Letter

New Second-Level-Discrete Zeroing Neural Network for Solving Dynamic Linear System

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Dear Editor,

This letter deals with a new second-level-discretization method with higher precision than the traditional first-level-discretization method. Specifically, the traditional discretization method utilizes the first-order time derivative information, and it is termed first-level-discretization method. By contrast, the new discretization method not only utilizes the first-order time derivative information, but also makes use of the second-order derivative information. By combining the new second-level-discretization method with zeroing neural network (ZNN), the second-level-discrete ZNN (SLDZNN) model is proposed to solve dynamic (i.e., time-variant or time-dependent) linear system. Numerical experiments and application to angle-of-arrival (AoA) localization show the effectiveness and superiority of the SLDZNN model.

Because of the real-time requirement, solving dynamic problems becomes more and more important [1]–[3], which has been applied in many fields such as robot control [4]–[6] and AoA localization [7]. ZNN [8]–[10] is a special type of recurrent neural network (RNN) [11]. In recent years, ZNN have shown its effectiveness for dynamic problems solving, such as for dynamic nonlinear inequalities solving, dynamic linear system solving, and dynamic quadratic minimization. For exmaple, a ZNN (CZNN) model with definable convergence time was developed for solving time-variant linear matrix equations in [2]. By performing the following straightforward steps: 1) Defining the error function; 2) Applying the design formula; and 3) Simplifying the result, one could eventually develop the continuous CZNN model for dynamic problems solving.

Furthermore, for convenient program implementation, the traditional first-level-discretization method [10], [12] is usually applied to CZNN discretization, and the first-level-discrete ZNN (FLDZNN) model could be developed. For instance, in time point $t_k = k\varsigma$ with ς being the sampling period, the FLDZNN model computes the dynamic solution $s(t_{k+1})$ on the basis of $s(t_k)$ and $\dot{s}(t_k)$. That is, the solution in time point t_{k+1} is computed by utilizing the information in time point t_k, and the precision is $O(\varsigma^2)$.

Generally, to improve the precision, the traditional way is to utilize more information at more time points. In recent years, many high-precision discrete methods utilizing information at several time points have been developed, and many high-precision discrete ZNN (DZNN) models have been obtained. For instance, a four-points first-level-discretization method is designed in [10]. A six-points first-level-discretization method is developed in [3]. However, for rapidly dynamic systems, the information before several time points may be valueless. Therefore, utilizing more information at a few time points to realize higher precision is meaningful. Note that the second-order derivative containing the rapidly changing information, which is very valuable. Upon careful study, we find there is another way to improve the precision. The traditional discrete method utilizes the first-order time derivative. If we can utilize the second-order time derivative, the model precision may be further improved. As a result, this letter proposes a new second-level-discretization method and further develops the SLDZNN model. In time point t_k, the SLDZNN model computes the dynamic solution $s(t_{k+1})$ on the basis of $s(t_k)$, $\dot{s}(t_k)$, and $\ddot{s}(t_k)$. As a result, the precision is improved as $O(\varsigma^3)$.

The main contributions of this letter are presented as below.

1) To our knowledge, the second-level-discretization method is proposed for CZNN discretization for the first time, and it is more effective than the traditional first-level-discretization method.

2) By utilizing the ZNN design formula twice and combining the second-level-discretization method, the SLDZNN model is proposed to solve dynamic linear system, and detailed theoretical analyses are presented.

3) Numerical experiments considering a specific example and application to AoA localization are displayed to illustrate the efficiency and superiority of the second-level-discretization method and SLDZNN model.

Problem statement: Consider dynamic linear system during time duration $[0, T]$

$$Y(t)\mathbf{x}(t) = \mathbf{z}(t) \tag{1}$$

in which coefficient matrix $Y(t) \in \mathbb{R}^{n\times n}$ and coefficient vector $\mathbf{z}(t) \in \mathbb{R}^n$. The vector $\mathbf{x}(t) \in \mathbb{R}^n$ is the dynamic solution to be attained. Besides, the matrix $Y(t)$ is always nonsingular to guarantee that (1) is solvable.

