(\tilde{\mathbf{X}}) = [\underline{f}^i(\tilde{\mathbf{X}}), \overline{f}^i(\tilde{\mathbf{X}})]$, which can be calculated as

$$\underline{f}^i(\tilde{\mathbf{X}}) = \wedge_{j=1}^p \big(\sup_{x_j \in U_j} \underline{\mu}_{\tilde{X}_j}(x_j) \wedge \underline{\mu}_{\tilde{X}_j^i}(x_j) \big), \quad (8)$$

$$\overline{f}^i(\tilde{\mathbf{X}}) = \wedge_{j=1}^p \big(\sup_{x_j \in U_j} \overline{\mu}_{\tilde{X}_j}(x_j) \wedge \overline{\mu}_{\tilde{X}_j^i}(x_j) \big). \quad (9)$$

where U_j is the universe of discourse of the input variable x_j.

Step 2: Aggregating the fired rules by the linguistic weighted average method

In the perceptual reasoning method [26]–[28], the linguistic weighted average algorithm [33], [37] is adopted to aggregate the consequents of the fired rules to obtain the output word (IT2 FS), i.e.

$$\tilde{Y} = \frac{\sum_{i=1}^M F^i(\tilde{\mathbf{X}}) \tilde{Y}^i}{\sum_{i=1}^M F^i(\tilde{\mathbf{X}})} \quad (10)$$

where $F^i(\tilde{\mathbf{X}})$ is an interval, $\tilde{Y}^i$ is the consequent IT2 FS of Rule i. This equation can be computed through the α-cuts method. Fig. 1 shows us an example on the α-cuts of

the IT2 FS $\widetilde{Y}$. The α-cut of its UMF can be represented as $[\widetilde{Y}_{Ll}(\alpha), \widetilde{Y}_{Rr}(\alpha)]$, while the α-cut of its LMF can be represented as $[\widetilde{Y}_{Lr}(\alpha), \widetilde{Y}_{Rl}(\alpha)]$.

From the results in [26]–[28], the left-end and right-end points of the α-cuts of the output IT2 FS can be computed as

$$
\begin{cases}
\widetilde{Y}_{Ll}(\alpha) = \displaystyle\min_{\forall f^i \in [\underline{f}^i(\widetilde{\mathbf{X}}), \overline{f}^i(\widetilde{\mathbf{X}})]} \frac{\sum_{i=1}^{M} \widetilde{Y}_{Ll}^i(\alpha) f^i}{\sum_{i=1}^{M} f^i}, \ \alpha \in [0,1], \\[3mm]
\widetilde{Y}_{Rr}(\alpha) = \displaystyle\max_{\forall f^i \in [\underline{f}^i(\widetilde{\mathbf{X}}), \overline{f}^i(\widetilde{\mathbf{X}})]} \frac{\sum_{i=1}^{M} \widetilde{Y}_{Rr}^i(\alpha) f^i}{\sum_{i=1}^{M} f^i}, \ \alpha \in [0,1], \\[3mm]
\widetilde{Y}_{Lr}(\alpha) = \displaystyle\min_{\forall f^i \in [\underline{f}^i(\widetilde{\mathbf{X}}), \overline{f}^i(\widetilde{\mathbf{X}})]} \frac{\sum_{i=1}^{M} \widetilde{Y}_{Lr}^i(\alpha) f^i}{\sum_{i=1}^{M} f^i}, \ \alpha \in [0, h_{\widetilde{Y}}], \\[3mm]
\widetilde{Y}_{Rl}(\alpha) = \displaystyle\max_{\forall f^i \in [\underline{f}^i(\widetilde{\mathbf{X}}), \overline{f}^i(\widetilde{\mathbf{X}})]} \frac{\sum_{i=1}^{M} \widetilde{Y}_{Rl}^i(\alpha) f^i}{\sum_{i=1}^{M} f^i}, \ \alpha \in [0, h_{\widetilde{Y}}],
\end{cases}
\tag{11}
$$

where $h_{\widetilde{Y}} = \min_{i=1}^{M} h^i$, and $h^i = h_{\widetilde{Y}^i}$ is the height of the LMF of the IT2 FS $\widetilde{Y}^i$. These equations can be computed by the Karnik-Mendel algorithm [35].

III. PROPERTY OF THE PERCEPTUAL REASONING METHOD

When the CWW paradigm is adopted as the modeling and/or control tool to deal with some complex systems or multivariable dynamic systems, we usually face the large number of fuzzy IF-THEN rules and the online computation tasks, and sometimes we need to theoretically analyze system performance or stability issues through mathematical expressions. The PR method is on the basis of α-cuts of the IT2 FSs which makes it difficult to directly depict the output IT2 FS by exact mathematical expressions. However, if special trapezoidal IT2 FSs are adopted in the consequent parts of the fuzzy rules, we have the following property which shows us how to compute the output word directly and makes us avoid the computation of the output word's α-cuts.

