Letter

Stability Analysis of a Class of Nonlinear Fractional Differential Systems With Riemann-Liouville Derivative

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Dear Editor,

This letter investigates the stability of n-dimensional nonlinear fractional differential systems with Riemann-Liouville derivative. By using the Mittag-Leffler function, Laplace transform and the Gronwall-Bellman lemma, one sufficient condition is attained for the asymptotical stability of a class of nonlinear fractional differential systems whose order lies in (0, 2). According to this theory, if the nonlinear term satisfies some conditions, then the stability condition for nonlinear fractional differential systems is the same as the ones for corresponding linear systems. Two examples are provided to illustrate the applications of our result.

Introduction: Fractional calculus is more than 300 years history, but its application to physics and engineering has attracted lots of attention only in the recent years. It has been found that many systems in interdisciplinary fields can be described by fractional differential equations, such as viscoelastic systems, dielectric polarization, electrode–electrolyte polarization, some finance systems and electromagnetic wave. Moreover, applications of fractional calculus have been reported in many areas such as signal processing, image processing, automatic control and robotics. These examples and many other similar samples perfectly clarify the importance of consideration and analysis of dynamical systems with fractional-order models. Significant contributions have been made to both the theory and applications of fractional differential equations (see [1] and references there in). Recently, the stability of fractional differential systems has attracted increasing interest due to its importance in control theory. In 1996, Matignon [2] firstly studied the stability of linear fractional differential systems. Since then, many researchers have studied further on the stability of linear fractional differential systems [3]−[5]. The stability analysis of nonlinear fractional differential systems is much more difficult and only a few available. For example, Li *et al*. investigated the Mittag-Leffler stability of fractional order nonlinear dynamic systems [6] and proposed Lyapunov direct method to check stability of fractional order nonlinear dynamic systems [7]. Wen *et al*. [8] and Zhou *et al*. [9] considered the stability of nonlinear fractional differential systems. Zhang and Yang [10] proposed a single state adaptive-feedback controller for stabilization of three-dimensional fractional-order chaotic systems. Based on the theory of linear matrix inequality (LMI), Faieghi *et al*. [11] proposed a simple controller for stabilization of a class of fractional-order chaotic systems. Wang and Li present the Ulam-Hyers stability for fractional Langevin equations [12], and Ulam-Hyers-Mittag-Leffer stability for fractional delay differential equations [13]. The methods which they proposed for stability of a class of fractional differential equations provide us with a very useful method for studying Hyers-Ulam stable system. That is, one does not have to reach the exact solution. What is required is to get a function which satisfies a suitable approximation inequality.

Problem statement: Note that these papers on the stability of the fractional differential systems mainly concentrated on fractional-order α lying in (0, 1). Recently, Zhang *et al*. [14] considered the stability of nonlinear fractional differential systems with Caputo derivative whose order lies in (0, 2). In this letter, we study the stability of the nonlinear fractional differential systems with Riemann-Liouville derivative whose order lies in (0, 2). By using the Mittag-Leffler function, Laplace transform and the Gronwall-Bellman lemma, a stability theorem is proven theoretically. The stability conditions have no restriction on the norm of the linear parameter matrix A.

Preliminaries:

Lemma 1 [8]: If $A\in\mathbb{C}^{n\times n}$, $0<\alpha<2$, β is an arbitrary real number, μ is such that $\frac{\pi\alpha}{2}<\mu<\min\{\pi,\pi\alpha\}$ and $C_1>0$ is real constant, then

$$\|E_{\alpha,\beta}(A)\|\le\frac{C_1}{1+\|A\|} \tag{1}$$

where $\mu\le\arg(\lambda(A))\le\pi$, $\lambda(A)$ denotes the eigenvalues of matrix A and $\|\cdot\|$ denotes the l_2-norm.

Lemma 2 [15] (Gronwall-Bellman lemma): If

$$\varphi(t)\le h(t)+\int_{t_0}^{t}g(\tau)\varphi(\tau)d\tau,\quad t_0\le t\le t_1$$

where $g(t)$, $h(t)$ and $\phi(t)$ are continuous on $[t_0,\ t_1]$, $t_1\to\infty$, and $g(t)\ge 0$. Then, $\varphi(t)$satisfies

$$\phi(t)\le h(t)+\int_{t_0}^{t}h(\tau)g(\tau)\exp[\int_{\tau}^{t}g(s)ds]d\tau,\ \ t_0\le t\le t_1. \tag{2}$$

In addition, if $h(t)$ is nondecreasing, then

$$\varphi(t)\le h(t)\exp[\int_{t_0}^{t}g(s)ds]d\tau,\qquad t_0\le t\le t_1. \tag{3}$$

Main results: In this section, based on the above definitions and lemmas, we present the stability theorem of a class of nonlinear fractional differential systems as follows.