CZNN model: To solve dynamic linear system (1), the CZNN model [9] is developed in the following theorem.

Theorem 1: Let $\gamma > 0$ be a ZNN design parameter. For nonsingular matrix $A(t)$, the CZNN model for solving dynamic linear system (1) is developed as

$$\dot{\mathbf{x}}(t) = Y^{-1}(t)\Big(\dot{\mathbf{z}}(t) - \dot{Y}(t)\mathbf{x}(t) - \gamma\big(Y(t)\mathbf{x}(t) - \mathbf{z}(t)\big)\Big) \tag{2}$$

and the synthesized $\mathbf{x}(t)$ converges to the theoretical continuous solution $\mathbf{x}^*(t)$ as $t >> 0$.

Proof: To make (1) Hold true, perform straightforward steps: 1) defining $\mathbf{e}(t) = Y(t)\mathbf{x}(t) - \mathbf{z}(t)$; 2) Applying $\dot{\mathbf{e}}(t) = -\gamma\mathbf{e}(t)$; and 3) Simplifying result. We get

$$Y(t)\dot{\mathbf{x}}(t) + \dot{Y}(t)\mathbf{x}(t) - \dot{\mathbf{z}}(t) = -\gamma\big(Y(t)\mathbf{x}(t) - \mathbf{z}(t)\big). \tag{3}$$

By rearranging the above result and doing an inverse operation, (2) is thus obtained. Evidently, CZNN model (2) is originated from ZNN design formula $\dot{\mathbf{e}}(t) = -\gamma\mathbf{e}(t)$. Choose Lyapunov functional candidate as $L(t) = 1/2\mathbf{e}^T(t)\mathbf{e}(t)$. Computing the derivative of $L(t)$ yields $\dot{L}(t) = \mathbf{e}^T(t)\dot{\mathbf{e}}(t) = -\gamma\mathbf{e}^T(t)\mathbf{e}(t)$. Evidently, we have $L(t) \geq 0$ and $\dot{L}(t) \leq 0$. If $\dot{L}(t) = 0$, $\mathbf{e}(t) = \mathbf{0}$. Therefore, when $t >> 0$, we have $\mathbf{e}(t) \to \mathbf{0}$. That is, $\mathbf{x}(t)$ synthesized by (2) converges to $\mathbf{x}^*(t)$ as $t >> 0$. ■

Traditional FLDZNN model: To generate discrete models, the traditional first-level-discretization method [12] and the corresponding FLDZNN model are developed.

Lemma 1: Let $\varsigma > 0$ be the sampling period. The traditional first-level-discretization method [12] is presented as

$$s(t_{k+1}) = s(t_k) + \varsigma\dot{s}(t_k) + O(\varsigma^2). \tag{4}$$

For simplicity, we use $\mathbf{x}_k$ to denote $\mathbf{x}(t_k)$, and other symbols have similar meanings. In addition, we use $\mathbf{O}(\varsigma^2)$ to denote the vector form of $O(\varsigma^2)$. By using (4) for model discretization, the following theorem develops the FLDZNN model.

Theorem 2: Let $\varsigma > 0$ be the sampling period and $\doteq$ denote the computational assignment operator. The FLDZNN model is developed as

$$\dot{\mathbf{x}}_k = Y_k^{-1}(\dot{\mathbf{z}}_k - \dot{Y}_k\mathbf{x}_k - \gamma(Y_k\mathbf{x}_k - \mathbf{z}_k)) \tag{5a}$$

$$\mathbf{x}_{k+1} \doteq \mathbf{x}_k + \varsigma\dot{\mathbf{x}}_k \tag{5b}$$

and the synthesized $\mathbf{x}_{k+1}$ converges to the theoretical discrete solu-