To begin, let us denote the slopes of the trapezoidal IT2 FS $\widetilde{Y}^i$ in the consequent part of rule R^i as

$$
\begin{cases}
k_{Ll}^i \triangleq \frac{1}{\widetilde{Y}_{Ll}^i(1) - \widetilde{Y}_{Ll}^i(0)}, \\[2mm]
k_{Rr}^i \triangleq \frac{1}{\widetilde{Y}_{Rr}^i(1) - \widetilde{Y}_{Rr}^i(0)}, \\[2mm]
k_{Lr}^i \triangleq \frac{h^i}{\widetilde{Y}_{Lr}^i(h^i) - \widetilde{Y}_{Lr}^i(0)}, \\[2mm]
k_{Rl}^i \triangleq \frac{h^i}{\widetilde{Y}_{Rl}^i(h^i) - \widetilde{Y}_{Rl}^i(0)}.
\end{cases}
\tag{12}
$$

Then, we have the following property:

Theorem 1: If the slopes of the consequent trapezoidal IT2 FSs satisfy the following constraints: $k_{Ll}^1 = \cdots = k_{Ll}^M = k_{Ll}$, $k_{Rr}^1 = \cdots = k_{Rr}^M = k_{Rr}$, $k_{Lr}^1 = \cdots = k_{Lr}^M = k_{Lr}$, and $k_{Rl}^1 = \cdots = k_{Rl}^M = k_{Rl}$, then, the output word $\widetilde{Y}$ is still a trapezoidal IT2 FS, whose slopes are $k_{Ll}(\widetilde{Y}) = k_{Ll}$, $k_{Rr}(\widetilde{Y}) = k_{Rr}$, $k_{Lr}(\widetilde{Y}) = k_{Lr}$, $k_{Rl}(\widetilde{Y}) = k_{Rl}$.

Proof: The proof of this Theorem is similar to that of Theorem 3 in [38]. The only differences are that:

1) Theorem 3 in [38] is described in the dynamic form for linguistic dynamic systems.

2) The IT2 FSs in Theorem 3 in [38] are trapezoidal IT2 fuzzy numbers whose LMFs' heights are equal to 1 while the

trapezoidal IT2 FSs in this theorem are more general with arbitrary LMFs' heights.

Hence, we omit the proof of this theorem here. $\square$

If the slopes of the consequent trapezoidal IT2 FSs satisfy the constraints in Theorem 1, then, to determine the output word $\widetilde{Y}$, we only need to compute its four vertexes $\widetilde{Y}_{Ll}(0)$, $\widetilde{Y}_{Rr}(0)$, $\widetilde{Y}_{Lr}(0)$, and $\widetilde{Y}_{Rl}(0)$. There is no need to compute the end points of the α-cuts of the output word.

In detail, for the output trapezoidal IT2 FS $\widetilde{Y}$, denoted as $\overline{\mu}_{\widetilde{Y}}(x) = \overline{\mu}_{\widetilde{Y}}(x, \overline{a}, \overline{b}, \overline{c}, \overline{d}), \underline{\mu}_{\widetilde{Y}}(x) = \underline{\mu}_{\widetilde{Y}}(x, \underline{a}, \underline{b}, \underline{c}, \underline{d}, h_{\widetilde{Y}})$, where $\overline{a} = \widetilde{Y}_{Ll}(0)$, $\underline{a} = \widetilde{Y}_{Lr}(0)$, $\underline{d} = \widetilde{Y}_{Rl}(0)$, and $\overline{d} = \widetilde{Y}_{Rr}(0)$, its parameters can be computed respectively as

$$
\begin{cases}
\overline{a} = \displaystyle\min_{\forall f^i \in [\underline{f}^i(\widetilde{\mathbf{X}}), \overline{f}^i(\widetilde{\mathbf{X}})]} \frac{\sum_{i=1}^{M} \overline{a}^i f^i}{\sum_{i=1}^{M} f^i}, \\[3mm]
\overline{b} = \overline{a} + \frac{1}{k_{Ll}}, \\[2mm]
\overline{d} = \displaystyle\max_{\forall f^i \in [\underline{f}^i(\widetilde{\mathbf{X}}), \overline{f}^i(\widetilde{\mathbf{X}})]} \frac{\sum_{i=1}^{M} \overline{d}^i f^i}{\sum_{i=1}^{M} f^i}, \\[3mm]
\overline{c} = \overline{d} + \frac{1}{k_{Rr}}, \\[2mm]
\underline{a} = \displaystyle\min_{\forall f^i \in [\underline{f}^i(\widetilde{\mathbf{X}}), \overline{f}^i(\widetilde{\mathbf{X}})]} \frac{\sum_{i=1}^{M} \underline{a}^i f^i}{\sum_{i=1}^{M} f^i}, \\[3mm]
\underline{b} = \underline{a} + \frac{h_{\widetilde{Y}}}{k_{Lr}}, \\[2mm]
\underline{d} = \displaystyle\max_{\forall f^i \in [\underline{f}^i(\widetilde{\mathbf{X}}), \overline{f}^i(\widetilde{\mathbf{X}})]} \frac{\sum_{i=1}^{M} \underline{d}^i f^i}{\sum_{i=1}^{M} f^i}, \\[3mm]
\underline{c} = \underline{d} + \frac{h_{\widetilde{Y}}}{k_{Rl}}.
\end{cases}
\tag{13}
$$

