Letter

Nonlinear Robust Stabilization of Ship Roll by Convex Optimization

Jiafeng Yu, Qinsheng Li, and Weijie Zhou

Dear Editor,

This letter presents a nonlinear robust controller design method for ship roll stabilization by combining the dual of Lyapunov's stability theorem with the sum of squares (SOS) technique. Varying initial metacentric height and ship speed are regarded as uncertainties, sea waves are considered as external disturbances, and then the robust nonlinear controller is designed. Taking a container ship as an example, simulations are performed to verify the effectiveness of the proposed design scheme.

Introduction: Robustness issues are among the most challenging research problems related to nonlinear control design, especially robustness with respect to uncertain parameters in a dynamic system. Take the ship roll stabilization as an example. During ship navigation, the model parameters for roll control are uncertain resulting from varying ship speed and initial metacentric height. Therefore, the ship roll controller with robustness is always expected [1], [2].

There exists voluminous literature on the subject of designing robust control schemes for ship roll motion. Reference [1] presented a robust adaptive fuzzy control approach for the problem of ship roll stabilization. A robust fin controller based on $L2$ gain design is proposed, in order to reduce the roll motion of surface ships [2]. Reference [3] proposed a robust fin controller based on functional link neural network for roll reduction. Reference [4] addressed application of a robust adaptive first-second-order sliding mode controller in stabilizing the uncertain fin roll dynamics.

Lyapunov's stability theorem has long been recognized as one of the most fundamental analytical tools for analysis and synthesis of nonlinear control systems [5], [6]. Stability analysis of nonlinear system has been a difficult problem since constructing Lyapunov function involves to verify positivity of a function and negativity of its derivative along the system trajectory.

Recent results on the SOS decomposition have transformed the verification of nonnegativity of polynomials into semidefinite programming; hence we can either check if a candidate function satisfies the condition of the Lyapunov's stability theorem or search directly for a polynomial Lyapunov function for stability [7].

There have been some results on the SOS method for controller design in existing literature. To metion a few, in [8], the SOS method is used to design event-based consensus controller for polynomial fuzzy multi-agent systems with Markovian switching signed topology. Reference [9] presents a systematic computational procedure for controller design of polynomial systems with uncertain parameters, combining the SOS techniques with some results on polynomial certificate in real algebraic geometry. Reference [10] studies nonlinear H_∞ control designs of an axisymmetric spacecraft using the SOS method.

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In this letter, combining SOS method and the dual to Lyapunov's stability theorem [11], we present a novel controller design method for the uncertain nonlinear system of ship roll. This combination allows the convex parameterization of the nonlinear controller. Motivated by [9], we explore the proposed method for the controller design of ship roll with uncertain parameters resulting from varying initial metacentric height and varying ship speed. Our objective is to find controller in worst case, in the sense that the system is stabilized for all the possible values of the uncertain parameter set. Therefore, the stability of the uncertain system is guaranteed.

Notations: Throughout the letter, the following notations will be used:

$$\nabla V = \left[\frac{\partial V}{\partial x_1} \cdots \frac{\partial V}{\partial x_n}\right],\ \ V:\mathbb{R}^n \longrightarrow \mathbb{R}$$

$$\nabla \cdot f = \frac{\partial f_1}{\partial x_1} + \cdots + \frac{\partial f_n}{\partial x_n},\ \ f:\mathbb{R}^n \longrightarrow \mathbb{R}^n.$$

Mathematical model of ship roll: In this letter, uncertainties of parameters are taken into account in ship roll mathematical model, which are originated from varying ship speed and initial metacentric height. Consider the following nonlinear model of ship roll motion by fin control [12], [13]:

$$(I_{xx}+J_{xx})\ddot{\theta} + N\dot{\theta} + M\dot{\theta}|\dot{\theta}| + Dh\theta(1-(\frac{\theta}{\theta_v})^2) = F_C + F_{SW} \qquad (1)$$