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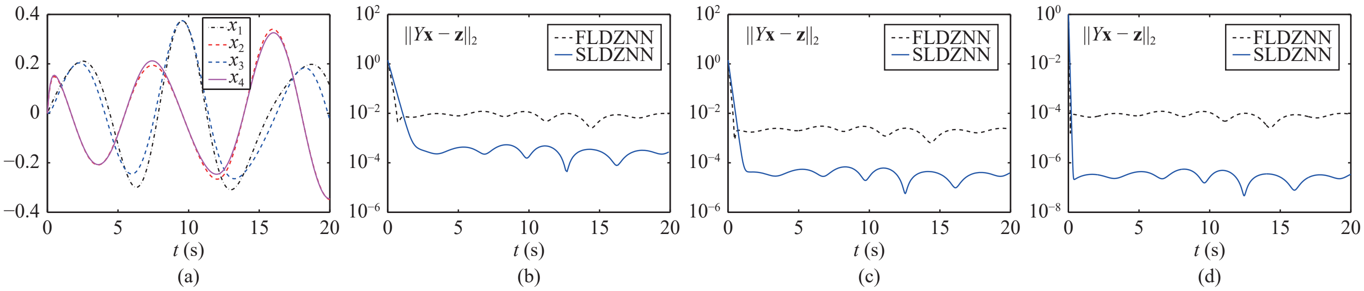

Fig. 1. Synthesized numerical results from FLDZNN model (5) and SLDZNN model (7) for dynamic linear system solving in Example 1. (a) Solution trajectories generated by SLDZNN model (7) with $\varsigma = 0.1$ s; (b) Residual errors with $\varsigma = 0.1$ s; (c) Residual errors with $\varsigma = 0.05$ s; (d) Residual errors with $\varsigma = 0.01$ s.

tion $\mathbf{x}^*_{k+1}$ with $\mathbf{O}(\varsigma^2)$ precision.

Proof: Evidently, (5a) is attained by sampling (2) in time point t_k. Meanwhile, (5b) is the vector form of (4). On the basis of Theorem 1 and Lemma 1, $\mathbf{x}_{k+1}$ synthesized by FLDZNN model (5) converges to $\mathbf{x}^*_{k+1}$ with $\mathbf{O}(\varsigma^2)$ precision. This completes the proof.

New SLDZNN model: On the basis of the previous analysis, the deeper-level information in time point t_k may be utilized to get the higher precision. The new second-level-discretization method is thus proposed in the following theorem.

Taylor expansion [12] is presented as

$$s(t_{k+1}) = s(t_k) + \varsigma\dot{s}(t_k) + \frac{1}{2}\varsigma^2\ddot{s}(t_k) + \frac{1}{6}\varsigma^3\dddot{s}(t_k) + O(\varsigma^4).$$

With $O(\varsigma^3)$ absorbing the remainder terms, the second-level-discretization method (6) is obtained as follows.

Lemma 2: Let $\varsigma > 0$ be the sampling period. The new second-level-discretization method is developed as

$$s(t_{k+1}) = s(t_k) + \varsigma\dot{s}(t_k) + \frac{1}{2}\varsigma^2\ddot{s}(t_k) + O(\varsigma^3). \tag{6}$$

By utilizing ZNN method twice and applying the second-level-discretization method (6), the SLDZNN model is proposed in the following theorem.

Theorem 3: Let $\varsigma > 0$ be the sampling period and $\doteq$ denote the computational assignment operator. The SLDZNN model is developed as

$$\dot{\mathbf{x}}_k = Y_k^{-1}(\dot{\mathbf{z}}_k - \dot{Y}_k\mathbf{x}_k - \gamma(Y_k\mathbf{x}_k - \mathbf{z}_k)) \tag{7a}$$

$$\ddot{\mathbf{x}}_k = Y_k^{-1}(\ddot{\mathbf{z}}_k - 2\dot{Y}_k\dot{\mathbf{x}}_k - \ddot{Y}_k\mathbf{x}_k + \gamma^2(Y_k\mathbf{x}_k - \mathbf{z}_k)) \tag{7b}$$

$$\mathbf{x}_{k+1} \doteq \mathbf{x}_k + \varsigma\dot{\mathbf{x}}_k + \frac{1}{2}\varsigma^2\ddot{\mathbf{x}}_k \tag{7c}$$

and the synthesized $\mathbf{x}_{k+1}$ converges to the theoretical discrete solution $\mathbf{x}^*_{k+1}$ with $\mathbf{O}(\varsigma^3)$ precision.