To simplify the PR method, the slopes of the trapezoidal IT2 FSs need to satisfy the conditions in Theorem 1. For ease of the following discussion, we first give the following definition on the slopes of different IT2 FSs.

Definition 1: Trapezoidal IT2 FSs $\widetilde{Y}^1, \widetilde{Y}^2, \cdots, \widetilde{Y}^M$ are said to be slope-consistent if they satisfy the constraints in Theorem 1, i.e. $k_{Ll}^1 = \cdots = k_{Ll}^M = k_{Ll}$, $k_{Rr}^1 = \cdots = k_{Rr}^M = k_{Rr}$, $k_{Lr}^1 = \cdots = k_{Lr}^M = k_{Lr}$, and $k_{Rl}^1 = \cdots = k_{Rl}^M = k_{Rl}$.

In some applications, the trapezoidal IT2 FSs in the word library may not be slope-consistent. In such case, we need to find the approximated trapezoidal IT2 FSs with consistent slopes for them.

IV. CONSTRUCTING SLOPE-CONSISTENT TRAPEZOIDAL IT2 FSS

Below, we will give detailed discussion on constructing slope-consistent trapezoidal IT2 FSs.

A. Problem Formulation

Suppose that the trapezoidal IT2 FSs $\widetilde{Y}^1, \cdots, \widetilde{Y}^M$ in the word library do not have consistent slopes. Then, we need to find M most approximated slope-consistent trapezoidal IT2 FSs $\widetilde{Z}^1, \cdots, \widetilde{Z}^M$ for $\widetilde{Y}^1, \cdots, \widetilde{Y}^M$. In other words, for $\forall i$, the trapezoidal IT2 FS $\widetilde{Z}^i$ is an approximation of $\widetilde{Y}^i$, and, $k_{Ll}(\widetilde{Z}^1) = \cdots = k_{Ll}(\widetilde{Z}^M) = k_{Ll}$, $k_{Rr}(\widetilde{Z}^1) = \cdots = k_{Rr}(\widetilde{Z}^M) = k_{Rr}$, $k_{Lr}(\widetilde{Z}^1) = \cdots = k_{Lr}(\widetilde{Z}^M) = k_{Lr}$, and $k_{Rl}(\widetilde{Z}^1) = \cdots = k_{Rl}(\widetilde{Z}^M) = k_{Rl}$.

To construct the slope-consistent trapezoidal IT2 FSs, we need to determine the parameters $k_{Ll}, k_{Rr}, k_{Lr}, k_{Rl}$, and the four end-points $\widetilde{Z}_{Ll}^i(0), \widetilde{Z}_{Lr}^i(0), \widetilde{Z}_{Rl}^i(0), \widetilde{Z}_{Rr}^i(0)$ of $\widetilde{Z}^i$. Below, we will study this issue in detail.

To measure the distance of the trapezoidal IT2 FS $\widetilde{Y}^i$ and its approximation $\widetilde{Z}^i$, we define the following metric, which is an extension of the Eclidean distance

$$
\begin{aligned}
&D(\widetilde{Y}^i, \widetilde{Z}^i) \\
&= \int_0^1 |\widetilde{Y}_{Ll}^i(\alpha) - \widetilde{Z}_{Ll}^i(\alpha)|^2 \mathrm{d}\alpha + \int_0^1 |\widetilde{Y}_{Rr}^i(\alpha) - \widetilde{Z}_{Rr}^i(\alpha)|^2 \mathrm{d}\alpha \\
&+ \int_0^{h^i} |\widetilde{Y}_{Lr}^i(\alpha) - \widetilde{Z}_{Lr}^i(\alpha)|^2 \mathrm{d}\alpha + \int_0^{h^i} |\widetilde{Y}_{Rl}^i(\alpha) - \widetilde{Z}_{Rl}^i(\alpha)|^2 \mathrm{d}\alpha
\end{aligned}
\tag{14}
$$

where h^i is the height of the lower MF of the trapezoidal IT2 FS $\widetilde{Y}^i$ and its approximation $\widetilde{Z}^i$.