where $\theta, \dot{\theta}$ and $\ddot{\theta}$ denote the roll angle, angular rate and angular acceleration of roll motion, respectively. I_{xx} and J_{xx} denote the inertia moment and added inertia moment, respectively. $N = N_{\theta 0} + \triangle N_\theta$, $M = M_{\theta 0} + \triangle M_\theta$. $N_{\theta 0}$ and $M_{\theta 0}$ denote the linear damping coefficient and the nonlinear damping coefficient, respectively. D is the displacement of ship. $h = h_0 + \triangle h$, h_0 is specified initial metacentric height. θ_v denotes the flooding angle. $F_C = -\rho_w(V_0+\Delta V)^2 A_F C_{L\alpha} \times l_{F\alpha}(\alpha_f + \frac{\dot{\theta} l_{F\alpha}}{V_0+\Delta V})$. F_C denotes the control moment of fin stabilizer. V_0 denotes nominal ship speed. $\triangle N_\theta$, $\triangle M_\theta$, $\triangle h$, and ΔV are the variation of the parameters resulting from ship loaded conditions, external environment and other reasons. A_F is the area of fin stabilizer, $C_{L\alpha}$ is the slope of lift coefficient, $l_{F\alpha}$ is the force arm of fin stabilizer, α_f is the rotation angle of fin stabilizer. $F_{SW} = F_W \sin\omega_e t$. F_W is the external wave amplitude. ω_e denotes the encounter frequency of the wave. $I_{xx}+J_{xx} = \frac{DB^2}{g}(0.3085 + \frac{0.0227B}{d} - \frac{0.0043L}{100})^2$, $N_{\theta 0} = \frac{2c_1\sqrt{Dh(I_{xx}+J_{xx})}}{\pi}$, $M_{\theta 0} = \frac{3c_2(I_{xx}+J_{xx})}{4}$, where g denotes gravitational acceleration, B denotes the ship breadth, d denotes the draft of the ship. L denotes the ship length. ρ_w is the density of sea water. c_1 and c_2 are the test coefficients for different ship types.

Preliminaries: 1) SOS method: It is well known that the problem of checking global nonnegativity of a polynomial of quartic (or higher) degree is computationally hard. For this reason, we need convenient sufficient conditions that guarantee nonnegativity and are not overly conservative. A particularly interesting sufficient condition is given by the existence of a sum of squares decomposition [14]: if the polynomial $G(x)$ can be written as $G(x) = \sum_i h_i^2(x)$ for some polynomials $h_i(x)$, then $G(x) \geq 0$.

In this respect, it is interesting to notice that many methods used in control theory for constructing Lyapunov functions (for example, backstepping [6]) use either implicitly or explicitly a SOS approach. In particular, the SOS technique provides a good method for constructing Lyapunov functions.

2) Dual to Lyapunov's stability theorem: A convergence criterion for nonlinear systems presented in [11] is viewed as a dual to Lyapunov's stability theorem.

Theorem 1 [11]: Given the system $\dot{x}(t) = f(x(t))$, where $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and $f(0) = 0$, suppose there exists a non-negative $\rho \in C^1(\mathbb{R}^n \backslash \{0\}, \mathbb{R})$ such that $\rho(x)f(x)/|x|$ is integrable on $\{x \in \mathbb{R}^n : |x| \geq 1\}$ and

$$\nabla \cdot (\rho f)(x) > 0, \text{ for almost all } x. \qquad (2)$$

Then, for almost all initial states $x(0)$ the trajectory $x(t)$ exists for $t \in [0, +\infty)$ and tends to zero as $t \to \infty$. Moreover, if the equilibrium $x = 0$ is stable, then the conclusion remains valid even if ρ takes negative values.

Controller design for polynomial systems with certain parameters: Consider the system

$$\dot{x} = f(x) + g(x)u \tag{3}$$

where $x \in \mathbb{R}^n$ is the state vector, $f(x)$ and $g(x)$ are polynomial vectors, and u is the control input vector. Inequality (2) for the polynomial nonlinear system (3) can be written as

$$\nabla \cdot [\rho(f + gu)](x) > 0. \tag{4}$$

Thus, the set $(\rho, u\rho)$ satisfying (4) is convex [15]. In order to search the density function ρ and the controller u jointly, consider the following parameterized representation for ρ and $u\rho$ [15]:

$$\rho = \frac{a(x)}{b(x)^s}, \quad u\rho = \frac{c(x)}{b(x)^s} \tag{5}$$

where $a(x)$, $b(x)$, and $c(x)$ are polynomials. $b(x)$ is positive. s is chosen large enough so as to satisfy the integrability condition in Theorem 1. Note that by choosing this particular representation, we presuppose that we will be searching for ρ and u that are rational. In particular, the controller will be

$$u = \frac{c(x)}{a(x)}. \tag{6}$$

In this case, (4) can be written as

$$\nabla \cdot [\rho(f + gu)] = \nabla \cdot \left[\frac{1}{b(x)^s}(fa + gc)\right]$$
$$= \frac{1}{b(x)^{s+1}}\left[b(x)\nabla \cdot (fa + gc) - s\nabla b(x) \cdot (fa + gc)\right] > 0.$$

