Finite-Time Stabilization for Constrained Discrete-time Systems by Using Model Predictive Control

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Abstract**—In this paper, a model predictive control (MPC) framework is proposed for finite-time stabilization of linear and nonlinear discrete-time systems subject to state and control constraints. The proposed MPC framework guarantees the finite-time convergence property by assigning the control horizon equal to the dimension of the overall system, and only penalizing the terminal cost in the optimization, where the stage costs are not penalized explicitly. A terminal inequality constraint is added to guarantee the feasibility and stability of the closed-loop system. Initial feasibility can be improved via augmentation. The finite-time convergence of the proposed MPC is proved theoretically, and is supported by simulation examples.**

Index Terms**—Constraints, deadbeat control, finite-time stabilization, model predictive control (MPC).**

I. Introduction

FOR discrete-time systems, finite-time (or deadbeat) control implies that the closed-loop state is identically zero after finite time steps. The essence of finite-time/deadbeat control for discrete-time linear systems was discovered in [1], where existence of linear finite-time control was proved to be equivalent to controllability. A well known design strategy for linear finite-time control is to assign all eigenvalues to the origin. Finite-time convergence is beneficial especially for systems requiring high accuracy and short transient process [2]. A typical situation is multi-layer (or multi-loop) system, where the lower-layer (inner-loop) subsystem is often required to track the desired input of the upper-layer (outer-loop) subsystem with sufficient accuracy [3]−[5]. Other potential applications include deadbeat observer [6], [7], adaptive model predictive control (MPC) [8], learning-based MPC [9], and adaptive identification [10], where the overall system performance can be improved by finite-time state estimation.

Fundamental and systematic design approaches of linear finite-time control are fairly mature, and can be found in [11], [12] and references therein. Apart from direct pole assignment, finite-time control can also be designed in the (unconstrained) linear quadratic regulator (LQR) framework, where the finite-time gain is calculated from optimizing the quadratic function with proper weighting matrices and boundary conditions. In [13], the finite-time control was solved in a recursive way from a singular Riccati equation, where only terminal cost was penalized. Analytical solution to singular Riccati equation was provided in [14], where the unconstrained finite-time control was considered as a special case. Another strategy for optimal finite-time control is to only penalize the state cost without penalizing the control costs [15]. In [16], a direct solution is proposed for singular Riccati equation to calculate the deadbeat gain for multi-input systems.

Comparatively, for nonlinear systems, finite-time control is a relatively open problem, and results are fairly limited. Some recent results are provided in [17], [18] and references therein. In [17], the deadbeat (or finite-time) controller is designed for fully-actuated nonlinear systems, where a specific Lyapunov function can be found for the design process and proof of deadbeat performance. In [18], a more general form of nonlinear system is considered, and a more general problem of deadbeat tracking is handled, where the nonlinear finite-time control is solved from intersections of sets. It is noted that either [17] or [18] considers state or control constraints.

In presence of constraints, if saturation is exerted directly on the previously designed unconstrained finite-time control, instability or sharp oscillation would possibly happen. Constrained deadbeat control problem was firstly investigated geometrically in [19], where the plant eigenvalues were required to be inside the closed unit circle to guarantee the global admissibility. An alternative strategy for constraint satisfaction is to apply model predictive control (MPC). MPC has been introduced to guarantee bounded finite-time convergence for some specific engineering systems, e.g., permanent-magnet synchronous motor [20], power converters [21], and vehicle charging systems [22], etc. Theoretical results on finite-time MPC are introduced in textbooks, e.g., [23] for canonical forms, [24] for input-output forms.

Currently, rigorous and systematic results on constrained finite-time MPC are somewhat limited. In [25] and [26], unconstrained finite-time constraints are applied to guarantee the finite-time convergence as extended special cases of traditional MPC for the input-output linear model. MPC via controller matching [27] can possibly be applied to achieve deadbeat performance; however, by setting weighting matrix for control to zero, exact matching is not always guaranteed. In [28] and [29], finite-time control is applied as theoretical basis

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to prove the feasibility of optimization in MPC. For (unconstrained) linear systems [28], the finite-time convergence can be achieved by properly selecting control parameters; for nonlinear systems [29], however, parameter settings are not provided to ensure finite-time convergence. An alternative strategy to implement finite-time convergence is minimum-time MPC [30], where a secondary optimization is applied to guarantee stability.

This paper is motivated by providing a solution (in the MPC framework) to the finite-time control problem, such that the finite-time convergence can be guaranteed in the presence of states and control constraints. Compared with existing results on finite-time (or deadbeat) control, main contributions of this paper include:

1) For linear discrete-time systems, results on finite-time control are abundant. However, in the presence of constraints, existing results would fail. The proposed MPC guarantees finite-time convergence for systems subject to constraints.

2) For nonlinear discrete-time systems, some recent existing results on finite-time control are often applicable for unconstrained scalar systems or fully actuated vector systems. The proposed MPC is applicable to finite-time stabilization for constrained under-actuated vector systems.

3) Feedback linearization can be applied to calculate finite-time controller. However, the diffeomorphism to achieve feedback linearization for discrete-time systems has to be computed through complicated algorithms, and it would fail in case of constraints. The proposed finite-time MPC is applicable for constrained vector systems, and calculation for diffeomorphism can be avoided.

4) The proposed finite-time MPC is applicable for multi-input systems, and its feasibility can be improved by augmentation.

This paper also provides a solution framework, where existing techniques can be directly applied for further studies, e.g., using set-theoretical methods to determine terminal region [31] and initial feasible region [32].

II. Preliminaries

A. Notations

In this paper, $\mathbb{Z}^+$ denotes the set of all positive integers; $\mathbb{C}$ denotes the set of all complex numbers; and $\mathbb{R}^n$ denotes the set of all n-dimensional real-value vectors, where $n \in \mathbb{Z}^+$.

The notation $\otimes$ denotes Cartesian product, and $\oplus$ denotes the Minkowski addition.

The time instant is denoted by $k \in \mathbb{Z}^+ \cup \{0\}$. Throughout this paper, $k = 0$ is the initial time.

B. Unconstrained Linear Finite-Time MPC

Consider the discrete-time linear system

$$x(k+1) = Ax(k) + Bu(k) \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the system state and control input, respectively; the state matrix A and the input matrix B have proper dimensions, and (A, B) is controllable.

For unconstrained single-input systems ($u \in \mathbb{R}^1$), deadbeat control can be obtained by inverse optimization [14], where only the terminal cost is penalized. In this regard, the cost function is constructed by

$$J(k) = x^T(N|k)Px(N|k) = x^T(n|k)Px(n|k)$$

where N denotes the control horizon, and it is set equal to the state dimension, namely $N = n$; the matrix P is positive definite. The terminal state $x(N|k) = x(n|k)$ is predicted by

$$x(i+1|k) = Ax(i|k) + Bu(i|k), \; i = 0, 1, \ldots, n-1 \tag{2}$$

where $x(i|k)$ and $u(i|k)$ denote the i-th step predictive state and control from time k, respectively. The deadbeat control can be solved from the following optimization:

$$[U^*(k), X^*(k)] = \arg \min_{U(k), X(k)} x^T(n|k)Px(n|k) \tag{3}$$

subject to (2) and $x(0|k) = x(k)$, where $U(k)$ denotes the predictive control sequence, namely

$$U(k) = [u(0|k), u(1|k), \ldots, u(n-1|k)]^T$$

and the corresponding state sequence is denoted by

$$X(k) = [x^T(1|k), x^T(1|k), \ldots, x^T(n|k)]^T.$$

The optimal control is exerted in a receding horizon scheme

$$u(k) = [1, 0, \ldots, 0]U^*(k) \tag{4}$$

where the superscript * denotes the optimal value.