Proof: According to the previous analyse, (3) is obtained by utilizing ZNN method [9]. Then, (7a) can be directly developed by sampling (3) in time point t_k. To get the second-order derivative, the another error function is defined according to (3)

$$\epsilon(t) = Y(t)\dot{\mathbf{x}}(t) + \dot{Y}(t)\mathbf{x}(t) - \dot{\mathbf{z}}(t) + \gamma(Y(t)\mathbf{x}(t) - \mathbf{z}(t)).$$

Utilizing ZNN design formula $\dot{\epsilon}(t) = -\gamma\epsilon(t)$ yields

$$\begin{aligned} &Y(t)\ddot{\mathbf{x}}(t) + \dot{Y}(t)\dot{\mathbf{x}}(t) + \dot{Y}(t)\dot{\mathbf{x}}(t) + \ddot{Y}(t)\mathbf{x}(t) - \ddot{\mathbf{z}}(t) \\ &\quad + \gamma\big(\dot{Y}(t)\mathbf{x}(t) + Y(t)\dot{\mathbf{x}}(t) - \dot{\mathbf{z}}(t)\big) \\ &= -\gamma\Big(\dot{Y}(t)\mathbf{x}(t) + Y(t)\dot{\mathbf{x}}(t) - \dot{\mathbf{z}}(t) + \gamma\big(Y(t)\mathbf{x}(t) - \mathbf{z}(t)\big)\Big). \end{aligned}$$

Simplifying the above result in time point t_k yields (7b). Furthermore, by applying new second-level-discretization method (6) to discretize CZNN model (2), (7c) is directly obtained. On the basis of ZNN knowledge [9] and Lemma 2, $\mathbf{x}_{k+1}$ synthesized by SLDZNN model (7) converges to $\mathbf{x}^*_{k+1}$ with $\mathbf{O}(\varsigma^3)$ precision. ■

Experiments verification: This section first provides a numerical example to illustrate the effectiveness of the proposed SLDZNN model (7). Then, an application to AoA localization is also conducted for illustrating the practicability of SLDZNN model (7).

Example 1: Consider dynamic linear system (1) during time duration $[0, 20]$ s with $Y(t) =$

$$\begin{bmatrix} \cos(0.4t)+4 & \sin(0.4t) & \sin(0.4t) & \sin(0.4t) \\ \cos(0.2t) & \cos(0.2t)+4 & \sin(0.2t) & \sin(0.2t) \\ \sin(0.4t) & \cos(0.4t) & \cos(0.4t)+4 & \sin(0.4t) \\ \sin(0.2t) & \cos(0.2t) & \cos(0.2t) & \sin(0.2t)+4 \end{bmatrix}$$

and $\mathbf{z}(t) = \begin{bmatrix} \sin(0.8t) & \cos(0.8t) & \sin(0.8t) & \cos(0.8t) \end{bmatrix}^T$.