Since $b(x)$ is positive, we only need to satisfy the inequality

$$G(x) = b(x)\nabla \cdot (fa + gc) - s\nabla b(x) \cdot (fa + gc) > 0. \tag{7}$$

Assuming that f and g in the above inequality are polynomials corresponding to either the original system or the recasted system obtained through algebraic transformations, the left-hand side of (7) will also be a polynomial. Instead of checking positivity in (7), we will resort to the relaxation to check if $G(x)$ is a SOS, and then the problem can be solved using semidefinite programming [14]. In particular, a free available software SOSTOOLS can be used for solving SOS programs [16].

Robust nonlinear control synthesis for polynomial systems with uncertain parameters: Consider the polynomial system

$$\dot{x} = f(x, \eta) + g(x, \eta)u \tag{8}$$

where $x \in \mathbb{R}^n$ is the state, $f(x, \eta) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $g(x, \eta) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n \times r}$, and $u \in \mathbb{R}^r$. $\eta = (\eta_1, ..., \eta_k)$ is unknown parameter vector but bounded as follows:

$$\underline{\eta_i} \le \eta_i \le \bar{\eta}_i, \ i = 1, \dots, k \tag{9}$$

Our objective is to find a controller u that stabilizes the system (8) for all possible values of the parameter η. Let

$$\zeta_1 = \eta_1 - \underline{\eta_1} \ge 0$$
$$\vdots$$
$$\zeta_{2k-1} = \eta_k - \underline{\eta_k} \ge 0$$
$$\zeta_{2k} = \bar{\eta}_k - \eta_k \ge 0.$$

Thus, define a set K for upcoming use as

$$K = \{(x, \eta) \in \mathbb{R}^n \times \mathbb{R}^m : \zeta_1(\eta) \ge 0, \dots, \zeta_{2k}(\eta) \ge 0\}.$$

Similar to (5), design the controller (6). In this case, the convergence criterion in Theorem 1 for the system (8) can be written as

$$\nabla \cdot [\rho(f + gu)]|_K$$
$$= \frac{1}{b(x)^{s+1}}\left[b(x)\nabla \cdot (fa + gc) - s\nabla b(x) \cdot (fa + gc)\right]|_K > 0.$$

Since $b(x)$ is positive, we only need to satisfy the inequality

$$G_K(x, \eta) = b(x)\nabla \cdot (fa + gc) - s\nabla b(x) \cdot (fa + gc) > 0$$
$$\forall (x, \eta) \in K. \tag{10}$$

Although the system contains uncertain parameters, the explicit expressions of the functions f and g are known, so inequality (10) can be computed.

We remark that (7) is different from (10). We check the global nonnegativity of $G(x)$ in (7) for all $x \in \mathbb{R}^n$, so (7) can be solved by the SOS method directly. However, the nonnegativity of $G_K(x, \eta)$ in (10) requires to be checked only on a local subset $K \in \mathbb{R}^n \times \mathbb{R}^m$, so (10) cannot be solved by SOS method directly.

In order to solve this problem we consider another kind of certificate using some concepts from the field of real algebraic geometry. The controller design procedure is based on the translation of the pointwise property of (10) into an algebraic property (polynomial certificate [9]) that can be directly checked by the SOS method. The specific constructive derivation of a polynomial certificate can be found in [9].

For our purposes, using Positivstellensatz theorem [17] and the polynomial certificate in [9], the pointwise property of inequality (10) can be satisfied if there exists $s_i(x, \eta) \in \Sigma^2$ such that

$$\vartheta(x, \eta) = G_K(x, \eta) - \sum_{i=1}^{2k} s_i(x, \eta)\zeta_i(x, \eta) \in \Sigma^2 \tag{11}$$

where Σ^2 denotes the SOS set. It is worth noting that membership test is sufficient condition to ensure nonnegativity of the polynomial $G_K(x, \eta)$ over the set K.