In the same way, the distance between the trapezoidal IT2 FSs $\widetilde{Y}^1, \cdots, \widetilde{Y}^M$ and their approximations $\widetilde{Z}^1, \cdots, \widetilde{Z}^M$ can be computed as

$$
D(\widetilde{Y}^1, \cdots, \widetilde{Y}^M, \widetilde{Z}^1, \cdots, \widetilde{Z}^M) = \sum_{i=1}^M D(\widetilde{Y}^i, \widetilde{Z}^i). \tag{15}
$$

Through minimizing this measure under the constraints on the slopes of the trapezoidal IT2 FSs, we can obtain one reasonable approximation. In other words, the construction of the slope-consistent trapezoidal IT2 FSs can be solved by the following optimization problem [39]

$$
\begin{cases}
\min & D(\widetilde{Y}^1, \cdots, \widetilde{Y}^M, \widetilde{Z}^1, \cdots, \widetilde{Z}^M), \\
s.t. & k_{Ll}(\widetilde{Z}^1) = \cdots = k_{Ll}(\widetilde{Z}^M) = k_{Ll}, \\
& k_{Rr}(\widetilde{Z}^1) = \cdots = k_{Rr}(\widetilde{Z}^M) = k_{Rr}, \\
& k_{Lr}(\widetilde{Z}^1) = \cdots = k_{Lr}(\widetilde{Z}^M) = k_{Lr}, \\
& k_{Rl}(\widetilde{Z}^1) = \cdots = k_{Rl}(\widetilde{Z}^M) = k_{Rl}
\end{cases}
\tag{16}
$$

Below, we will show how to solve this optimization problem.

B. Problem Transformation

To compute the distance measure in (15), we just need to respectively compute

$$
D_{Ll} = \sum_{i=1}^M \int_0^1 |\widetilde{Y}_{Ll}^i(\alpha) - \widetilde{Z}_{Ll}^i(\alpha)|^2 \mathrm{d}\alpha, \tag{17}
$$

$$
D_{Lr} = \sum_{i=1}^M \int_0^{h^i} |\widetilde{Y}_{Lr}^i(\alpha) - \widetilde{Z}_{Lr}^i(\alpha)|^2 \mathrm{d}\alpha, \tag{18}
$$

$$
D_{Rl} = \sum_{i=1}^M \int_0^{h^i} |\widetilde{Y}_{Rl}^i(\alpha) - \widetilde{Z}_{Rl}^i(\alpha)|^2 \mathrm{d}\alpha, \tag{19}
$$

$$
D_{Rr} = \sum_{i=1}^M \int_0^1 |\widetilde{Y}_{Rr}^i(\alpha) - \widetilde{Z}_{Rr}^i(\alpha)|^2 \mathrm{d}\alpha. \tag{20}
$$

Let us take the computation of D_{Ll} for example. Notice that

$$
\widetilde{Y}_{Ll}^i(\alpha) = \frac{1}{k_{Ll}^i}\alpha + \widetilde{Y}_{Ll}^i(0), \tag{21}
$$

$$
\widetilde{Z}_{Ll}^i(\alpha) = \frac{1}{k_{Ll}}\alpha + \widetilde{Z}_{Ll}^i(0). \tag{22}
$$

where k_{Ll}^i, $\widetilde{Y}_{Ll}^i(0)$ are known, and k_{Ll}, $\widetilde{Z}_{Ll}^i(0)$ are the parameters need to be determined.

If we denote $\widetilde{k}_{Ll}^i = \frac{1}{k_{Ll}^i}$, $\widetilde{k}_{Ll} = \frac{1}{k_{Ll}}$, $b_{Ll}^i = \widetilde{Y}_{Ll}^i(0)$, and $\widetilde{b}_{Ll}^i = \widetilde{Z}_{Ll}^i(0)$, then we have

$$
\widetilde{Y}_{Ll}^i(\alpha) = \widetilde{k}_{Ll}^i\alpha + b_{Ll}^i, \tag{23}
$$

$$
\widetilde{Z}_{Ll}^i(\alpha) = \widetilde{k}_{Ll}\alpha + \widetilde{b}_{Ll}^i. \tag{24}
$$

And, to determine k_{Ll}, $\widetilde{Z}_{Ll}^i(0)$, we can firstly determine $\widetilde{k}_{Ll}$ and $\widetilde{b}_{Ll}^i$.

From (23) and (24), we have

$$
\begin{aligned}
&\int_0^1 |\widetilde{Y}_{Ll}^i(\alpha) - \widetilde{Z}_{Ll}^i(\alpha)|^2 \mathrm{d}\alpha \\
&= \int_0^1 [(\widetilde{k}_{Ll}^i\alpha + b_{Ll}^i) - (\widetilde{k}_{Ll}\alpha + \widetilde{b}_{Ll}^i)]^2 \mathrm{d}\alpha \\
&= \frac{1}{3}\widetilde{k}_{Ll}^2 + \widetilde{k}_{Ll}\widetilde{b}_{Ll}^i + (\widetilde{b}_{Ll}^i)^2 + p_{Ll}^i\widetilde{k}_{Ll} + q_{Ll}^i\widetilde{b}_{Ll}^i + r_{Ll}^i, \quad (25)
\end{aligned}
$$

where

$$
\begin{cases}
p_{Ll}^i &= -\frac{2}{3}\widetilde{k}_{Ll}^i - b_{Ll}^i, \\
q_{Ll}^i &= -\widetilde{k}_{Ll}^i - 2b_{Ll}^i, \\
r_{Ll}^i &= \frac{2}{3}\widetilde{k}_{Ll}^i + \widetilde{k}_{Ll}^i b_{Ll}^i + (b_{Ll}^i)^2.
\end{cases}
\tag{26}
$$