On the basis of the previous experience [1], [3], [10], $h = \gamma\varsigma$ is usually set as a constant, e.g., $h = 0.5$ in this letter. For comparison, FLDZNN model (5) and SLDZNN model (7) are both applied, and the corresponding numerical results are presented in Fig. 1. By setting the sampling period $\varsigma = 0.1$ s, the trajectories of the dynamic solution synthesized by SLDZNN model (7) are shown in Fig. 1(a). Meanwhile, Fig. 1(b) illustrates that the residual error synthesized by SLDZNN model (7) is between 10^{-3} and 10^{-4}. By contrast, Fig. 1(b) also shows that the residual error synthesized by FLDZNN model (5) is just about 10^{-2}. Meanwhile, when setting sampling period $\varsigma =$ 0.05 s and 0.01 s, the residual errors attained from two DZNN models are displayed in Figs. 1(c) and 1(d), respectively. For instance, with $\varsigma = 0.01$ s, the residual error generated by SLDZNN model (7) is between 10^{-6} and 10^{-7}. That is, the precision will be increased by 10^3 times if the sampling period decreases by 10 times. In summary, the precision of SLDZNN model (7) is indeed $\mathbf{O}(\varsigma^3)$ while the precision of FLDZNN model (5) is $\mathbf{O}(\varsigma^2)$, which illustrates the superiority of SLDZNN model (7).

Example 2: The AoA localization system consists of a moving object and two moving stations (simply termed MS1 and MS2). Note that $(u_1(t), v_1(t))$ and $(u_2(t), v_2(t))$ represent the positions of MS1 and MS2, respectively, which are both known. AoA system uses array antenna to detect direction of arrival in real time. Specifically, $\theta_1(t)$ and $\theta_2(t)$ mean the arrival angles of the object to two stations, respectively. The target of AoA localization is to compute (estimate) the real-time position on the basis of two arrival angles. According to the previous work [7], the problem of AoA localization system can be formulated as dynamic linear system (1) with

$$Y(t) = \begin{bmatrix} \tan(\theta_1(t)) & -1 \\ \tan(\theta_2(t)) & -1 \end{bmatrix}, \quad \mathbf{x}(t) = \begin{bmatrix} u_3(t) \\ v_3(t) \end{bmatrix}$$

$$\mathbf{z}(t) = \begin{bmatrix} u_1(t)\tan(\theta_1(t)) - v_1(t) \\ u_2(t)\tan(\theta_2(t)) - v_2(t) \end{bmatrix}$$

where $(u_3(t), v_3(t))$ represents the position of the moving object. Consider the AoA localization during time duration $[0, 30]$ s.

By setting $h = 0.5$ and $\varsigma = 0.1$ s, Fig. 2(a) through Fig. 2(d) display some important results generated by SLDZNN model (7). The AoA localization is successfully completed. Specifically, Fig. 2(a) presents the profiles of two arrival angles. Fig. 2(b) illustrates the real-time positions of two moving stations and the predicted localization of the moving object. We can conclude that the predicted trajectory of the moving object basically coincides with the actual trajectory. For details, Fig. 2(c) shows the predicted position of the moving object, which converges to the actual position quickly. To further analyze the localization precision, the residual error

$$e_{\mathrm{r}} = \left\| \begin{bmatrix} \tan(\theta_1(t)) & -1 \\ \tan(\theta_2(t)) & -1 \end{bmatrix} \begin{bmatrix} u_3(t) \\ v_3(t) \end{bmatrix} - \begin{bmatrix} u_1(t)\tan(\theta_1(t)) - v_1(t) \\ u_2(t)\tan(\theta_2(t)) - v_2(t) \end{bmatrix} \right\|_2$$

is used as the index. With $\varsigma = 0.1$ s, 0.05 s, and 0.01 s, Figs. 2(d)−2(f)