It can be proved by direct calculation that, in unconstrained finite-time MPC, the control is linear state feedback

$$u(k) = [1, 0, \ldots, 0]U^*(k) = -K_{db}x(k) \tag{5}$$

where K_{db} is the feedback gain such that all eigenvalues of $A - BK_{db}$ are zero.

C. Proof of Finite-Time Convergence by "Tail-Method"

The finite-time convergence of (5) can also be proved in the framework of MPC by using the "tail method" without explicitly calculating K_{db}.

Theorem 1: Consider the linear system (1) with single input (namely, $u \in \mathbb{R}^1$ and $m = 1$), and suppose no constraints are exerted, and (A, B) is controllable. The MPC is calculated from optimization (3) and implemented by (4), where the control horizon is equal to the system dimension $N = n$. Then, the closed-loop state satisfies $x(k) = 0$ for all $k > n$.

Proof: In the absence of constraints, the optimal terminal state is $x^*(n|k) = 0$. It is always feasible since

$$x^*(n|k) = A^n x(0|k) + S\,U^*(k) \tag{6}$$

where $S = [A^{n-1}B \; A^{n-2}B \; \cdots \; B]$ is invertible (due to that (A, B) is controllable, and B is a vector for single-input systems). Consequently, $U^*(k) = -S^{-1}A^n x(k)$ can be solved uniquely from (6) with $x^*(n|k) = 0$ and $x(0|k) = x(k)$ at time k. Consequently, receding horizon implementation at time k can be obtained by

$$u(k) = -[1, 0, \ldots, 0]S^{-1}A^n x(k) = -K_{db}x(k).$$

At time $k = 0$, the optimal control sequence is unique, i.e.,

$$U^*(0) = [u^*(0|0), \ldots, u^*(n-1|0)]^T$$

where $u(0) = u^*(0|0)$ is implemented, thus $x(1) = x^*(1|0)$. The corresponding optimal state sequence is also unique

$$X^*(0) = [x^{*T}(1|0), \ldots, x^{*T}(n-1|0), 0]^T. \tag{7}$$

At time $k = 1$, the optimal terminal state is $x^*(n|1) = 0$. There exists a feasible control sequence

$$U^*(1) = [u^*(1|0), \ldots, u^*(n-1|0), 0]^T$$

and the corresponding state sequence

$$X^*(1) = [x^{*T}(2|0), \ldots, x^{*T}(n-1|0), 0, 0]^T.$$

The above feasible solution $U^*(1)$ is also unique due to invertibility of S, such that it has to be the optimal solution. Hence, $u(1) = u^*(0|1) = u^*(1|0)$ is implemented, and $x(2) = x^*(1|1) = x^*(2|0)$.

Similar proof of unique control and state sequence can be repeated till $k = n-1$, where the control sequence and the optimal state are unique, and they have to be

$$U^*(n-1) = [u^*(n-1|0), 0, \ldots, 0]^T$$
$$X^*(n-1) = [0, \ldots, 0]^T$$

where $u(n-1) = u^*(0|n-1) = u^*(n-1|0)$ is implemented, and $x(n) = 0$. It implies that, $x(k) = 0$ for all $k > n$. ■

III. Problem Statement

Suppose the linear system (1) is subject to constraints

$$x \in \mathcal{X}, \ \ u \in \mathcal{U} \tag{8}$$

where $\mathcal{X}$ and $\mathcal{U}$ are convex sets containing the origin in their interior. The *first objective* is to design $u(k)$ in the constrained linear MPC framework (subject to (8)), such that the system state converges to the origin in finite time, or mathematically

$$x(k) = 0, \quad \forall\, k > T \tag{9}$$

where $T \in \mathbb{Z}^+$ is a finite positive integer.

Moreover, consider the nonlinear discrete-time system

$$x(k+1) = f(x(k), u(k)) \tag{10}$$

where $f: \mathbb{R}^n \times \mathbb{R}^1 \to \mathbb{R}^n$ is continuously differentiable with respect to x and u, and it satisfies $f(0,0) = 0$. The second objective is to design constrained nonlinear control $u(k)$ in MPC framework (subject to (8)) such that the finite-time convergence (9) is guaranteed.

Remark 1: In the case of constraints (8), the terminal state $x(n|k) = 0$ that ideally minimizes $x(n|k)^T Px(n|k)$ is possibly infeasible, and hence the proof of unconstrained finite-time MPC is inapplicable to the constrained finite-time MPC.

IV. Constrained Linear Finite-Time MPC

In this section, finite-time MPC for linear constrained systems is investigated. We start from the single input case, and the result is extended to multi-input case. Integral action can be incorporated to improve the feasibility of optimization.

A. Constrained Finite-Time MPC for Single-Input Linear Systems

For single-input discrete-time linear systems, i.e., $u \in \mathbb{R}^1$, key settings of the constrained finite-time MPC include:

1) The control horizon is set equal to the dimension of system state, i.e., $N = n$;

2) The stage cost is set to zero, and only the terminal cost is penalized, i.e., $J(k) = x^T(n|k)Px(n|k)$;

3) The positive definite weighting matrix P is solved from the Lyapunov equation

$$A_K^T PA_K - P = -Q - K^T RK \tag{11}$$

where Q and R are positive definite matrices; and the control gain K is selected such that all eigenvalues of $A_K = A - BK$ are inside the unit circle.

The optimization is constructed by

$$\begin{aligned}
&[U^*(k), X^*(k)] = \arg\min_{U(k), X(k)} x^T(N|k)Px(N|k)\\
&\text{s.t.} \quad x(i+1|k) = Ax(i|k) + Bu(i|k), \ \text{ for } i = 0, \ldots, N-1\\
&\qquad\ \ x(i+1|k) \in \mathcal{X}, \ u(i|k) \in \mathcal{U}, \ \text{ for } i = 0, \ldots, N-1\\
&\qquad\ \ x(0|k) = x(k), \quad x(N|k) \in \mathcal{X}_f
\end{aligned} \tag{12}$$

where the control horizon is set to $N = n$; the set $\mathcal{X}_f$ denotes the terminal constraint containing the origin in its interior, and satisfying

$$\mathcal{X}_f \subset \mathcal{X}, \quad -K\mathcal{X}_f \subset \mathcal{U}, \quad (A - BK)\mathcal{X}_f \subset \mathcal{X}_f. \tag{13}$$

The MPC is implemented in receding horizon scheme (4).

Remark 2: The gain K in (11) does not have to be K_{db}.

Remark 3: Conditions in (13) indicate that $\mathcal{X}_f$ is an invariant set of $x(k+1) = (A - BK)x(k)$, and it can be given by $\mathcal{X}_f = \{x \in \mathbb{R}^n | x^T Px \le \epsilon\}$, where ϵ is calculated by

$$\epsilon = \min\left[\min_{-Kz \in \partial\mathcal{U}} z^T Pz, \ \min_{-z \in \partial\mathcal{X}} z^T Pz\right] \tag{14}$$

where $\partial\mathcal{U}$ and $\partial\mathcal{X}$ denote the boundaries of $\mathcal{U}$ and $\mathcal{X}$, respectively. Alternatively, a conservatively small set $\mathcal{X}_f$ in simpler form (e.g., a small neighborhood of $x = 0$) can be chosen to satisfy (13).

Theorem 2: Consider system (1) with single input $u \in \mathbb{R}^1$ subject to constraints (8). The MPC is calculated by constrained optimization (12), and implemented by (4), where

i) the control horizon is set to $N = n$;

ii) the stage cost is set to zero;

iii) the terminal weighting matrix P is calculated from (11).