Theorem 1: Consider the following systems of nonlinear fractional differential equation:

$${}_{0}^{RL}D_t^{\alpha}x(t)=Ax(t)+f(x(t)) \tag{4}$$

where $x(t)\in\mathbb{R}^{n\times 1}$ is the state vector, $A\in\mathbb{R}^{n\times n}$ is the constant parameter matrix, $f(x(t))\in\mathbb{R}^{n\times 1}$ is a nonlinear function vector and $0<\alpha<2$. If

1) The matrix A such that $|\lambda(A)|\ne 0$, $|\arg(\lambda(A))|>\alpha\pi/2$;

2) The function $f(x(t))$ satisfies $f(0)=0$ and

$$\lim_{x(t)\to 0}\frac{\|f(x(t))\|}{\|x(t)\|}=0. \tag{5}$$

Then, the zero solution of (11) is asymptotically stable.

Proof: 1) The case $0<\alpha<1$.

In this case, the initial condition is

$${}_{0}^{RL}D_t^{\alpha-1}x(t)|_{t=0}=x_0. \tag{6}$$

Taking Laplace transform on (11), we have

$$X(s)=(Is^{\alpha}-A)^{-1}(x_0+L[f(x(t))]) \tag{7}$$

where I is an $n\times n$ identity matrix.

Then, taking Laplace inverse transform for (7), it yields

$$x(t)=x_0t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(A(t-\tau)^{\alpha})f(x(\tau))d\tau. \tag{8}$$

By the condition (5), there exists $C_1>0$ and $\delta>0$, such that

$$\|f(x(t))\|<\frac{\alpha\|A\|}{C_1}\|x(t)\|\ \text{ as }\ \|x(t)\|<\delta. \tag{9}$$

From (9) and Lemma 1, (8) gives

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$$\|x(t)\| \le \frac{C_1\|x_0\|t^{\alpha-1}}{1+\|At^{\alpha}\|} + \int_0^t \frac{\|(t-\tau)^{\alpha-1}\|C_1}{1+\|A(t-\tau)^{\alpha}\|}\frac{\alpha\|A\|}{C_1}\|x(\tau)\|d\tau$$
$$= \frac{C_1\|x_0\|t^{\alpha-1}}{1+\|A\|t^{\alpha}} + \int_0^t \frac{\alpha\|A\|(t-\tau)^{\alpha-1}}{1+\|A\|(t-\tau)^{\alpha}}\|x(\tau)\|d\tau.$$

According to Lemma 2.2, we obtain

$$\|x(t)| \le \frac{C_1\|x_0\|t^{\alpha-1}}{1+\|A\|t^{\alpha}} + \int_0^t \frac{C_1\|x_0\|\tau^{\alpha-1}}{1+\|A\|\tau^{\alpha}} \times \frac{\alpha\|A\|(t-\tau)^{\alpha-1}}{1+\|A\|(t-\tau)^{\alpha}}$$
$$\times \exp\left(\int_\tau^t \frac{\alpha\|A\|(t-s)^{\alpha-1}}{1+\|A\|(t-s)^{\alpha}}ds\right)d\tau$$
$$= \frac{C_1\|x_0\|}{t^{1-\alpha}+\|A\|t} + \int_0^t \frac{\alpha C_1\|x_0\|\tau^{\alpha-1}\|A\|(t-\tau)^{\alpha-1}}{1+\|A\|\tau^{\alpha}}d\tau$$
$$\le \frac{C_1\|x_0\|}{t^{1-\alpha}+\|A\|t} + \alpha C_1\|x_0\|\int_0^t \tau^{\alpha-1}(t-\tau)^{\alpha-1}d\tau$$
$$\le \frac{C_1\|x_0\|}{t^{1-\alpha}+\|A\|t} + \alpha C_1\|x_0\|\Gamma(\alpha)\Gamma(\alpha)t^{2(\alpha-1)} \to 0 \text{ as } t\to\infty.$$

So, the zero solution of (4) is asymptotically stable.

2) The case $1<\alpha<2$.

