Target Controllability of Multi-Layer Networks With High-Dimensional Nodes

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***Abstract*—This paper studies the target controllability of multi-layer complex networked systems, in which the nodes are high-dimensional linear time invariant (LTI) dynamical systems, and the network topology is directed and weighted. The influence of inter-layer couplings on the target controllability of multi-layer networks is discussed. It is found that even if there exists a layer which is not target controllable, the entire multi-layer network can still be target controllable due to the inter-layer couplings. For the multi-layer networks with general structure, a necessary and sufficient condition for target controllability is given by establishing the relationship between uncontrollable subspace and output matrix. By the derived condition, it can be found that the system may be target controllable even if it is not state controllable. On this basis, two corollaries are derived, which clarify the relationship between target controllability, state controllability and output controllability. For the multi-layer networks where the inter-layer couplings are directed chains and directed stars, sufficient conditions for target controllability of networked systems are given, respectively. These conditions are easier to verify than the classic criterion.**

***Index Terms*—High-dimensional nodes, inter-layer couplings, multi-layer networks, target controllability.**

I. Introduction

COMPLEX network is a powerful tool for modeling complex systems [1]−[3]. A large number of complex systems can be described by complex networks. The ultimate goal of studying complex network is to understand and control it to serve human beings. Fully controllability is the prerequisite for achieving control of complex networks [4], [5]. Controllability of complex network has been extensively studied and developed in the past few decades, and many controllability conditions have been established.

At first, the research on controllability of complex networks mainly focuses on the structural controllability [6]−[8] and state controllability of the networks that nodes were denoted by one-dimensional state variables. In this case, graph theory and linear algebra are the main theoretical tools. However, with the deepening of understanding of networks, it has been found that many real complex networks cannot be perfectly described by one-dimensional nodes, which usually need to be described by high-dimensional nodes [9]. For example, in power systems, generator units have many physical parameters that affect the operation of the same units, similar situations also exist in social networks [10], ecological networks [11], communication networks [12], and transportation networks [13], which also require high-dimensional nodes describing the situations. The research on controllability of networks with high-dimensional node dynamics is mainly based on the Kalman rank criterion [14] and the PBH (Popov-Belevitch-Hautus) criterion [15]. Wang *et al.* [9] studied the controllability of networked systems with high-dimensional homogeneous node dynamics, and found the controllability of the overall network is the integrated result of node-system dynamic and network topology. Jiang *et al.* [16] studied the controllability of multi-relational networks, and gave some controllability conditions based on the geometric multiplicity of the eigenvalues of matrix. Hao *et al.* [17] established the relationship between decentralized fixed modes and controllability in multi-input-multi-output (MIMO) systems and single-input-single-output (SISO) systems separately. Kong *et al.* [18] considered the situation that the inner-coupling matrices among all nodes are nonidentical, and proposed controllability conditions suitable for networked systems with high-dimensional heterogeneous node dynamics. Hao *et al.* [19], [20] derived lower-dimensional conditions by generalized left eigenvectors to verify the controllability of networked systems with high-dimensional node dynamics.

The above works mainly focus on single-layer networks and do not take into account the interaction between networks. However, in the real world, complex networks are usually not independent of each other, but interact with each other. For example, subways and buses on different routes typically have multiple public stops, which form a multi-layer transportation network; In daily life, people often have different types of relationships with others, such as kinship and friendship, which form a multi-layer social network [21]. Multi-layer networks are very common in reality, but the methods applicable to single-layer networks may not be applicable to multi-layer networks. Therefore, in recent years, people have begun to study methods suitable for multi-layer networks [22]−[26]. For general multi-layer networks, Wu *et al.* [22] studied the state controllability of multi-layer homogeneous networks, and provided sufficient and necessary controllability conditions for two types of multi-layer networks with special inter-layer couples. For Kronecker product networks, Doostmoham-

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madian and Khan [23] proposed the minimum sufficient condition for the controllability of the networked systems based on S-rank and strong connectivity; Hao *et al.* [24] proposed the necessary and sufficient conditions for verifying the controllability of the Kronecker product network based on generalized eigenvectors. For Cartesian product networks, Yang *et al.* [25] studied the controllability of multilayer sampled-data network and presented some easier verified methods; She *et al.* [26] studied the controllability of multilayer networks from the perspective of energy dependence and provided conditions for determining the controllability.