If constrained optimization (12) is feasible initially, then,

1) constrained optimization (12) is feasible recursively;

2) the closed-loop system is asymptotically stable;

3) there exists a finite time $T > 0$, such that the state of the closed-loop system satisfies $x(k) = 0$ for all $k > T$.

Proof: 1) Suppose, at time k, the optimization is feasible, i.e., $U^*(k)$ exists such that all constraints are satisfied, and $u(k) = u^*(0|k)$ is implemented.

It then follows that, at time $k+1$, the state satisfies:

$$x(k+1) = x(0|k+1) = x^*(1|k) = Ax(0|k) + Bu^*(0|k)$$

and at least one feasible control sequence exists:

$$u(i|k+1) = u^*(i+1|k), \ \ i = 0, \ldots, n-2$$
$$u(n-1|k+1) = -Kx^*(n|k)$$

where $u^*(i|k)$, $i = 0, \ldots, n-1$ denote the optimal control sequence at time k, and $x^*(n|k)$ denotes the corresponding optimal terminal state at time k.

It is clear that $u(i|k+1)$, $i = 0, \ldots, n-2$ satisfy the control constraint, and $u(n-1|k+1) = -Kx^*(n|k)$ satisfies the control constraint due to (13). It is also clear that $x(i|k+1)$, $i = 1, \ldots, n-1$ satisfy the state constraint, and $x(n|k+1) = (A - BK)x(n-$

$1|k+1) = (A-BK)x^*(n|k)$ satisfies the state constraint due to (13).

Consequently, at time $k+1$, the constrained optimization is feasible, provided that it is feasible at time k. This completes the proof of recursive feasibility.

2) Take $J^*(k) = x^{*T}(n|k)Px^*(n|k)$ as the Lyapunov candidate for $x^*(n|k)$. It follows that:

$$\begin{aligned} J^*(k+1)-J^*(k) &\le J(k+1)-J^*(k) \\ &= x^T(n|k+1)Px(n|k+1) - x^{*T}(n|k)Px^*(n|k) \\ &= x^{*T}(n|k)A_K^T P A_K x^*(n|k) - x^{*T}(n|k)Px^*(n|k) \\ &= x^{*T}(n|k)\left(A_K^T P A_K - P\right)x^*(n|k) \\ &= -\|x^*(n|k)\|_Q^2 - \|Kx^*(n|k)\|_R^2 \end{aligned} \tag{15}$$

where $\|x^*(n|k)\|_Q^2 = x^{*T}(n|k)Qx^*(n|k)$, $\|Kx^*(n|k)\|_R^2 = x^{*T}(n|k)\times K^T RKx^*(n|k)$. It indicates that $x^*(n|k) \to 0$ as $k \to \infty$.

The terminal state $x^*(n|k)$ at each time k satisfies

$$x^*(n|k) = A^n x(k) + S\,U^*(k) \tag{16}$$

where $S = [A^{n-1}B, A^{n-2}B, \ldots, B]$ is invertible. It follows from (16) that, for each $x^*(n|k)$, the corresponding constrained control sequence is $U^*(k) = S^{-1}(x^*(n|k) - A^n x(k))$, and the control is implemented by

$$u^*(k) = [1,0,\ldots,0]S^{-1}(x^*(n|k) - A^n x(k)) \tag{17}$$

$$= -K_{db}x(k) + S_n^T x^*(n|k) \tag{18}$$

where $K_{db} = [1,0,\ldots,0]S^{-1}A^n$ is the unconstrained deadbeat gain, and $S_n^T = [1,0,\ldots,0]S^{-1}$ is the first row of S^{-1}.

Substituting the control (18) into the original system yields

$$x(k+1) = (A-BK_{db})x(k) + BS_n^T x^*(n|k) \tag{19}$$

which is the closed-loop finite-time control system perturbed by the vanishing term $BS_n^T x^*(n|k)$. Its solution is

$$x(k) = A_{db}^k x(0) + [A_{db}^{k-1}B, \ldots, B]\begin{bmatrix} S_n^T x^*(n|0) \\ \vdots \\ S_n^T x^*(n|k) \end{bmatrix}$$

where $A_{db} = A - BK_{db}$. For $k \ge n$, it holds that $A_{db}^k = 0$, and

$$x(k) = [A_{db}^{n-1}B, \ldots, B]\begin{bmatrix} S_n^T x^*(n|k-n+1) \\ \vdots \\ S_n^T x^*(n|k) \end{bmatrix}. \tag{20}$$

Since $x^*(n|k) \to 0$ as $k \to \infty$, it follows that $x(k) \to 0$ as $k \to \infty$, i.e., the closed-loop system is asymptotically stable.

3) Based on 1) and 2), $x(k)$ converges to the origin asymptotically, and enters the non-empty set $\mathcal{X}_{db}$ satisfying

$$\mathcal{X}_{db} \in \mathcal{X},\ -K_{db}\mathcal{X}_{db} \in \mathcal{U},\ (A-BK_{db})\mathcal{X}_{db} \in \mathcal{X}_{db}$$

after a finite time. Inside the set $\mathcal{X}_{db}$, the state and control constraints are inactive, and the optimization is actually unconstrained. Consequently, the behavior of $x(k)$ follows from that of the unconstrained finite-time MPC, and it reaches the origin in finite time and maintains zero thereafter. ■

The implementation procedure of the proposed constrained finite-time MPC for single-input discrete-time linear systems can be summarized by Algorithm 1.

Algorithm 1 Constrained finite-time MPC for single-input discrete-time linear systems

Offline:
1: Set the control horizon equal to the system dimension, i.e., $N = n$; set Q and R to be positive definite matrices.
2: Find a feedback gain K such that eigenvalues of $A-BK$ are inside the unit circle.
3: Solve the terminal weighting matrix P from Lyapunov equation (11).
4: Calculate the terminal constraint via (14).
Online:
5: **for** $k = 0$ to the end of task **do**
6: Measure the system state $x(k)$.
7: Solve the optimization (12) to obtain the optimal control sequence $U^*(k)$.
8: Apply $u(k) = [1,0,\ldots,0]U^*(k)$ to system (1).
9: **end for**

B. Constrained Finite-Time MPC for Multi-Input Linear Systems

The proposed constrained finite-time MPC can be extended to address multi-input systems

$$x(k+1) = Ax(k) + Bu(k) = Ax(k) + \sum_{i=1}^{m} b_i u_i(k) \tag{21}$$

where m denotes the number of inputs, and b_i is the i-th column of B. States and inputs are subject to constraints (8).

Remark 4: For multi-input systems, control constraints can be coupled, such that it is inappropriate to express control constraints in element-wise form, i.e., $u_i \in \mathcal{U}_i$.

Provided that (A,B) is controllable, system (21) can be transformed to Wonham controllable canonical form [33]:

$$z(k+1) = Fz(k) + Gu(k) \tag{22}$$

where $z = Mx$ with M being the transformation matrix; and $F = MAM^{-1}$ and $G = MB$ are in the form of

$$F = \begin{bmatrix} F_{11} & F_{12} & \cdots & F_{1q} \\ 0 & F_{22} & \cdots & F_{2q} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & F_{qq} \end{bmatrix},\ G = \left[\begin{array}{c|c|c|c} g_1 & & & * \\ \hline 0 & \ddots & & * \\ \hline 0 & 0 & g_q & * \end{array}\right]$$

where, for $i = 1,\ldots,q$, $F_{ii} \in \mathbb{R}^{n_i \times n_i}$, $g_i \in \mathbb{R}^{n_i \times 1}$, $\sum_{i=1}^q n_i = n$, and (F_{ii}, g_i) are controllable.