In this case, the initial condition is

$$ {}_0^{RL}D_t^{\alpha-k}x(t)|_{t=0} = x_{k-1},\ (k=1,\ 2). \tag{10}$$

We can get the solution of (4) with the initial condition (10) by using the Laplace transform and Laplace inverse transform

$$x(t) = x_0t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha}) + t^{\alpha-2}x_1E_{\alpha,\alpha-1}(At^{\alpha})$$
$$+ \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(A(t-\tau)^{\alpha})f(x(\tau))d\tau. \tag{11}$$

By the condition (5), there exists $C_1>0$ and $\delta>0$, such that

$$\|f(x)\| < \frac{\alpha\|A\|}{2C_1}\|x(t)\| as \|x(t)\| < \delta. \tag{12}$$

From (12) and Lemma 1, (11) gives

$$\|x(t)\| \le \frac{C_0\|x_0\|t^{\alpha-1}}{1+\|At^{\alpha}\|} + \frac{C_1\|x_1\|t^{\alpha-2}}{1+\|At^{\alpha}\|}$$
$$+ \int_0^t \frac{\|(t-\tau)^{\alpha-1}\|C_1}{1+\|A(t-\tau)^{\alpha}\|}\frac{\alpha\|A\|}{2C_1}\|x(\tau)\|d\tau$$
$$= \frac{C_0\|x_0\|t^{\alpha-1}}{1+\|A\|t^{\alpha}} + \frac{C_1\|x_1\|t^{\alpha-2}}{1+\|A\|t^{\alpha}}$$
$$+ \int_0^t \frac{\|(t-\tau)^{\alpha-1}\|}{1+\|A\|(t-\tau)^{\alpha}}\frac{\alpha\|A\|}{2}\|x(\tau)\|d\tau$$
$$\le \frac{C_0\|x_0\|t^{\alpha-1}}{(1+\|A\|t^{\alpha})^{0.5}} + \frac{C_1\|x_1\|t^{\alpha-2}}{1+\|A\|t^{\alpha}}$$
$$+ \int_0^t \frac{\|(t-\tau)^{\alpha-1}\|}{1+\|A\|(t-\tau)^{\alpha}}\frac{\alpha\|A\|}{2}\|x(\tau)\|d\tau.$$

According to Lemma 2, we obtain

$$\|x(t)| \le \frac{C_0\|x_0\|t^{\alpha-1}}{(1+\|A\|t^{\alpha})^{0.5}} + \frac{C_1\|x_1\|t^{\alpha-2}}{1+\|A\|t^{\alpha}}$$
$$+ \int_0^t\Big(\frac{C_0\|x_0\|t^{\alpha-1}}{(1+\|A\|t^{\alpha})^{0.5}} + \frac{C_1\|x_1\|t^{\alpha-2}}{1+\|A\|t^{\alpha}}\Big)$$
$$\times \frac{\alpha\|A\|(t-\tau)^{\alpha-1}}{2(1+\|A\|(t-\tau)^{\alpha})}\exp\Big(\int_\tau^t \frac{\alpha\|A\|(t-s)^{\alpha-1}}{2(1+\|A\|(t-s)^{\alpha})}ds\Big)d\tau$$
$$= \frac{C_0\|x_0\|t^{\alpha-1}}{(1+\|A\|t^{\alpha})^{0.5}} + \frac{C_1\|x_1\|t^{\alpha-2}}{1+\|A\|t^{\alpha}}$$
$$+ \int_0^t\Big(\frac{C_0\|x_0\|t^{\alpha-1}}{(1+\|A\|t^{\alpha})^{0.5}} + \frac{C_1\|x_1\|t^{\alpha-2}}{1+\|A\|t^{\alpha}}\Big)\times\frac{\alpha\|A\|(t-\tau)^{\alpha-1}}{2(1+\|A\|(t-\tau)^{\alpha})^{0.5}}d\tau$$
$$= \frac{C_0\|x_0\|t^{\alpha-1}}{(1+\|A\|t^{\alpha})^{0.5}} + \frac{C_1\|x_1\|t^{\alpha-2}}{1+\|A\|t^{\alpha}} + \int_0^t \frac{C_0\|x_0\|t^{\alpha-1}}{(1+\|A\|t^{\alpha})^{0.5}}$$
$$\times \frac{\alpha\|A\|(t-\tau)^{\alpha-1}}{2(1+\|A\|(t-\tau)^{\alpha})^{0.5}}d\tau + \int_0^t \frac{C_1\|x_1\|t^{\alpha-2}}{1+\|A\|t^{\alpha}}$$
$$\cdot \frac{\alpha\|A\|(t-\tau)^{\alpha-1}}{2(1+\|A\|(t-\tau)^{\alpha})^{0.5}}d\tau \le \frac{C_0\|x_0\|t^{\alpha-1}}{(1+\|A\|t^{\alpha})^{0.5}} + \frac{C_1\|x_1\|t^{\alpha-2}}{1+\|A\|t^{\alpha}}$$
$$+ \frac{\alpha C_0\|x_0\|}{2\|A\|^{0.5}}\int_0^t \tau^{0.5\alpha-1}(t-\tau)^{0.5\alpha-1}d\tau$$
$$+ \frac{\alpha C_1\|x_1\|}{2\|A\|}\int_0^t \tau^{\alpha-2}(t-\tau)^{0.5\alpha-1}d\tau$$
$$\le \frac{C_0\|x_0\|}{\|A\|^{0.5}t^{1-0.5\alpha}} + \frac{C_1\|x_1\|}{\|A\|t^2} + \frac{\alpha C_0\|x_0\|}{2\|A\|^{0.5}}\frac{\Gamma(0.5\alpha)\Gamma(0.5\alpha)}{\Gamma(\alpha)t^{2-\alpha}}$$
$$+ \frac{\alpha C_1\|x_1\|}{2\|A\|}\frac{\Gamma(2-\alpha)\Gamma(0.5\alpha)}{\Gamma(2-0.5\alpha)t^{3-1.5\alpha}} \to 0 \text{ as } t\to\infty.$$