With the deepening of research on the controllability of complex networks, it has been found that for a large complex network with thousands of nodes, achieving complete controllability of each node is very difficult and often unnecessary. For example, in fields such as biology and economic networks, it is usually not necessary to guide the state set of all nodes to the expected value, but only a portion of nodes (state subset) to the expected values [27]. Therefore, target controllability has become a concern in recent years [21], [28]–[32]. Ding *et al.* [21] proposed a target controllability algorithm with minimum cost and maximum traffic. Song *et al.* [31] solved the problem of adding the least control input to ensure the controllability of each target node in a two-layer network with one-dimensional node by the target path coverage algorithm. Hao *et al.* [32] considered the target controllability of single-layer network with high-dimensional node dynamics and provided sufficient and necessary conditions based on generalized left eigenvectors. In summary, most of the researches on the target controllability of networked systems with high-dimensional node dynamics mainly focus on single-layer networks, and few on multi-layer networks.

Based on the above discussion, this paper investigates the problem of target controllability in multi-layer networks, in which the nodes are high-dimensional dynamical systems. The main work and contributions are summarized as follows: 1) The influence of inter-layer coupling on the target controllability of the entire network is revealed. The target controllability of the entire network is not solely determined by the target controllability of each layer, but also related to inter-layer coupling. 2) The sufficient and necessary criteria for target controllability of multi-layer networks with general structure are given by establishing the relationship between the uncontrollable subspace and the output matrix of the networked system. Besides, the relationship between target controllability, state controllability and output controllability is clarified. 3) For multi-layer networks with special inter-layer structure (directed chains and directed stars), sufficient conditions are provided for verifying the target controllability. Compared to the conditions in [32], these conditions are more applicable.

II. Preliminaries

A. Notation and Mathematical Preliminaries

In this paper, $\mathbb{R}$ ($\mathbb{C}$) represents the set of real (complex) numbers, $\mathbb{R}^n$ ($\mathbb{C}^n$) represents n-dimensional real (complex) vector space, $\mathbb{R}^{n\times m}$ ($\mathbb{C}^{n\times m}$) represents the set of $n\times m$ real (complex) matrices. $\otimes$ represents Kronecker product operation. I_n represents a $n\times n$ identity matrix. $\sigma(A)$ denotes the set of eigenvalues of matrix A. $\text{diag}\{a_1,a_2,\dots,a_n\}$ represents a diagonal matrix with diagonal entries $a_1,a_2,\dots,a_n$. $\text{diag}\{A_1,A_2,\dots,A_n\}$ denotes a diagonal matrix with diagonal blocks $A_1,A_2,\dots,A_n$. $\text{span}\{v_1,v_2,\dots,v_k\} = \{\sum_{i=1}^{c} c_iv_i|c_i\in\mathbb{C}\}$ represents the linear combination of row vectors $v_1,v_2,\dots,v_k$. $\text{span}(A)$ denotes the linear combination of row vectors of matrix A. $\text{Lker}(A)$ denotes the left null space of matrix A. $\Upsilon(\sigma|A)$ denotes the left eigenspace of A related to the eigenvalue σ. Without specified, the operations between matrices are considerd to be compatible. $e_i^T\in\mathbb{R}^N$ is a column vector with all zero entries except for $[e_i]_i = 1$.

Definition 1 [33]: x_k is a k-th-order generalized left eigenvector (LE) of matrix A associated with the eigenvalue σ if $x_k(A-\sigma I)^k = 0$ and $x_k(A-\sigma I)^{k-1}\neq 0$. If a set of vectors $\{x_1, x_2,\dots,x_k\}$ satisfies

$$\begin{cases} x_1\in Lker(A-\sigma I)\setminus\{0\}\\ x_{i+1}(A-\sigma I) = x_i,\ i = 1,\dots,k-1\end{cases}$$

then it is called a left Jordan chain (LJC) about x_1 of matrix A.