Denote $z = [z_1^T, \ldots, z_q^T]^T$, where $z_i \in \mathbb{R}^{n_i}$ for $i = 1,\ldots,q$. Denote $u = [u_1,\ldots,u_q,u_{q+1},\ldots,u_m]^T$. The controllable multi-input system can be regarded as q subsystems, with u_i ($i = 1,\ldots,q$) being the single input for the i-th subsystem, and u_j ($j = q+1,\ldots,m$) can be set to zero.

Set the control horizons for decoupled subsystems to be $N_i = n_i$ ($i = 1,\ldots,q$), and the predictive horizon for the overall system to be $N_p = \max_{i=1,\ldots,q}[n_i]$. The cost function can be designed by

$$J_m(k) = \sum_{i=1}^{q} z_i(N_i|k)^T P_i z_i(N_i|k) = \sum_{i=1}^{q} z_i(n_i|k)^T P_i z_i(n_i|k)$$

where P_i is the solution to Lyapunov equation

$$(F_{ii} - g_i K_i)^T P_i (F_{ii} - g_i K_i) - P_i = -Q_i - K_i^T R_i K_i \tag{23}$$

where Q_i and R_i are positive definite; and K_i is selected such that all eigenvalues of $F_{ii} - g_i K_i$ are inside the unit circle.

Denote $U_i(k) = [u_i(1|k), \ldots, u_i(n_i|k)]^T$ for $i = 1, \ldots, q$. Set $u_j = 0$ for $j = q+1, \ldots, m$.

The optimization is designed by

$$\begin{aligned}
&[U_1^*(k), \ldots, U_q^*(k)] = \arg \min_{U_1(k),\ldots,U_q(k)} J_m(k) \\
&\text{s.t.} \quad z(i+1|k) = Fz(i|k) + Gu(i|k), \ \text{for } i = 0, \ldots, N_p - 1 \\
&\qquad z(i+1|k) \in M\mathcal{X}, \ u(i|k) \in \mathcal{U}, \ \text{for } i = 0, \ldots, N_p - 1 \\
&\qquad z(0|k) = Mx(k) \\
&\qquad z_i(N_i|k) \in \mathcal{Z}_{f,i}, \ \text{for } i = 1, \ldots, q
\end{aligned} \tag{24}$$

where $\mathcal{Z}_{f,i}$ denotes the terminal constraint satisfying

$$\bigotimes_{i=1}^{q} \mathcal{Z}_{f,i} \subset M\mathcal{X}, \ \bigotimes_{i=1}^{q} -K_i \mathcal{Z}_{f,i} \subset \mathcal{U} \tag{25}$$

$$(F_{ii} - g_i K_i)\mathcal{Z}_{f,i} \in \mathcal{Z}_{f,i}. \tag{26}$$

The finite-time MPC for multi-input discrete-time linear systems can be implemented in receding horizon scheme

$$u_i(k) = [1, 0, \ldots, 0]U_i^*(k), \ \text{for } i = 1, \ldots, q \tag{27}$$

$$u_j(k) = 0, \ \text{for } j = q+1, \ldots, m. \tag{28}$$

Remark 5: Set-theoretic methods (e.g., [31], [34]) can be applied to calculate the terminal sets satisfying (25) and (26). Practically, it is unnecessary to always explicitly calculate maximum terminal sets; instead, for controllable systems, conservatively small sets $\mathcal{Z}_{f,i}$ can be chosen such that they are subsets of the largest terminal sets. By using conservative terminal sets, recursive feasibility and finite-time stability are guaranteed with some sacrifice of initial feasibility.

Theorem 3: For multi-input discrete-time linear system (21) subject to state and control constraints (8), if the deadbeat MPC is designed by (24)–(28), then the optimization is recursively feasible, and there exists a finite $T > 0$ such that $x(k) = 0$ for all $k > T$.

Proof: The recursive feasibility can be proved by using "tail method", since the overall model is applied to prediction, and the terminal sets are invariant.

It follows from (28) that the q-th subsystem in transformed decoupled system (22) becomes:

$$z_q(k+1) = F_{qq} z_q(k) + g_q u_q(k)$$

where (F_{qq}, g_q) is controllable. Then its finite-time convergence follows from Theorem 2, and it holds that $z_q(k) = 0$ for all $k > T_q$, where $T_q > 0$ is finite.

After the finite T_q steps, the $(q-1)$-th subsystem becomes:

$$z_{q-1}(k+1) = F_{q-1,q-1} z_{q-1}(k) + g_{q-1} u_{q-1}(k)$$

where $(F_{q-1,q-1}, g_{q-1})$ is controllable. It then holds that $z_{q-1}(k) = 0$ for all $k > T_q + T_{q-1}$, where $T_{q-1} > 0$ is finite.

The above analysis can be processed recursively, and it can be proved that all z_i $(i = 1, \ldots, q)$ converge to the origin within finite times steps, indicating that $z(k) = 0$ for some finite $k > T > 0$. Consequently, $x(k) = M^{-1} z(k) = 0$ for some finite $k > T$. ■

The implementation procedure of the proposed constrained finite-time MPC for multi-input discrete-time linear systems can be summarized by Algorithm 2.

Algorithm 2 Constrained finite-time MPC for multi-input discrete-time linear systems

Offline:
1: Find $z = Mx$ to transform (21) to decoupled form (22).
2: Set the control horizons to be $N_i = n_i$ $(i = 1, \ldots, q)$, where n_i is the dimension of each subsystem after decoupling; set Q_i and R_i to be positive definite matrices.
3: Find K_i such that all eigenvalues of $F_{ii} - g_i K_i$ are inside the unit circle.
4: Solve the terminal weighting matrices P_i from Lyapunov equation (23).
5: Calculate the terminal constraint via (25) and (26).
Online:
6: **for** $k = 0$ to the end of task **do**
7: Measure the system state $z(k) = Mx(k)$.
8: Solve the optimization (24) to obtain the optimal control sequence $U_i^*(k)$.
9: Set $u_i(k) = [1, 0, \ldots, 0]U_i^*(k)$ for $i = 1, \ldots, q$.
10: Apply $u(k) = [u_1(k), \ldots, u_q(k), 0, \ldots, 0]^T$ to system (21).
11: **end for**

C. Improving Initial Feasibility by Augmentation

For the proposed constrained linear finite-time MPC, at $k = 0$, it holds that

$$x^*(n|0) = A^n x(0|0) + SU^*(0) = A^n x(0) + SU^*(0) \tag{29}$$

where $x^*(n|0) \in \mathcal{X}_f$, $x(0) \in \mathcal{X}$, and $U^*(0) \in \bar{\mathcal{U}} = \mathcal{U} \otimes \cdots \otimes \mathcal{U}$. It follows from (29) that:

$$A^n x(0) = -SU^*(0) + x^*(n|0)$$

and the initial feasible region can be calculated explicitly by

$$\mathcal{X} \cap \left\{ x \in \mathbb{R}^n | A^n x \in -S\bar{\mathcal{U}} \oplus \mathcal{X}_f \right\} = \mathcal{X}_0. \tag{30}$$

A limitation of the proposed finite-time MPC is that, if the system dimension is small, then the control horizon $N = n$ is short, such that the initial feasibility would be unsatisfactory. In this section, augmentation is applied to improve the initial feasibility. The core idea is to increase the system dimension by using "integral-like" action, such that the control horizon increases, and the initial feasibility can be improved.