Hence, from (17) and (25), we can derive that

$$
D_{Ll} = \boldsymbol{w}_{Ll}^{\mathrm{T}}\mathbf{Q}_{Ll}\boldsymbol{w}_{Ll} + \mathbf{R}_{Ll}^{\mathrm{T}}\boldsymbol{w}_{Ll} + c_{Ll} \tag{27}
$$

in which

$$
\begin{cases}
\boldsymbol{w}_{Ll} &= [\widetilde{k}_{Ll}, \widetilde{b}_{Ll}^1, \widetilde{b}_{Ll}^2, \cdots, \widetilde{b}_{Ll}^M]^{\mathrm{T}}, \\
\mathbf{Q}_{Ll} &= \begin{bmatrix} \frac{M}{3} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & 1 & 0 & \cdots & 0 \\ \frac{1}{2} & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \frac{1}{2} & 0 & 0 & \cdots & 1 \end{bmatrix}, \\
\mathbf{R}_{Ll} &= \left[\sum_{i=1}^M p_{Ll}^i, q_{Ll}^1, q_{Ll}^2, \cdots, q_{Ll}^M\right]^{\mathrm{T}}.
\end{cases}
\tag{28}
$$

where $\boldsymbol{w}_{Ll}$ is a vector of the parameters to be determined in $\widetilde{Z}_{Ll}^1, \cdots, \widetilde{Z}_{Ll}^M$ which represent the left sides of the UMFs of the IT2 FSs $\widetilde{Z}^1, \cdots, \widetilde{Z}^M$.

In the same way, define the parameters to be determined in $\widetilde{Z}_{Lr}^1, \cdots, \widetilde{Z}_{Lr}^M$ as the vector $\boldsymbol{w}_{Lr}$, the parameters to be determined in $\widetilde{Z}_{Rl}^1, \cdots, \widetilde{Z}_{Rl}^M$ as the vector $\boldsymbol{w}_{Rl}$, and the parameters to be determined in $\widetilde{Z}_{Rr}^1, \cdots, \widetilde{Z}_{Rr}^M$ as the vector $\boldsymbol{w}_{Rr}$, i.e.

$$
\begin{cases}
\boldsymbol{w}_{Lr} &= [\widetilde{k}_{Lr}, \widetilde{b}_{Lr}^1, \widetilde{b}_{Lr}^2, \cdots, \widetilde{b}_{Lr}^M]^{\mathrm{T}}, \\
\boldsymbol{w}_{Rl} &= [\widetilde{k}_{Rl}, \widetilde{b}_{Rl}^1, \widetilde{b}_{Rl}^2, \cdots, \widetilde{b}_{Rl}^M]^{\mathrm{T}}, \\
\boldsymbol{w}_{Rr} &= [\widetilde{k}_{Rr}, \widetilde{b}_{Rr}^1, \widetilde{b}_{Rr}^2, \cdots, \widetilde{b}_{Rr}^M]^{\mathrm{T}}.
\end{cases}
\tag{29}
$$

Similarly, we can derive the following equations:

$$
\begin{cases}
D_{Lr} &= \boldsymbol{w}_{Lr}^{\mathrm{T}}\mathbf{Q}_{Lr}\boldsymbol{w}_{Lr} + \mathbf{R}_{Lr}^{\mathrm{T}}\boldsymbol{w}_{Lr} + c_{Lr}, \\
D_{Rl} &= \boldsymbol{w}_{Rl}^{\mathrm{T}}\mathbf{Q}_{Rl}\boldsymbol{w}_{Rl} + \mathbf{R}_{Rl}^{\mathrm{T}}\boldsymbol{w}_{Rl} + c_{Rl}, \\
D_{Rr} &= \boldsymbol{w}_{Rr}^{\mathrm{T}}\mathbf{Q}_{Rr}\boldsymbol{w}_{Rr} + \mathbf{R}_{Rr}^{\mathrm{T}}\boldsymbol{w}_{Rr} + c_{Rr},
\end{cases}
\tag{30}
$$

where $\mathbf{Q}_{Lr}, \mathbf{R}_{Lr}, c_{Lr}, \mathbf{Q}_{Rl}, \mathbf{R}_{Rl}, c_{Rl}, \mathbf{Q}_{Rr}, \mathbf{R}_{Rr}, c_{Rr}$ can be obtained as the derivations of $\mathbf{Q}_{Ll}, \mathbf{R}_{Ll}, c_{Ll}$ in D_{Ll} and are omitted here.