B. LTI System and Controllability

Consider a linear time invariant (LTI) system

$$\begin{cases}\dot{x} = Ax+Bu\\ y = Cx\end{cases} \tag{1}$$

where x is the n-dimension state vector, u is the p-dimension input vector, y is the m-dimension output vector, A, B and C are $n\times n$ state matrix, $n\times p$ input matrix and $m\times n$ output matrix, respectively.

Definition 2 [6]: The LTI system (1) is controllable if there is a piecewise continuous input u that can make it transfer from any initial state $x(t_0)$ to final state $x(t) = 0$ within a finite time $[t_0;t]$.

Lemma 1 [34]: Considering the system (1), the following are equivalent:

1) The system (A, B) is controllable;

2) For any eigenvalues λ_i of A, there is no non-zero eigenvector α simultaneously satisfying $\alpha^TA = \lambda_i\alpha^T$ and $\alpha^TB = 0$;

3) $rank(sI_n-A,B) = n$, $\forall s\in\mathbb{C}$.

$\langle A|B\rangle$ denotes the controllable subspace of the system (A,B), where controllable states are all in this subspace. The orthogonal complementary space $\langle A|B\rangle^\perp$ denotes the uncontrollable subspace of the system (A,B), which is equal to the left zero space of the controllability matrix $[B,AB,\dots,A^{n-1}B]$ [32]. Any state in this subspace is uncontrollable and any uncontrollable state belongs to this subspace.

The eigenvalue of A are $\mu_1,\dots,\mu_s$. For any eigenvalue μ_i, the corresponding LJC is $p_i^1,\dots,p_i^{\theta_i}$. Let $\Omega = \{p_i^j|p_i^jB = 0,\dots, p_i^1B = 0, 1\le i\le s, 1\le j\le\theta_i\}$. The uncontrollable subspace of the system (A,B) is $\langle A|B\rangle^\perp = \text{span}\,(\Omega)$ [32].

Definition 3 [35]: The system (1) is said to be completely output controllable on $[t_0;t_f]$, if for given t_0 and t_f, any output $y(t_f)$ can be attained starting with arbitrary initial conditions at $t = t_0$.

Lemma 2 [35]: The system (1) is output controllable if and only if the output controllability matrix $Q = [CB,CAB,\dots, CA^{n-1}B]$ has full rank, i.e., $rankQ = m$.

III. Multi-Layer Network System

A. Model Description

A weighted and directed M-layer networked system is presented below. Each layer consisting of N node systems can be described as follows:

$$\begin{cases} \dot{x}_i^K = A^K x_i^K + \sum_{j=1, j\neq i}^{N} w_{ij}^K H^K x_j^K \\ \qquad + \sum_{J=1, J\neq K}^{M} \sum_{j=1}^{N} d_{ij}^{KJ} H^{KJ} x_j^J + \delta_i^K B^K u_i^K \\ y_i^K = \tau_i^K x_i^K,\ i = 1, 2, \ldots, N \end{cases} \tag{2}$$

where $x_i^K \in \mathbb{R}^n$, $u_i^K \in \mathbb{R}^p$ and $y_i^K \in \mathbb{R}^m$ denote the state vector, input vector and output vector of node i in layer K, respectively. $A^K \in \mathbb{R}^{n\times n}$ and $B^K \in \mathbb{R}^{n\times p}$ denote the state matrix and input matrix of nodes, respectively in layer K. $H^K \in \mathbb{R}^{n\times n}$ is the inner-coupling matrix, which describes the coupling mode between components of nodes in layer K. $H^{KJ} \in \mathbb{R}^{n\times n}$ is the inter-layer coupling matrix, which describes the coupling mode between components of nodes in layer K and layer J, $H^{KJ} = 0$ if $K = J$. The weighted adjacency matrix $W^K = [w_{ij}^K] \in \mathbb{R}^{N\times N}$ and $D^{KJ} = [d_{ij}^{KJ}] \in \mathbb{R}^{N\times N}$ denote intra-layer network topology and inter-layer network topology, respectively. There is a directed edge from node j to node i in layer K if $w_{ij}^K \neq 0$ and no edge from node j to node i if $w_{ij}^K = 0$. Generally, $w_{ii}^K = 0$. Similarly, there is a directed edge from node j in layer J to node i in layer K if $d_{ij}^{KJ} \neq 0$ and no edge from node j in layer J to node i in layer K if $d_{ij}^{KJ} = 0$. $\delta_i^K = 1$ if the node i in layer K is a controlled node, otherwise, $\delta_i^K = 0$. $\tau_i^K = 1$ if the node i in layer K is selected as a target node, otherwise, $\tau_i^K = 0$.