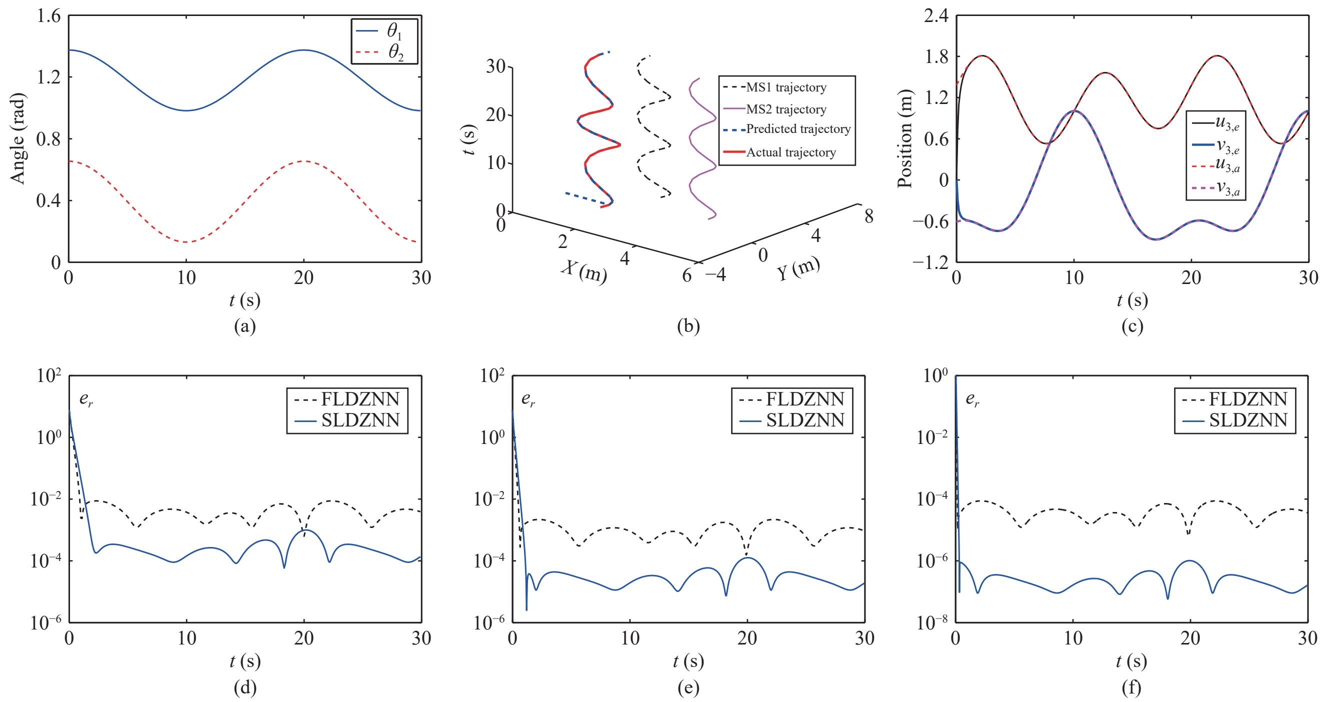

Fig. 2. Synthesized numerical results from FLDZNN model (5) and SLDZNN model (7) for AoA localization in Example 2. (a) Two arrival angles; (b) Predicted localization of moving object generated by SLDZNN model (7) with $\varsigma = 0.1$ s; (c) Predicted trajectory of moving object computed by SLDZNN model (7) when $\varsigma = 0.1$ s; (d) Residual errors when $\varsigma = 0.1$ s; (e) Residual errors when $\varsigma = 0.05$ s; (f) Residual errors when $\varsigma = 0.01$ s.

respectively display the corresponding residual errors generated by two discrete models. SLDZNN model (7) indeed has $\mathbf{O}(\varsigma^3)$ precision for dynamic linear system solving, which is one-order higher than FLDZNN model (5).

Conclusion: This letter has investigated the problem of dynamic linear system (1). Considering that the traditional discretization methods only utilize the first-order time derivative, new second-level-discretization method (6) with higher precision has been developed by utilizing the second-order derivative. Aided with second-level-discretization method (6), SLDZNN model (7) has been further proposed. Theoretical analyses and simulations have been conducted to validate the efficacy and superiority of second-level-discretization method (6) and SLDZNN model (7). Developing more second-level-discretization methods with higher precision is a future research direction. Moreover, Combining second-level-discretization methods with zeroing neural network for solving more kinds of dynamic problems is also a significant future research direction.

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