From previous discussion, the distance measure $D(\widetilde{Y}^1, \cdots, \widetilde{Y}^M, \widetilde{Z}^1, \cdots, \widetilde{Z}^M)$ can be computed as

$$D(\widetilde{Y}^1, \cdots, \widetilde{Y}^M, \widetilde{Z}^1, \cdots, \widetilde{Z}^M)$$
$$= D_{Ll} + D_{Lr} + D_{Rl} + D_{Rr} = \boldsymbol{w}^{\mathrm{T}}\mathbf{Q}\boldsymbol{w} + \mathbf{R}^{\mathrm{T}}\boldsymbol{w} + c, \quad (31)$$

where

$$\begin{cases} \boldsymbol{w} &= [\boldsymbol{w}_{Ll}^{\mathrm{T}}, \boldsymbol{w}_{Rr}^{\mathrm{T}}, \boldsymbol{w}_{Lr}^{\mathrm{T}}, \boldsymbol{w}_{Rl}^{\mathrm{T}}]^{\mathrm{T}}, \\ \mathbf{Q} &= \begin{bmatrix} \mathbf{Q}_{Ll} & & & \\ & \mathbf{Q}_{Rr} & & \\ & & \mathbf{Q}_{Lr} & \\ & & & \mathbf{Q}_{Rl} \end{bmatrix}, \\ \mathbf{R} &= [\mathbf{R}_{Ll}^{\mathrm{T}}, \mathbf{R}_{Rr}^{\mathrm{T}}, \mathbf{R}_{Lr}^{\mathrm{T}}, \mathbf{R}_{Rl}^{\mathrm{T}}]^{\mathrm{T}}, \\ c &= c_{Ll} + c_{Rr} + c_{Lr} + c_{Rl}. \end{cases} \quad (32)$$

in which $\boldsymbol{w}$ is the vector of the parameters to be determined for $\widetilde{Z}^1, \cdots, \widetilde{Z}^M$, and c is a constant.

The approximations $\widetilde{Z}^1, \cdots, \widetilde{Z}^M$ are trapezoidal IT2 FSs too. So, the vertexes of these IT2 FSs still need to satisfy the following constraints:

$$\begin{cases} \widetilde{Z}_{Ll}^i(0) \le \widetilde{Z}_{Ll}^i(1), \\ \widetilde{Z}_{Rr}^i(1) \le \widetilde{Z}_{Rr}^i(0), \\ \widetilde{Z}_{Lr}^i(0) \le \widetilde{Z}_{Lr}^i(h^i), \\ \widetilde{Z}_{Rl}^i(h^i) \le \widetilde{Z}_{Rl}^i(0), \\ \widetilde{Z}_{Ll}^i(1) \le \widetilde{Z}_{Rr}^i(1), \\ \widetilde{Z}_{Ll}^i(h^i) \le \widetilde{Z}_{Lr}^i(h^i), \\ \widetilde{Z}_{Lr}^i(h^i) \le \widetilde{Z}_{Rl}^i(h^i), \\ \widetilde{Z}_{Rl}^i(h^i) \le \widetilde{Z}_{Rr}^i(h^i), \\ \widetilde{Z}_{Ll}^i(0) \le \widetilde{Z}_{Lr}^i(0), \\ \widetilde{Z}_{Rl}^i(0) \le \widetilde{Z}_{Rr}^i(0), \\ i = 1, 2, \cdots, M \end{cases} \Leftrightarrow \begin{cases} \tilde{k}_{Ll} \ge 0, \\ \tilde{k}_{Rr} \le 0, \\ \tilde{k}_{Lr} \ge 0, \\ \tilde{k}_{Rl} \le 0, \\ \tilde{k}_{Ll} + \tilde{b}_{Ll}^i \le \tilde{k}_{Rr} + \tilde{b}_{Rr}^i, \\ \tilde{k}_{Ll}h^i + \tilde{b}_{Ll}^i \le \tilde{k}_{Lr}h^i + \tilde{b}_{Lr}^i, \\ \tilde{k}_{Lr}h^i + \tilde{b}_{Lr}^i \le \tilde{k}_{Rl}h^i + \tilde{b}_{Rl}^i, \\ \tilde{k}_{Rl}h^i + \tilde{b}_{Rl}^i \le \tilde{k}_{Rr}h^i + \tilde{b}_{Rr}^i, \\ \tilde{b}_{Ll}^i \le \tilde{b}_{Lr}^i, \\ \tilde{b}_{Rl}^i \le \tilde{b}_{Rr}^i, \\ i = 1, 2, \cdots, M \end{cases} \quad (33)$$

where $h^i = h_{\widetilde{Z}^i} = h_{\widetilde{Y}^i}$. We should set the height of trapezoidal IT2 FS $\widetilde{Z}^i$ to be the height of $\widetilde{Y}^i$.