$x = [(x^1)^T, \ldots, (x^M)^T]^T$ is the vector describing the whole state of the networked system, where $x^K = [(x_1^K)^T, \ldots, (x_N^K)^T]^T$ is the state vector of layer K. $u = [(u^1)^T, \ldots, (u^M)^T]^T$ is the vector describing the input of the whole networked system, in which $u^K = [(u_1^K)^T, \ldots, (u_N^K)^T]^T$ is the input vector of layer K. $y = [(y^1)^T, \ldots, (y^M)^T]^T$ is the vector describing the output of the whole networked system, where $y^K = [(y_{i1}^K)^T, \ldots, (y_{im_K}^K)^T]^T$ is the output vector of layer K. $\Phi = [\Phi^{KJ}] \in \mathbb{R}^{MNn\times MNn}$ denotes the state matrix of the whole networked system, where $K = 1, \ldots, M$, $J = 1, \ldots, M$. $\Phi^{KK} = I_N \otimes A^K + W^K \otimes H^K$ if $K = J$ and $\Phi^{KJ} = D^{KJ} \otimes H^{KJ}$ if $K \neq J$. $\Psi = \mathrm{diag}\{\Psi^1, \ldots, \Psi^M\}$ is the input matrix of the whole networked system, in which $\Psi^K = \Delta^K \otimes B^K$ is the input matrix of layer K. $\Delta^K = \mathrm{diag}\{\delta_1^K, \ldots, \delta_N^K\}$ describes the connection relationship between external control inputs and nodes in layer K. $\Xi = T \otimes I_n$ denotes the output matrix of the whole networked system, where $T = \mathrm{diag}\{T^1, \ldots, T^M\}$ describes the output channel of the entire networked system. $T^K = [(e_{i1}^K)^T, \ldots, (e_{im_K}^K)^T]^T$ describes the output channel of layer K, and m_K represents the number of target nodes in the K-th layer. System (2) can be written as the following equations:

$$\begin{cases} \dot{x} = \Phi x + \Psi u \\ y = \Xi x. \end{cases} \tag{3}$$

If the system output is represented by a subset of states (the target set), target controllability is a special case of output controllability [32]. Let $Z = \{x_{i1}^1, \ldots, x_{im_1n}^1, \ldots, x_{i1}^M, \ldots, x_{im_Mn}^M\} \subset \{x_1^1, \ldots, x_{Nn}^1, \ldots, x_1^M, \ldots, x_{Nn}^M\}$ represent the set of target nodes. According to the notion of output controllability, the definition of target controllability is presented.

Definition 4 [32]: Consider networked system (3), where output matrix $\Xi = [e_{i1}^T, \ldots, e_{i(m_1+\cdots+m_M)n}^T]^T \in \mathbb{R}^{(m_1+\cdots+m_M)n\times MNn}$ (this means the output is the subset of state of target nodes). System (Φ, Ψ, Ξ) is said to be target controllable if it is output controllable.

It can be seen that target controllability is a special case of output controllability, which means it is feasible to verify the target controllability by the output controllability criteria.

Lemma 3 [35]: The system (Φ, Ψ, Ξ) is target controllable, if and only if

$$rank(\Xi\Psi, \Xi\Phi\Psi, \ldots, \Xi\Phi^{MNn-1}\Psi) = (m_1 + \cdots + m_M)n.$$

For single-layer networks, a method for calculating the eigenvalues of the state matrix and the LJC corresponding to each eigenvalue of the networks by lower-dimensional matrices is provided in [24]. Therefore, there is the following lemma if only one layer is considered (such as the K-th layer).