The original system can be augmented by

$$\begin{aligned}
\sigma_1(k+1) &= \alpha_1 \sigma_1(k) + C_1 x(k) \\
x(k+1) &= Ax(k) + Bu(k)
\end{aligned} \tag{31}$$

where α_1 is a constant to be chosen (preferably $-1 \le \alpha_1 \le 1$). The augmented system (31) can be rewritten into

$$\xi(k+1) = \mathcal{A}\xi(k) + \mathcal{B}u(k) \tag{32}$$

where $\xi = [\sigma_1^T, x^T]^T$, and

$$\mathcal{A} = \begin{bmatrix} \alpha_1 I_{p_1 \times p_1} & C_1 \\ 0 & A \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} 0 \\ B \end{bmatrix}.$$

The matrix $C_1 \in \mathbb{R}^{p_1 \times n}$, and $0 < p_1 \leq m$ are selected such that

$$Rank \begin{bmatrix} C_1 & 0 \\ \alpha_1 I_n - A & B \end{bmatrix} = n + p_1 \tag{33}$$

to guarantee the controllability of the augmented system (31). Proof of controllability of (31) is given in Appendix.

The proposed constrained finite-time MPC can be applied to the augmented system (31) with a longer control horizon $N = n + p_1$, such that the initial feasibility is improved.

The implementation procedure of finite-time MPC with integral action can be summarized by Algorithm 3.

Algorithm 3 Constrained finite-time MPC with integral action

1: Choose C_1 and α_1 such that (33) is satisfied.
2: Calculated $\mathcal{A}$ and $\mathcal{B}$ in (31) and (32).
3: Apply Algorithm 1 or 2 to (32) with A and B replaced by $\mathcal{A}$ and $\mathcal{B}$, respectively.

Remark 6: A typical example of C_1 satisfying (33) is one such that (A, C_1) is observable, and $p_1 \leq m$.

Remark 7: If initial feasibility of (32) is still unsatisfactory, the system can be augmented further by

$$\begin{aligned} \sigma_2(k+1) &= \alpha_2 \sigma_2(k) + C_2 \xi(k) \\ \xi(k+1) &= \mathcal{A}\xi(k) + \mathcal{B}u(k) \end{aligned} \tag{34}$$

where $C_2 \in \mathbb{R}^{p_2 \times (p_1+n)}$ and $0 < p_2 \leq m$, such that

$$Rank \begin{bmatrix} C_2 & 0 \\ \alpha_2 I - \mathcal{A} & \mathcal{B} \end{bmatrix} = n + p_1 + p_2.$$

Apply the proposed finite-time MPC to (34) with greater control horizon $N = n + p_1 + p_2$, such that the states have deadbeat performance, and constraints can be satisfied. This augmentation by integration can be applied recursively, such that the control horizon can be set sufficiently large, and the initial feasibility can be improved further.

V. Nonlinear Finite-Time MPC

In this section, we consider the single-input nonlinear plant (10) subject to convex state and control constraints (8). The result is straightforward to be extended for multi-input case by using the technique similar to Section IV-B.

A. Finite-Time MPC via Feedback Linearization

A sufficient condition for deadbeat controllability of discrete nonlinear systems is that the plant is feedback linearizable. Sufficient conditions for feedback linearization can be found in [35]−[37].

If the nonlinear plant is feedback linearizable, then there exists a (local) diffeomorphism $z = T(x)$ and a feedback control $u = \gamma(z, v) = \gamma(T(x), v)$ for all $x \in \mathcal{D}$, such that $T(0) = 0$ and $\gamma(0, 0) = 0$, and the nonlinear system can be linearized to

$$z(k+1) = Az(k) + Bv(k) \tag{35}$$

for some controllable (A, B), and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$. For the linearized system (35), linear finite-time MPC in Section IV can be applied to achieve deadbeat performance.

Specifically, set $N = n$. The optimization can be formulated by

$$[V^*(k), Z^*(k)] = \arg \min_{V(k), Z(k)} z^T(N|k) P z(N|k) \tag{36}$$

$$\text{s.t.} \quad z(i+1|k) = Az(i|k) + Bv(i|k), \ \text{ for } i = 0, 1, \ldots, N-1 \tag{37}$$

$$z(i|k) \in T(\mathcal{X} \cap \mathcal{D}), \ \text{ for } i = 1, \ldots, N \tag{38}$$

$$\gamma(z(i|k), v(i|k)) \in \mathcal{U}, \ \text{ for } i = 0, 1, \ldots, N-1 \tag{39}$$

$$z(0|k) = T(x(k)) \tag{40}$$

$$z(N|k) \in \mathcal{Z}_f \tag{41}$$

where $\mathcal{Z}_f$ is an invariant set such that $\mathcal{Z}_f \subset T(\mathcal{X} \cap \mathcal{D})$, and

$$(A - BK)z \in \mathcal{Z}_f, \ \ \gamma(z, -Kz) \in \mathcal{U}, \ \ \forall z \in \mathcal{Z}_f$$

with all eigenvalues of $(A - BK)$ inside the unit circle; the matrix P is the solution to the discrete-time Lyapunov equation (11). It then follows that the finite-time control can be implemented by

$$u(k) = \gamma(T(x(k)), v^*(0|k))$$

where $v^*(0|k) = [1, 0, \ldots, 0]V^*(k)$.

According to Theorem 2, provided that the optimization (36)–(41) is feasible initially, it is feasible recursively, and there exists finite $T > 0$ such that $z(k) = 0$ for all $k > T$. It indicates that $x(k) = T^{-1}(z(k)) = 0$ for all $k > T$.

The above optimization and receding horizon control are only for theoretical purpose. They are capable of guaranteeing the existence of nonlinear finite-time control. However, it is unsuitable for finite-time MPC design, since the diffeomorphism $z = T(x)$ is often difficult, or sometimes prohibitive, to find.

B. Finite-Time MPC for Unconstrained Nonlinear Systems

For unconstrained nonlinear system (10), suppose that it is globally feedback linearizable.

In unconstrained nonlinear finite-time MPC, the optimization can be formulated by

$$[U^*(k), X^*(k)] = \arg \min_{U(k), X(k)} x^T(n|k) P x(n|k) \tag{42}$$

$$\text{s.t.} \quad x(i+1|k) = f(x(i|k), u(i|k)), \ \text{ for } i = 0, 1, \ldots, n-1 \tag{43}$$

$$x(0|k) = x(k). \tag{44}$$

where the control horizon is set equal to the system dimension, and P is a positive definite matrix. The MPC can be implemented by the receding horizon way (4).

Theorem 4: Consider unconstrained nonlinear system (10), and suppose it is globally feedback linearizable. The MPC is calculated from optimization (42), where the control horizon is set equal to the system dimension. Then the receding horizon implementation (4) guarantees $x(k) = 0$ for all $k > T$, where $T > 0$ is a finite constant.

Proof: Global feedback linearizability guarantees that the unconstrained system is capable of reaching the origin in no more than n steps. It indicates that (42) has the ideal solution $[U^*(k), X^*(k)]$ satisfying $x(n|k) = 0$.

The ideal solution is unique, since the ideal $[V^*(k), Z^*(k)]$

for the linearized system (35) is unique, and $z = T(x)$ is a global diffeomorphism.