Obviously, the above inequality constraints are linear and can be rewritten as the following matrix form

$$\mathbf{A}\boldsymbol{w} \le \mathbf{0}. \quad (34)$$

From above discussion, the construction of the slope-consistent trapezoidal IT2 FSs can be transformed to the following constrained linear-quadratic optimization problem

$$\begin{cases} \min_{\boldsymbol{w}} & \boldsymbol{w}^{\mathrm{T}}\mathbf{Q}\boldsymbol{w} + \mathbf{R}^{\mathrm{T}}\boldsymbol{w} + c \\ s.t. & \mathbf{A}\boldsymbol{w} \le \mathbf{0} \end{cases} \quad (35)$$

C. Algorithm

From above discussion, we can use the algorithm shown in Fig. 3 to construct the slope-consistent trapezoidal IT2 FSs.

After solving the constrained linear-quadratic optimization problem, we can get the approximation $\widetilde{Z}^i$ for $\widetilde{Y}^i$ as

$$\begin{cases} \overline{\mu}_{\widetilde{Z}^i}(x) &= \overline{\mu}_{\widetilde{Z}^i}(x, \tilde{b}_{Ll}^i, \tilde{k}_{Ll} + \tilde{b}_{Ll}^i, \tilde{k}_{Rr} + \tilde{b}_{Rr}^i, \tilde{b}_{Rr}^i), \\ \underline{\mu}_{\widetilde{Z}^i}(x) &= \underline{\mu}_{\widetilde{Z}^i}(x, \tilde{b}_{Lr}^i, \tilde{k}_{Lr}h^i + \tilde{b}_{Lr}^i, \tilde{k}_{Rl}h^i + \tilde{b}_{Rl}^i, \tilde{b}_{Rl}^i, h^i) \end{cases} \quad (36)$$
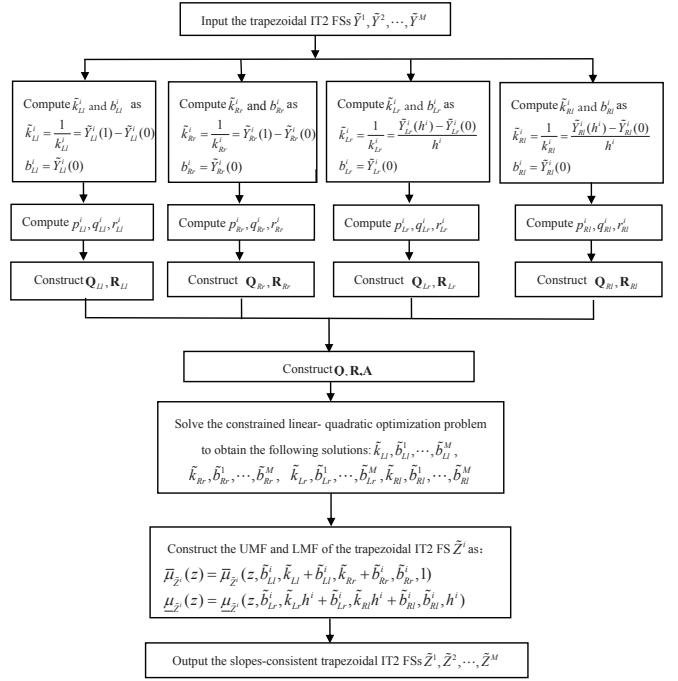

Fig. 3. The algorithm for constructing the slope-consistent trapezoidal IT2 FSs

V. EXAMPLE AND DISCUSSIONS

This section first gives an example to verify the effectiveness of the proposed algorithm. And then, discussions will be made.

A. Example

Consider three trapezoidal IT2 FSs $\widetilde{Y}^1, \widetilde{Y}^2, \widetilde{Y}^3$ as shown in Fig. 4 (grey areas) with the following UMFs and LMFs:

$$\widetilde{Y}^1 : \begin{cases} \overline{\mu}_{\widetilde{Y}^1}(x) &= \overline{\mu}_{\widetilde{Y}^1}(x, 0, 1, 3, 4) \\ \underline{\mu}_{\widetilde{Y}^1}(x) &= \underline{\mu}_{\widetilde{Y}^1}(x, 0.5, 1, 2.5, 3, 0.8) \end{cases}$$

$$\widetilde{Y}^2 : \begin{cases} \overline{\mu}_{\widetilde{Y}^2}(x) &= \overline{\mu}_{\widetilde{Y}^2}(x, 3, 5, 5.5, 7.5) \\ \underline{\mu}_{\widetilde{Y}^2}(x) &= \underline{\mu}_{\widetilde{Y}^2}(x, 4, 5, 5, 6.5, 0.7) \end{cases}$$

$$\widetilde{Y}^3 : \begin{cases} \overline{\mu}_{\widetilde{Y}^3}(x) &= \overline{\mu}_{\widetilde{Y}^3}(x, 6, 7, 9, 10) \\ \underline{\mu}_{\widetilde{Y}^3}(x) &= \underline{\mu}_{\widetilde{Y}^3}(x, 7, 8, 9, 9.5, 0.6) \end{cases}$$