Robust controller design for ship roll: Here, using the proposed method, we design the controller for a container ship [18], whose length is 175 m, breadth 25.4 m, full-load draft 8.5 m, displacement 21 222 t, fin area 10.2 m^2, fin arm 14.88 m, slope of the fin lift coefficient 3.39, flooding angle 42°, initial metacentric height 1 m, and nominal speed 7.71 m/s. Uncertainties of parameters and environmental disturbances are taken into account in ship roll mathematical model.

Choose the state variables $x_1 = \theta, \ x_2 = \dot{\theta}$. Consider the mathematics model (1) of ship roll with the uncertain parameters

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \eta_1 x_2 + \eta_2 x_2 |x_2| + \eta_3 x_1 + \eta_4 x_1^3 + \eta_5 \alpha_f + f_w \end{cases} \tag{12}$$

where $x_1 = \theta$ denotes the ship roll angle, $x_2 = \dot{\theta}$ denotes angular rate and $v = \eta_5 \alpha_f, \ u = \alpha_f$ is the control input. $\eta_1 = \eta_{10} + \triangle\eta_1$, $\eta_2 = \eta_{20} + \triangle\eta_2, \ \eta_3 = \eta_{30} + \triangle\eta_3, \ \eta_4 = \eta_{40} + \triangle\eta_4, \ \eta_5 = \eta_{50} + \triangle\eta_5$. $\eta_1 = -\frac{N_{\theta 0} + \triangle N_\theta + \rho_w (V_0 + \Delta V) A_F C_{L\alpha} l_{F\alpha}^2}{I_{xx} + J_{xx}}$, $\eta_2 = -\frac{M_{\theta 0} + \triangle M_\theta}{I_{xx} + J_{xx}}$, $\eta_3 = -\frac{D(h_0 + \triangle h)}{I_{xx} + J_{xx}}$, $\eta_4 = \frac{D(h_0 + \triangle h)}{(I_{xx} + J_{xx})\theta_v}$, $\eta_5 = -\frac{\rho_w (V_0 + \Delta V)^2 A_F C_{L\alpha} l_{F\alpha}}{I_{xx} + J_{xx}}$, $f_w = \frac{F_W}{I_{xx} + J_{xx}} \sin \omega_e$. $\eta_{10} = -\frac{N_{\theta 0} + \rho_w V_0 A_F C_{L\alpha} l_{F\alpha}^2}{I_{xx} + J_{xx}}$, $\eta_{20} = -\frac{M_{\theta 0}}{I_{xx} + J_{xx}}$, $\eta_{30} = -\frac{D h_0}{I_{xx} + J_{xx}}$, $\eta_{40} = \frac{D h_0}{(I_{xx} + J_{xx})\theta_v}$, $\eta_{50} = -\frac{\rho_w {V_0}^2 A_F C_{L\alpha} l_{F\alpha}}{I_{xx} + J_{xx}}$. Ship parameters are obtained by nominal speed and initial metacentric height: $\eta_{10} = -0.0106$, $\eta_{20} = -0.0131$, $\eta_{30} = -0.1117$, $\eta_{40} = 0.1983$, $\eta_{50} = -0.0433$. $f_w = \frac{F_W}{I_{xx} + J_{xx}} \sin \omega_e$ is regarded as external disturbance. During ship navigation, model parameters are uncertain resulting from varing initial metacentric height and ship speed. Parameters are uncertain but assumed to be bounded, i.e., $|\triangle\eta_1| \le \psi_1$, $|\triangle\eta_2| \le \psi_2$, $|\triangle\eta_3| \le \psi_3$, $|\triangle\eta_4| \le \psi_4$, $|\triangle\eta_5| \le \psi_5$. ψ_1, ψ_2, ψ_3, ψ_4 and ψ_5 denote the maximum absolute value of the estimated parameters variation, respectively. The bounds can be estimated as $\psi_1 = 0.01$, $\psi_2 = 0.02$, $\psi_3 = 0.06$, $\psi_4 = 0.1$ and $\psi_5 = 0.06$.