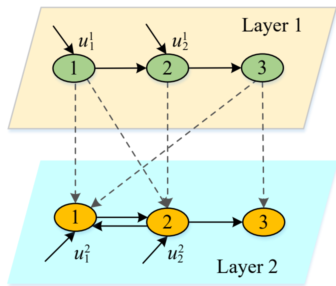
Lemma 4 [24]: Assume W^K is diagonalizable. $\sigma(W^K) = \{\mu_1^K, \mu_2^K, \ldots, \mu_N^K\}$ and $\sigma(A^K + \mu_i^K H^K) = \{\lambda_{i1}^K, \lambda_{i2}^K, \ldots, \lambda_{ir_i^K}^K\}$, then $\sigma(\Phi^{KK}) = \{\lambda_{11}^K, \ldots, \lambda_{1r_1^K}^K, \ldots, \lambda_{N1}^K, \ldots, \lambda_{Nr_N^K}^K\}$. The LJC about eigenvalue λ_{ij}^K of Φ^{KK} is $\eta_{ij}^K(1) = p_i^K \otimes \phi_{ij}^K(1)$, $\eta_{ij}^K(2) = p_i^K \otimes \phi_{ij}^K(2), \ldots, \eta_{ij}^K(\theta_{ij}^K) = p_i^K \otimes \phi_{ij}^K(\theta_{ij}^K)$, where p_i^K is the left eigenvector associated with eigenvalue μ_i^K of W^K, and $\phi_{ij}^K(1), \phi_{ij}^K(2), \ldots, \phi_{ij}^K(\theta_{ij}^K)$ is the LJC about the eigenvalue λ_{ij}^K of $A^K + \mu_i^K H^K$, for all $K = 1, \ldots, M$, $i = 1, \ldots, N$, $j = 1, \ldots, r_i^K$.

Lemmas 3 and 4 will be used in subsequent proofs.

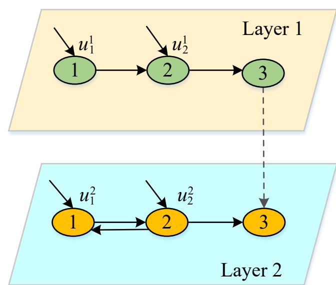
B. The Impact of Inter-Layer Couplings on the Target Controllability of Networks

In this section, an example is provided to illustrate the impact of inter-layer couplings on the target controllability of multi-layer networked systems.

Example 1: Consider a two-layer networked system, in which each layer has $N = 3$ nodes, as shown in Fig. 1(a). In Layer 1, the topology matrix is $W^1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Nodes 1 and 2 are selected as both controlled nodes and target nodes (i.e., $\Delta^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $T^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$). The state matrix of node is $A^1 = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$, the input matrix is $B^1 = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$ and the inner-coupling matrix is $H^1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$. After calculation, it can be got $rank(T^1 \otimes I_2[\Psi^1, \Phi^{11}\Psi^1, \ldots, (\Phi^{11})^5\Psi^1]) = 4$. According to Lemma 3, if only considering the network structure of Layer



(a)



(b)

Fig. 1. Two two-layer networks with different inter-layer coupling modes ((a) The original multi-layer network; (b) The multi-layer network with varying inter-layer couplings).

1, it is target controllable. In Layer 2, the topology matrix is $W^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Nodes 1 and 2 are selected as controlled nodes (i.e., $\Delta^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$). The state matrix of node is $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, the input matrix is $B^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and the inner-coupling matrix is $H^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Node 2 is selected as target node (i.e., $T^2 = [0\ 1\ 0]$). After calculating, it is easy to get that $rank(T^2 \otimes I_2[\Psi^2, \Phi^{22}\Psi^2, \ldots, (\Phi^{22})^5\Psi^2]) = 1 < 2$. According to Lemma 3, if only considering the network structure of Layer 2, it is target uncontrollable.

The topology matrix from Layer 1 to the Layer 2 is $D^{21} = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and the inter-coupling matrix is $H^{21} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. After calculation, $rank(\Xi[\Psi, \Phi\Psi, \ldots, \Phi^{11}\Psi]) = 6$, one can get that the system (Φ, Ψ, Ξ) is target controllable. However, if the topology matrix from Layer 1 to Layer 2 is $D^{21} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, the network is as shown in Fig. 1(b). After calculating, it can be get that $rank(\Xi[\Psi, \Phi\Psi, \ldots, \Phi^{11}\Psi]) = 5 < 6$, which indicates the system (Φ, Ψ, Ξ) is not target controllable.