Consequently, at time $k+1$, the optimal solution is unique, and it satisfies $x(n-1|k+1) = x(n|k) = 0$ and $x(n|k+1) = 0$. The rest of the proof follows from that of Theorem 2. ■

Remark 8: If the diffeomorphism $z = T(x)$ and control $u = \gamma(x,v)$ can be found explicitly, then the unconstrained finite-time MPC can be written in explicit feedback form given by

$$u^*(k) = \gamma\Big(T(x(k)), [1,0,\dots,0]S^{-1}A^n T(x(k))\Big)$$

where $S = [B, AB, \dots, A^{n-1}B]$. Otherwise we have to calculate the optimization (42) numerically.

C. Finite-Time MPC for Constrained Nonlinear Systems

Suppose that the nonlinear plant (10) is feedback linearizable for all $x \in \mathcal{D}$.

In the presence of constraints (8), the optimization for nonlinear finite-time MPC can be formulated by

$$[U^*(k), X^*(k)] = \arg\min_{U(k),X(k)} x^T(N|k)Px(N|k) \tag{45}$$

$$\text{s.t.}\quad x(i+1|k) = f(x(i|k), u(i|k)),\ \ \text{for } i = 0,1,\dots,N-1 \tag{46}$$

$$x(0|k) = x(k) \tag{47}$$

$$x(i|k) \in \mathcal{X}\cap\mathcal{D},\ \ \text{for } i = 1,\dots,N \tag{48}$$

$$u(i|k) \in \mathcal{U},\ \ \text{for } i = 0,1,\dots,N-1 \tag{49}$$

$$x(N|k) \in T^{-1}(\mathcal{Z}_f) \tag{50}$$

where the control horizon is set equal to the system dimension, i.e., $N = n$; and $T^{-1}(\mathcal{Z}_f)$ is an invariant set such that $T^{-1}(\mathcal{Z}_f) \subset \mathcal{X}\cap\mathcal{D}$, and for all $x \in T^{-1}(\mathcal{Z}_f)$ it holds that

$$f(x, \gamma(T(x), -KT(x))) \in T^{-1}(\mathcal{Z}_f),\ \gamma(T(x), -KT(x)) \in \mathcal{U}.$$

The constrained finite-time MPC is implemented in receding horizon way (4).

Theorem 5: Suppose that the nonlinear plant (10) is feedback linearizable for all $x \in \mathcal{D}$, and it is subject to constraints (8). The constrained optimization is formulated by (45)–(50), and the MPC is implemented by (4). If the optimization (45)–(50) is feasible initially, then

1) the optimization (45)–(50) is feasible recursively;

2) there exists a finite time $T > 0$, such that the state of the closed-loop system satisfies $x(k) = 0$ for all $k > T$.

Proof: 1) Recursive feasibility is guaranteed by the terminal constraint (50).

Specifically, suppose that optimization (45)–(50) is feasible at time k. It follows that, at time $k+1$, there exists a feasible control sequence:

$$\begin{aligned} u(i|k+1) &= u^*(i+1|k),\quad \text{for } i = 0,1,\dots,n-2 \\ u(n-1|k+1) &= \gamma(T(x^*(n|k)), -KT(x^*(n|k))) \end{aligned}$$

such that the corresponding state sequence can be obtained by

$$\begin{aligned} x(i|k+1) &= x^*(i+1|k),\quad \text{for } i = 0,\dots,n-1 \\ x(n|k+1) &= f(x(n-1|k+1), \gamma(T(x(n-1|k+1)) \\ &\qquad - KT(x(n-1|k+1)))). \end{aligned}$$

where $x(n-1|k+1) \in \mathcal{X}_f$.

It is clear that $u(i-1|k+1) \in \mathcal{U}$ and $x(i|k+1) \in \mathcal{X}\cap\mathcal{D}$ for $i = 1,\dots,n-1$. It is guaranteed by the terminal constraint (50) that $u(n-1|k) \in \mathcal{U}$ and $x(n|k) \in \mathcal{X}_f$. This completes the proof of recursive feasibility.

2) Since $z = T(x)$ is a diffeomorphism in $\mathcal{D}$ with $T(0) = 0$, it indicates that optimizing $x^T Px$ is equivalent to optimizing $T^T(x)PT(x) = z^T Pz$. Consequently, optimization (45)–(50) is equivalent to optimization (36)–(41), and it follows from Theorem 2 that $z(k) = 0$ for all $k > T$. Finally, $x(k) = T^{-1}(z(k)) = 0$ for all $k > T$. ■

The implementation procedure of the proposed constrained finite-time MPC for discrete-time nonlinear systems can be summarized by Algorithm 4.

Algorithm 4 Constrained finite-time MPC for discrete-time nonlinear systems

Offline:

1: Determine the region $\mathcal{D}$ where the nonlinear system (10) is feedback linearizable.

2: Set the control horizon to be $N = n$, where n is the system dimension; set P to be a positive definite matrix.

3: Calculate a control invariant set for (10) via some nonlinear design techniques, and use this control invariant set as the terminal constraint.

Online:

4: **for** $k = 0$ to the end of task **do**

5: Measure the system state $x(k)$.

6: Solve the optimization (45) to obtain the optimal control sequence $U_i^*(k)$.

7: Set $u_i(k) = [1,0,\dots,0]U_i^*(k)$ to system (10).

8: **end for**

Remark 9: The terminal constraint (50) can be replaced by a smaller control invariant set, and explicit calculation of $z = T(x)$ can be avoided.

VI. Simulation

Simulation examples are provided to illustrate the proposed theoretical results on constrained linear and nonlinear systems.

A. Constrained Finite-Time MPC for Single-Input Linear Systems

The plant to be controlled is given by

$$A = \begin{bmatrix} 1.1 & 2 & 0 \\ 0 & 0.95 & 1 \\ 0 & 0 & 1.2 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0.079 \\ 0.1 \end{bmatrix}$$

where its dimension is $n = 3$. Suppose that the states and control input are subject to hard constraints given by

$$-2 \le x_2 + x_3 \le 2,\ \ -6 \le u \le 6.$$

Algorithm 1 is applied to guarantee the finite-time convergence. The control horizon is set to $N = 3$ which equals the system dimension. Set the weighting matrices $Q = I_{3\times3}$ and $R = 0.1$. A linear feedback $K = [2.2150\ 15.0471\ 14.6128]$ is selected to solve the weighting matrix P from the Lyapunov equation (11)

$$P = \begin{bmatrix} 6.1590 & 19.4637 & 5.8132 \\ 19.4637 & 96.8173 & 40.0964 \\ 5.8132 & 40.0964 & 29.9407 \end{bmatrix}.$$

The terminal constraint is $\mathcal{X}_f = \{x^T P x \le 4.6151\}$, which is calculated from (14). The initial feasibility region can be calculated by (30).

Suppose the initial state is $x(0) = [1.9, -1.1, 0.8]^T$. By testing (30), it can be verified that the constrained optimization (12) is feasible initially. Closed-loop performances are illustrated by Fig. 1, where states and control input are capable of converging to zero in finite time. The states and control input are bounded within their constraints. The finite transient process takes 6 steps.