So, the zero solution of (4) is asymptotically stable. ■

Remark 1: The nonlinear term of many fractional order chaotic systems satisfy (5). For example, fractional-order Lorenz system [15], fractional-order Chen system [16], fractional-order Lu system [17], fractional-order Liu system [18], fractional-order Arneodo system [19], fractional-order Chua system [20] and fractional-order hyperchaotic Chen system [21], etc. So, Theorem 1 can be applicable to control chaos in a large class of generalized fractional-order chaotic or hyperchaotic systems via a linear feedback controller.

Remark 2: Theorem 1 provides us with a simple procedure for determining the stability of the fractional order nonlinear systems with Riemann-Liouville derivative with order $0<\alpha<2$. If the nonlinear term $f(x(t))$ satisfies (10), then one does not have to reach the exact solution. What is required is to calculate the eigenvalues of the matrix A, and test their arguments. If $|\arg(\lambda_i(A))| > \alpha\pi/2$ for all i, we conclude that the origin is asymptotically stable.

Two illustrative examples: The following illustrative examples are provided to show the effectiveness of the stability theorem. When numerically solving fractional differential equations, we adopt the method introduced in [22].

Example 1: Consider the nonlinear fractional differential systems

$$ {}_0^{RL}D_t^{\alpha}x_1 = x_1+x_2+x_3+x_2x_3$$
$$ {}_0^{RL}D_t^{\alpha}x_2 = -x_1+x_2-x_3+x_2^2$$
$$ {}_0^{RL}D_t^{\alpha}x_3 = x_1x_2-x_3. \tag{13}$$

System (13) can be rewritten as (1), in which

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix},\quad f(x(t)) = \begin{pmatrix} x_2x_3 \\ x_2^2 \\ x_1x_2 \end{pmatrix}. \tag{14}$$

Obviously, it is easy to verify that

$$\lim_{\|x(t)\|\to 0}\frac{\|f(x(t))\|}{\|x(t)\|} = \lim_{\|x(t)\|\to 0}\frac{\sqrt{(x_2x_3)^2+x_2^4+(x_1x_2)^2}}{\sqrt{x_1^2+x_2^2+x_3^2}}$$
$$\le \lim_{\|x(t)\|\to 0}\frac{\sqrt{(x_2x_3)^2+x_2^4+(x_1x_2)^2}}{\sqrt{x_2^2}} \le \lim_{\|x(t)\|\to 0}\sqrt{x_3^2+x_2^2+x_1^2} = 0$$

which implies that $f(x(t)$ satisfies Conditions (2) in Theorem 1. By using simple calculation, the eigenvalues of A are $\lambda_{1,2} = 1\pm i$ and. $\lambda_3 = -1$. According to Theorem 1, if $\alpha<0.5$, the zero solution of (13) is asymptotically stable. Simulation results are displayed in Fig. 1. Fig. 1(a) shows the zero solution of the system (13) is asymptotically stable with $\alpha = 0.49$, Fig. 1(b) shows the zero solution of the system (13) is not stable with $\alpha = 0.5$.