Remark 1: It is revealed that the target controllability of multi-layer networks will vary with the changes of the inter-layer couplings by Example 1, that is, the target controllability of the multi-layer networks is not only related to the structure and dynamic characteristics of each single-layer network, but also to the inter-layer coupling relationships. This makes it impossible to infer the target controllability of the entire multi-layer networks by verifying the target controllability of each layer separately.

Regarding multi-layer networks, it also concludes that it is impossible to verify the state controllability of the entire multi-layer networks by the state controllability of each single-layer network in [22]. This paper demonstrates the conclusion is still valid for target controllability of multi-layer networks by Example 1, i.e., it is also impossible to infer the target controllability of the entire networks based on the target controllability of each single-layer network.

IV. Target Controllability of Multi-Lakyer Networks

For a multi-layer network, achieving target controllability and state controllability is related, but still has differences. To explore the essential differences between them, in this section, the general multi-layer networks and special multi-layer networks with directed chain and directed star structure are studied, and several conditions are established for verifying the target controllability.

A. Target Controllability of General Inter-Layer Structured Multi-Layer Networks

A necessary and sufficient condition for verifying the target controllability of multi-layer networks with general inter-layer couplings is given.

Theorem 1: Consider system (3). The following equations:

$$\begin{aligned}\sum_{J=1,J\neq K}^{M} (D^{JK})^T[(\alpha_1^K)^T, \ldots, (\alpha_N^K)^T]^T H^{JK} &= [(\alpha_1^K)^T, \ldots, (\alpha_N^K)^T]^T (sI_n - A^K) \\ &\quad - (W^K)^T[(\alpha_1^K)^T, \ldots, (\alpha_N^K)^T]^T H^K \end{aligned} \tag{4}$$

$$\Delta^K[(\alpha_1^K)^T, \ldots, (\alpha_N^K)^T]^T B^K = 0 \tag{5}$$

have solutions $\alpha^K = (\alpha_1^K, \ldots, \alpha_N^K)$ (for all $K = 1, \ldots, M$). The system (Φ, Ψ, Ξ) is target controllable, if and only if span(Ω) $\bigcap$ span(Ξ) $=$ $\{0_{MNn}\}$, where $\Omega = \{\alpha \in \mathbb{C}^{1\times MNn} | \alpha = (\alpha^1, \ldots, \alpha^M)\}$.

Proof: There exists a vector α that simultaneously satisfies $\alpha(sI_{MNn} - \Phi) = 0$ and $\alpha\Psi = 0$. According to $\alpha(sI_{MNn} - \Phi) = 0$, it is easy to get

$$\alpha(sI_{MNn} - \Phi) = (\alpha^1, \ldots, \alpha^M)(sI_{MNn} - \Phi) = 0$$

which is equivalent to

$$\alpha^K sI_{Nn} - \alpha^K(I_N \otimes A^K) - \alpha^K(W^K \otimes H^K) - \sum_{J=1,J\neq K}^{M} \alpha^J(D^{JK} \otimes H^{JK}) = 0$$

for all $K = 1, \ldots, M$.

Further, one can get

$$\begin{aligned}&(\alpha_1^K, \ldots, \alpha_N^K)sI_{Nn} - (\alpha_1^K A^K, \ldots, \alpha_N^K A^K) \\ &\quad - \left(\sum_{j=1,j\neq i}^{N} w_{j1}^K \alpha_j^K H^K, \ldots, \sum_{j=1,j\neq i}^{N} w_{jN}^K \alpha_j^K H^K\right) \\ &\quad - \sum_{J=1,J\neq K}^{M} \left(\sum_{j=1,j\neq i}^{N} d_{j1}^{JK} \alpha_j^J H^{JK}, \ldots, \sum_{j=1,j\neq i}^{N} d_{jN}^{JK} \alpha_j^J H^{JK}\right) \\ &= 0\end{aligned}$$

for all $K = 1, \dots, M$.