In order to satisfy the condition $f(0) = 0$ in Theorem 1, external disturbance f_w is temporarily ignored while designing the controller. In this way, system model is rewritten as

$$\dot{x} = f(x, \eta) + g(x, \eta)v$$

where $f(x, \eta) = \begin{bmatrix} x_2 \\ \eta_1 x_2 + \eta_2 x_2 |x_2| + \eta_3 x_1 + \eta_4 x_1^3 \end{bmatrix}$ and $g(x, \eta) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

To find a controller $u(x)$, we only need to satisfy (10), that is

$$G_K(x,\eta) = b\nabla\cdot(fa+gc) - s\nabla b\cdot(fa+gc) > 0$$
$$\forall(x,\eta)\in K$$

where

$$K = \left\{(x,\eta)\in\mathbb{R}^2\times\mathbb{R}^5 : \zeta_i(\eta)\geq 0, i=1,\ldots,10\right\}$$
$$= \left\{\begin{array}{l} (x,\eta)\in\mathbb{R}^2\times\mathbb{R}^5 : \\ \eta_1+0.0206\geq 0,\ -0.0006-\eta_1\geq 0 \\ \eta_2+0.0331\geq 0,\ 0.0069-\eta_2\geq 0 \\ \eta_3+0.1717\geq 0,\ -0.0517-\eta_3\geq 0 \\ \eta_4-0.0983\geq 0,\ 0.2983-\eta_4\geq 0 \\ \eta_5+0.1033\geq 0,\ 0.0167-\eta_5\geq 0 \end{array}\right\}.$$

Choose $b = 3x_1^2+2x_1x_2+2x_2^2$, a cubic polynomial for $c(x)$, and a constant for $a(x)=1$ and let $s=4$ to satisfy the integrability condition in Theorem 1. Now, we apply for polynomial certificate of nonnegativity of (10) in the form $\vartheta(x,\eta)$, and translate (10) into the algebraic property that can be directly checked by the SOS method. Controller can be found by SOS programming as follows: find s_i, $i=1,\ldots,10$, such that

$$G_K(x,\eta) - \sum_{i=1}^{10} s_i(x,\eta)\zeta_i(x,\eta)\in\Sigma^2.$$

After solving the above SOS problem, $v=-1.3714x_1-x_1^3-x_2|x_2|-0.72219x_2$. The nonlinear controller is obtained as

$$u = \frac{v}{\eta_5} = \frac{-1.3714x_1-x_1^3-x_2|x_2|-0.72219x_2}{\eta_5}.$$

Fig. 1 shows the phase plot of the close-loop system illustrating the convergence to the origin with the initial state $(x_1,x_2)=(0.3,2)$ for five different values for the parameters $(\eta_1,\eta_2,\eta_3,\eta_4)$. For external disturbance, time responses of ship roll angle under the action of fin control and without the fin are shown in Fig. 2.

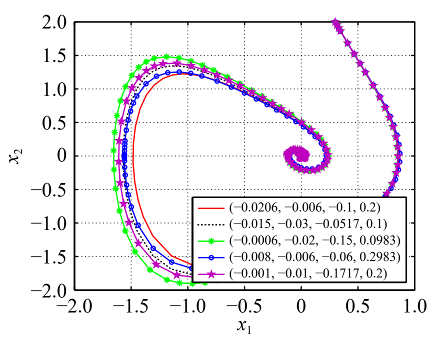

Fig. 1. Phase plot of the closed-loop system about ship roll stabilization. The curves are trajectories with the initial state $(x_1,x_2)=(0.3,2)$.

Conclusion: The focus of this letter is to develop a nonlinear controller design method for a ship roll using the SOS method with the dual of the Lyapunov's stability theorem. We consider the model with nonlinear damping moment and nonlinear restoring moment. Especially, our method can be applied to the ship large rolling angle motion. As shown in the letter, the controller has good robustness to uncertain parameters. Combination of the SOS method and dual of the Lyapunov's stability theorem is a promising method in the problem of ship roll control.

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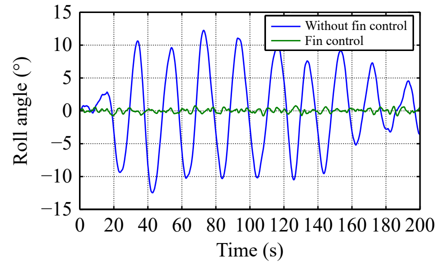

Fig. 2. Time response of ship roll angle.

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