Example 2: Consider the nonlinear fractional differential systems

$$ {}_0^{\pi L}D_t^{\alpha}x_1 = -x_1+x_2x_3$$
$$ {}_0^{a}D_t^{\alpha}x_2 = x_3$$
$$ {}_0^{\pi L}D_t^{\alpha}x_3 = x_1-x_2-x_3-x_1x_2 \tag{15}$$

(15) can be rewritten as (1), in which

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix},\quad f(x(t)) = \begin{pmatrix} x_2x_3 \\ 0 \\ -x_1x_2 \end{pmatrix}. \tag{16}$$

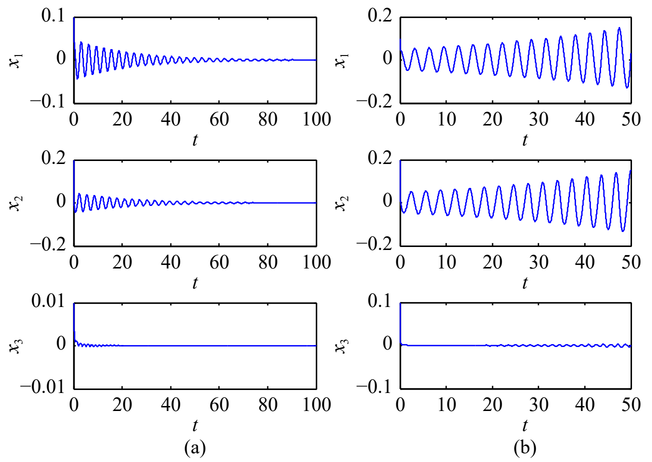

Fig. 1. The stability of the zero solution of system (13). (a) System (13) is asymptotically stable with $\alpha = 0.49$; (b) System (13) is not stable with $\alpha = 0.50$.

Obviously, it is easy to verify that $\lim_{\|x(t)\| \to 0} \frac{\|f(x(t))\|}{\|x(t)\|} = 0$, which implies that $f(x(t)$ satisfies Condition (2) in Theorem 1. By using simple calculation, the eigenvalues of A are $\lambda_{1,2} = -1/2 \pm \sqrt{3}i/2$ and $\lambda_3 = -1$. According to Theorem 1, if $\alpha < 4/3$, the zero solution of (13) is asymptotically stable. Simulation results are displayed in Fig. 2. Fig. 2(a) shows the zero solution of the system (15) is asymptotically stable with $\alpha = 1.30$, Fig. 2(b) shows the zero solution of the system (15) is not stable with $\alpha = 1.34$.

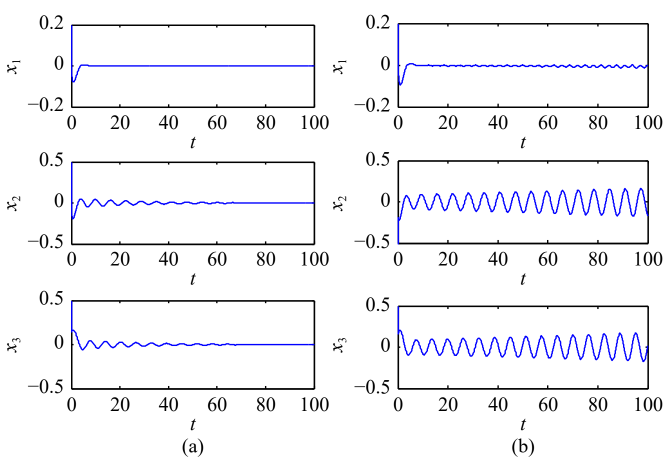

Fig. 2. The stability of the zero solution of system (15). (a) System (15) is asymptotically stable with $\alpha = 1.30$; (b) System (15) is not stable with $\alpha = 1.34$.

Conclusions: In this letter, we have studied the local asymptotic stability of the zero solution of n-dimensional nonlinear fractional differential systems with Riemann-Liouville derivative. The results are obtained in terms of the Mittag-Leffler function, Laplace transform and the Gronwall-Bellman lemma. Compare the current results with the results in [14], it shows the stability condition of Riemann-Liouville fractional differential systems, is same as one of Caputo fractional differential systems. Three numerical examples are given to demonstrate the effectiveness of the proposed approach.

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