Then it is easy to get

$$[(\alpha_1^K)^T, \dots, (\alpha_N^K)^T]^T(sI_n - A^K) - (W^K)^T[(\alpha_1^K)^T, \dots, (\alpha_N^K)^T]^T H^K = \sum_{J=1,J\neq K}^{M} (D^{JK})^T[(\alpha_1^K)^T, \dots, (\alpha_N^K)^T]^T H^{JK}$$

for all $K = 1, \dots, M$, i.e., (4). After calculation for $\alpha\Psi = 0$, it is easy to get

$$(\alpha^1, \dots, \alpha^M)\begin{bmatrix} \Delta^1 \otimes B^1 & & \\ & \ddots & \\ & & \Delta^M \otimes B^M \end{bmatrix} = 0$$

i.e., $\Delta^K[(\alpha_1^K)^T, \dots, (\alpha_N^K)^T]^T B^K = 0$, for all $K = 1, \dots, M$, i.e., (5).

Ω is the subspace generated by all the vectors α satisfying both (4) and (5). For any vector $\beta \in \Omega$, one has $\beta\Phi = \lambda\beta$, $\beta\Psi = 0$, where $\lambda \in \mathbb{C}$. Then, $\beta[\Psi, \Phi\Psi, \dots, \Phi^{MNn-1}\Psi] = 0$. It is concluded that all the elements of Ω are also the elements of $\langle\Phi|\Psi\rangle^\perp$, which means Ω is the subspace of $\langle\Phi|\Psi\rangle^\perp$. For any vector $\gamma \in \langle\Phi|\Psi\rangle^\perp$, one knows $\gamma[\Psi, \Phi\Psi, \dots, \Phi^{MNn-1}\Psi] = 0$, that is, $\gamma\Psi = 0, \gamma\Phi\Psi = 0, \dots, \gamma\Phi^{MNn-1}\Psi = 0$. In other words, $\gamma\Phi = \mu\gamma, \gamma\Psi = 0$, where $\mu \in \mathbb{C}$. $\langle\Phi|\Psi\rangle^\perp$ is the subspace of Ω. One can get $\langle\Phi|\Psi\rangle^\perp$ is equivalent to Ω.

Sufficiency: If $\mathrm{span}(\Omega) \bigcap \mathrm{span}(\Xi) \neq \{0_{MNn}\}$, there exists $\alpha \in \mathbb{C}^{1\times(m_1+\cdots+m_M)n}\backslash\{0_{(m_1+\cdots+m_M)n}\}$ satisfying $\alpha\Xi \in \Omega$. Then $\alpha\Xi[\Psi, \Phi\Psi, \dots, \Phi^{MNn-1}\Psi] = 0$, that is, $\alpha[\Xi\Psi, \Xi\Phi\Psi, \dots, \Xi\Phi^{MNn-1}\Psi] = 0$. One can get $rank(\Xi\Psi, \Xi\Phi\Psi, \dots, \Xi\Phi^{MNn-1}\Psi) < (m_1 + \cdots + m_M)n$. According to Lemma 3, one can get the system (Φ, Ψ, Ξ) is target uncontrollable.

Necessity: According to Lemma 3, if the system (Φ, Ψ, Ξ) is target uncontrollable, then $rank(\Xi\Psi, \Xi\Phi\Psi, \dots, \Xi\Phi^{MNn-1}\Psi) < (m_1 + \cdots + m_M)n$, which means that there exists a non-zero vector satisfying that $\alpha[\Xi\Psi, \Xi\Phi\Psi, \dots, \Xi\Phi^{MNn-1}\Psi] = \alpha\Xi[\Psi, \Phi\Psi, \dots, \Phi^{MNn-1}\Psi] = 0$. One can get $\beta = \alpha\Xi \in \Omega$. It can be concluded that $\mathrm{span}(\Omega) \bigcap \mathrm{span}(\Xi) \neq \{0_{MNn}\}$. ■

Remark 2: Theorem 1 provides a necessary and sufficient condition for verifying the target controllability of multi-layer networks. Differing from Lemma 3, Theorem 1 does not verify the rank condition of the system (Φ, Ψ, Ξ) directly, but rather determines the target controllability of system (Φ, Ψ, Ξ) by verifying the relationship between the uncontrollable subspace (Φ, Ψ) and the output matrix Ξ. By Theorem 1, it is easy to find that the system (Φ, Ψ, Ξ) may be target controllable even if the system (Φ, Ψ) is not complete state controllable. Therefore, if a networked system is found to be state uncontrollable by the conditions in [22], it is not possible to determine the target controllability of the multi-layer networks.