Using the proposed algorithm, the constructed slope-consistent trapezoidal IT2 FSs $\widetilde{Z}^1, \widetilde{Z}^2, \widetilde{Z}^3$ have the following UMFs and LMFs:

$$\widetilde{Z}^1 : \begin{cases} \overline{\mu}_{\widetilde{Z}^1}(x) &= \overline{\mu}_{\widetilde{Z}^1}(x, -0.17, 1.17, 2.83, 4.17) \\ \underline{\mu}_{\widetilde{Z}^1}(x) &= \underline{\mu}_{\widetilde{Z}^1}(x, 0.31, 1.19, 2.29, 3.21, 0.8) \end{cases}$$

$$\widetilde{Z}^2 : \begin{cases} \overline{\mu}_{\widetilde{Z}^2}(x) &= \overline{\mu}_{\widetilde{Z}^2}(x, 3.33, 4.67, 5.83, 7.17) \\ \underline{\mu}_{\widetilde{Z}^2}(x) &= \underline{\mu}_{\widetilde{Z}^2}(x, 4.12, 4.88, 5.35, 6.15, 0.7) \end{cases}$$

$$\widetilde{Z}^3 : \begin{cases} \overline{\mu}_{\widetilde{Z}^3}(x) &= \overline{\mu}_{\widetilde{Z}^3}(x, 5.83, 7.17, 8.83, 10.17) \\ \underline{\mu}_{\widetilde{Z}^3}(x) &= \underline{\mu}_{\widetilde{Z}^3}(x, 7.17, 7.83, 8.90, 9.60, 0.6) \end{cases}$$

The constructed IT2 FSs $\widetilde{Z}^1, \widetilde{Z}^2, \widetilde{Z}^3$ are also shown in Fig. 4 using dotted areas. From this figure, we can observe that the approximated IT2 FSs are trapezoidal and have consistent slopes.
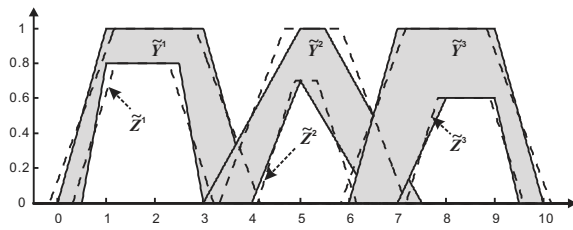
Fig. 4. Trapezoidal IT2 FSs without and with consistent slopes

B. Discussion

In applications, if the word library is manually constructed, the slope-consistent trapezoidal IT2 FSs can be easily achieved. Sometimes, the IT2 FSs in the word library are obtained by data-driven method. Such IT2 FSs are usually not slope-consistent. In this case, we should use this algorithm to modify or approximate them. One thing to be mentioned is that only the IT2 FSs in the consequents of the fuzzy rules, rather than all the IT2 FSs in the word library, need to be modified. This can be done separately before the running of the perceptual reasoning process so that it does not burden or complicate the reasoning process.

As shown in [27], when all IT2 FSs are trapezoidal, the PR output is approximately trapezoidal, so we can compute $\alpha = 0$ and $\alpha = 1$, and then connect them to obtain the complete trapezoidal IT2 FS. This is effective enough for decision-making or judgement problems, as the perceptual reasoning will run only once in such applications and this approximation error will not be large. However, when we adopt the CWW paradigm to model complex dynamic systems (e.g. the linguistic dynamic systems proposed in [9], [10]), the perceptual reasoning will run recursively, so the approximation error will be transported and can not be predicted and controlled. Using the proposed method, errors will occur before the reasoning and can be eliminated through mapping the output trapezoidal IT2 FS to the original word in the library.

VI. CONCLUSIONS

This paper studied how to simplify the reasoning of the PR method by avoiding the α-cuts computation of the output word. First, we gave a novel property of the PR method, which shows that trapezoidal output II2 FSs can be obtained if trapezoidal IT2 FSs with consistent slopes are adopted in the consequent parts of the fuzzy IF-THEN rules. Then, we explored how to derive slope-consistent trapezoidal IT2 FSs from trapezoidal IT2 FSs without consistent slopes. We found that this approximation problem can be transformed as a constrained linear-quadratic optimization problem which can be solved by popular optimization algorithms. An example was also given to show how to obtain the slope-consistent trapezoidal IT2 FSs. In the future, we will apply the simplified PR method to construct the linguistic dynamic systems for some real-world complex applications.

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