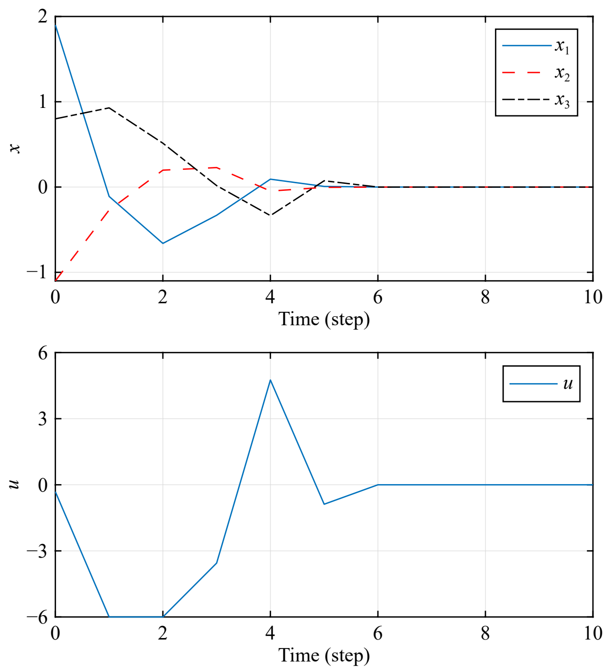

Fig. 1. States and control of the single-input system with constrained finite-time MPC: the constraints $-2 \le x_2 + x_3 \le 2$ and $-6 \le u \le 6$ are satisfied, and the transient process is completed in 6 steps.

As expected, the finite-time convergence and constraint satisfaction are in well accordance with Theorem 2. The transient process (6 steps) is longer than that of the unconstrained case (less than 3 steps). The transient performance varies with different choices of Q, R and K. However, currently no explicit clues are found on how to select Q, R and K to improve the transient performance.

B. Improving Initial Feasibility by Augmentation

Augmentation in Section IV-C can be applied to improve the initial feasibility of the proposed finite-time MPC. For the example in Section VI-A, we apply Algorithm 3 with $\alpha = -0.5$ and $C_1 = [0.5, 0, 0]$. The control horizon is equal to the augmented system dimension, i.e., $N = 4$.

The initial feasible regions of the finite-time MPC and the augmented finite-time MPC are illustrated by Fig. 2, where it can be seen that the initial feasible region is significantly enlarged via augmentation.

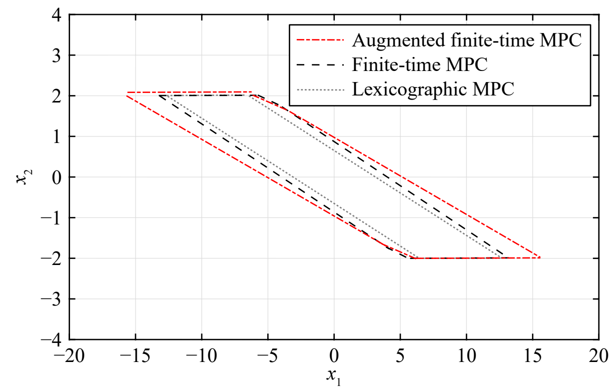

Fig. 2. Initial feasibility regions intersected by $x_1 - x_2$ plane: the proposed finite-time MPC with integral control (red dot-dashed); the proposed finite-time MPC (black dashed); lexicographic control (grey dotted).

For comparative study, the initial feasible region of an existing finite-time MPC, namely lexicographic MPC [30], is displayed in Fig. 2, where the key control parameters are set the same as those in the proposed finite-time MPC. In lexicographic MPC, the terminal equality constraint is applied to guarantee closed-loop stability. Comparatively, in the proposed finite-time MPC, inequality terminal constraints are applied. It can be seen that the proposed finite-time MPC demonstrate superior performance in initial feasibility. If initial states are feasible, and the key control parameters are with same values, then transient and steady-state performances of the proposed finite-time MPC are very similar with those of the lexicographic MPC.

C. Constrained Finite-Time MPC for Multi-Input Linear Systems

The multi-input linear system to be controlled is given by

$$A = \begin{bmatrix} 1.47 & 1.44 & -0.06 & 0.68 & 0.08 \\ -0.17 & 1.44 & 1.06 & -0.43 & 0.50 \\ -0.42 & 1.04 & 1.33 & -0.91 & 0.51 \\ -0.99 & 0.12 & -0.02 & -0.77 & 0.39 \\ 1.18 & -3.38 & -0.41 & 2.62 & -0.41 \end{bmatrix}$$
$$B = [b_1^T, b_2^T]$$
$$b_1 = \begin{bmatrix} -0.0075 & 0.085 & 0.116 & 0.023 & -0.048 \end{bmatrix}^T$$
$$b_2 = \begin{bmatrix} 0.29 & -0.27 & -0.61 & -0.51 & 1.91 \end{bmatrix}^T$$

subject to decoupled and coupled control constraints

$$-6 \le u_1 \le 6,\ -2 \le u_2 \le 2,\ -4 \le 0.5u_1 + u_2 \le 4. \tag{51}$$

The system can be decoupled to

$$z(k+1) = \begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix} z(k) + \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} u(k)$$
$$F_{11} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1.25 & -3.5 & 3.25 \end{bmatrix},\ F_{12} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix},\ g_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$F_{22} = \begin{bmatrix} 0 & 1 \\ 0.99 & -0.2 \end{bmatrix},\ g_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Algorithm 2 is applied in this example, where the weighting matrices are set by $Q_i = I$ and $R_i = 0.1$ with $i = 1,2$. Feedback gains $K_1 = [2.2150, 15.0471, 14.6128]$ and $K_2 = [0.9750, -0.0120]$ are applied to calculate P_1 and P_2 from (23). The terminal constraints are assigned conservatively by $-0.1 \le z_i \le 0.1$ with $i = 1,\ldots,5$. The result is displayed in Fig. 3, where the closed-loop multi-input system is stabilized with deadbeat performance, and the transient process is completed within 6 steps. It is straightforward to judge that constraints in (51) are all satisfied.

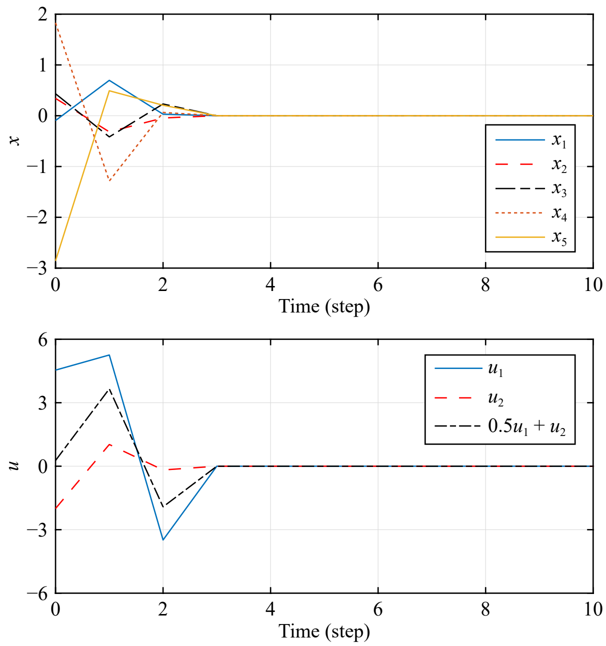

Fig. 3. States and control of the multi-input closed-loop system with constrained finite-time MPC: the transient process ends within 6 steps; the control u_1 satisfies the constraint $-6 \le u_1 \le 6$; the control u_2 satisfies the constraint $-2 \le u_2 \le 2$; the coupled constraint $-4 \le 0.5u_1 + u_2 \le 4$ is satisfied.

The finite-time convergence and constraint satisfaction are with expectation. Some decoupling works have to be processed before calculating the finite-time MPC. The initial feasibility can be improved via augmentation proposed in Section IV-C.

D. Finite-Time MPC for Nonlinear Systems

The nonlinear plant to be controlled is given by

$$\begin{aligned} x_1(k+1) &= -1.1x_1(k) + 2\sin x_2(k) + w_1(k) \\ x_2(k+1) &= 0.12x_1(k)x_2(k) + 0.79x_3(k) \\ x_3(k+1) &= x_3(k) + u + w_2(k) \end{aligned}$$

where $w_1(k)$ and $w_2(k)$ are un-matched and matched disturbances. The nonlinear plant is subject to constraints

$$-2 \le x_2 + x_3 \le 2, \quad -2 \le u \le 2.$$

To test the finite-time convergence, we have to assume that the nonlinear system is disturbance-free, i.e., $w_1 = 0$ and $w_2 = 0$. By checking conditions for feedback linearization in [37], it can be shown that the nonlinear plant is feedback linearizable in $\mathcal{D} = \{x \in \mathbb{R}^3 | -\pi/2 \le x_2 \le \pi/2\}$. The nonlinear finite-time MPC is processed by Algorithm 4, where the control horizon is set to the system dimension $N = 3$. The simulated results are displayed in Fig. 4, where it can be seen that states reach the origin with finite steps, and stay at the origin for all future time instants. The states and nonlinear control input are bounded within their constraints.

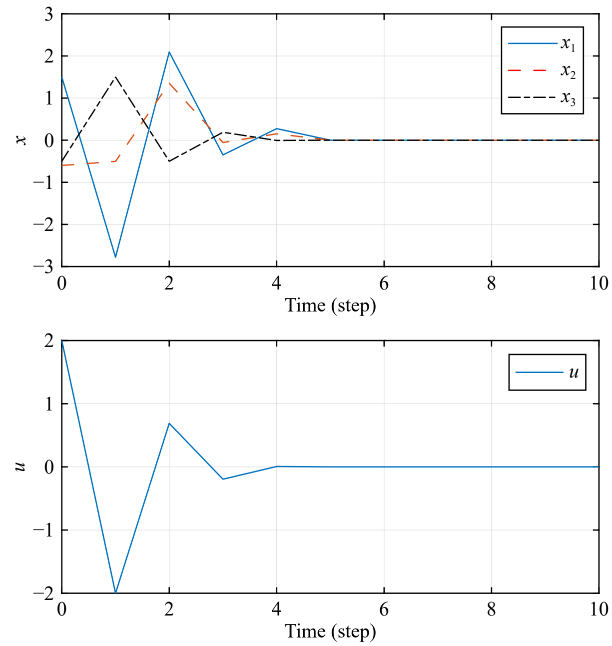

Fig. 4. States and control of the closed-loop nonlinear system with constrained finite-time MPC in the case of no disturbance: the transient process ends within 5 steps, and states satisfy constraints $-2 \le x_2 + x_3 \le 2$ and $-\pi/2 \le x_2 \le \pi/2$; the control u satisfies the constraint $-2 \le u \le 2$.

In implementation of the proposed nonlinear finite-time MPC, we do not have to explicitly calculate the diffeomorphism for feedback linearization. The existence of diffeomorphism is only used in theoretically proving feasibility of optimization. If the finite-time solution exists, it can be calculated numerically by the proposed finite-time nonlinear MPC.

To test the robustness of the proposed finite-time MPC against non-vanishing disturbances, we assume that $|w_1(k)| \le 0.2$ and $|w_2(k)| \le 0.1$ are random numbers. The results are shown in Fig. 5, where it can be seen that the closed-loop states are ultimately bounded due to the inherent robustness of MPC. Finite-time convergence property is no longer ensured, since the closed-loop poles cannot be assigned exactly to the origin in the presence of disturbances.

VII. Conclusion

In this paper, it is shown that finite-time control for discrete-time systems can be designed in MPC framework, such that the finite-time convergence can be achieved in the case of state and control constraints. The key settings in the proposed finite-time MPC include: set the control horizon equal to the system dimension, and only penalize the terminal cost. Recursive feasibility, closed-loop stability and finite-time convergence are proved theoretically. Initial feasibility can be improved by integral control. The proposed finite-time MPC is applicable in not only single-input linear systems, but multi-input and nonlinear systems.

Limitations of the proposed finite-time MPC include: 1)

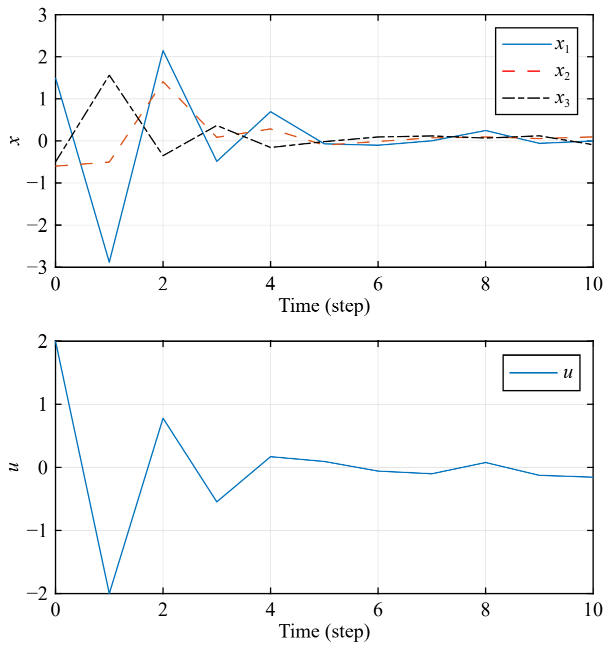

Fig. 5. States and control of the closed-loop nonlinear system with constrained finite-time MPC in the presence of non-vanishing disturbance: closed-loop states are ultimately bounded; constraints $-2 \le x_2 + x_3 \le 2$, $-\pi/2 \le x_2 \le \pi/2$ and $-2 \le u \le 2$ are satisfied.

although it is proved that the transient process can be completed before a finite time, however, the finite time itself cannot be accurately calculated in prior; 2) in the presence of disturbances, the finite-time (deadbeat) performance is not robust, and only ultimate boundedness is guaranteed.

Future works include: 1) explicit calculation of the finite convergence time, 2) robust finite-time MPC with respect to noise and disturbances, and 3) applying the proposed finite-time MPC to engineering systems, e.g., UAV (unmanned aerial vehicle) [38] and UGV (unmanned ground vehicle) [39].

Appendix
Proof of Controllability of $(\mathcal{A}, \mathcal{B})$

The controllability of the augmented system (31) can be proved by Popov-Belevitch-Hautus (PBH) criterion. For all $s \in \mathbb{C}$, test

$$Rank[sI_{n+p_1} - \mathcal{A}, \mathcal{B}] = Rank\begin{bmatrix} (s-\alpha_1)I_{p_1} & -C_1 & 0 \\ 0 & sI_n - A & B \end{bmatrix}.$$

If $s \neq \alpha_1$, then $Rank[(s-\alpha_1)I_{p_1}] = p_1$, and $Rank[sI_n - A, B] = n$, such that $Rank[sI_{n+p_1} - \mathcal{A}, \mathcal{B}] = n + p_1$.

If $s = \alpha_1$, then

$$Rank\begin{bmatrix} (s-\alpha_1)I_{p_1} & -C_1 & 0 \\ 0 & sI_n - A & B \end{bmatrix} = Rank\begin{bmatrix} 0 & -C_1 & 0 \\ 0 & \alpha_1 I_n - A & B \end{bmatrix} = n + p_1$$

which is given by the rank condition (33). Consequently, for all $s \in \mathbb{C}$, it holds that $Rank[sI_{n+p_1} - \mathcal{A}, \mathcal{B}] = n + p_1$, and $(\mathcal{A}, \mathcal{B})$ is controllable.

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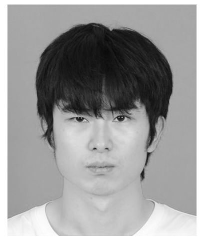

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