Remark 3: Based on the PBH criterion, the uncontrollable subspace of the system (Φ, Ψ) is calculated in Theorem 1 by lower-dimensional matrix operations. The scale of the matrices involved in the operations is smaller than the scale of the entire network system matrices. In addition, for many actual networked systems, the topology between different layers of networks is sparse (for example, each subway line in the subway networks often has a few public stations), that is, the matrix D^{JK} is usually sparse matrix, which makes Theorem 1 easier to verify in real multi-layer networks.

Theorem 1 considers the existence of non-zero vectors in the uncontrollable subspace of system (Φ, Ψ). If only zero vector exists in the uncontrollable subspace of system (Φ, Ψ) (i.e., the system (Φ, Ψ) is complete state controllable), the following condition can be derived.

Corollary 1: Consider system (Φ, Ψ, Ξ). If the following equations:

$$\sum_{J=1,J\neq K}^{M} (D^{JK})^T[(\alpha_1^K)^T, \dots, (\alpha_N^K)^T]^T H^{JK} = [(\alpha_1^K)^T, \dots, (\alpha_N^K)^T]^T(sI_n - A^K) - (W^K)^T[(\alpha_1^K)^T, \dots, (\alpha_N^K)^T]^T H^K \tag{6}$$

$$\Delta^K[(\alpha_1^K)^T, \dots, (\alpha_N^K)^T]^T B^K = 0 \tag{7}$$

have a unique solution $\alpha_i^K = 0$, for all $K = 1, \dots, M$, $i = 1, \dots, N$, then the system (Φ, Ψ, Ξ) is target controllable for all selected target subsets.

Proof: According to Lemma 1, the system (Φ, Ψ) is complete state controllable if and only if $\alpha(sI_{MNn} - \Phi) = 0$ and $\alpha\Psi = 0$ have a unique solution $\alpha = 0$. According to $\alpha(sI_{MNn} - \Phi) = 0$, one can get

$$(\alpha^1, \dots, \alpha^M)\begin{bmatrix} sI_{Nn} - \Phi^{11} & -\Phi^{12} & \cdots & -\Phi^{1M} \\ -\Phi^{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ -\Phi^{M1} & \cdots & \cdots & sI_{Nn} - \Phi^{MM} \end{bmatrix} = 0$$

which is equivalent to

$$\alpha^K(sI_{Nn} - \Phi^{KK}) - \sum_{J=1,J\neq K}^{M} \alpha^J\Phi^{JK} = 0$$

for all $K = 1, \dots, M$.

One can further obtain

$$(\alpha_1^K, \dots, \alpha_N^K)\begin{bmatrix} sI_n - A^K & -w_{12}^K H^K & \cdots & -w_{1N}^K H^K \\ -w_{21}^K H^K & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ -w_{N1}^K H^K & \cdots & \cdots & sI_n - A^K \end{bmatrix} - \sum_{J=1,J\neq K}^{M} (\alpha_1^J, \dots, \alpha_N^J)\begin{bmatrix} d_{11}^{JK}H^{JK} & \cdots & d_{1N}^{JK}H^{JK} \\ \vdots & \ddots & \vdots \\ d_{N1}^{JK}H^{JK} & \cdots & d_{NN}^{JK}H^{JK} \end{bmatrix} = 0$$

for all $K = 1, \dots, M$.

Then

$$\alpha_i^K(sI_n - A^K) - \sum_{j=1,j\neq i}^{N} w_{ji}^K\alpha_j^K H^K - \sum_{J=1,J\neq K}^{M}\sum_{j=1,j\neq i}^{N} d_{ji}^{JK}\alpha_j^J H^{JK} = 0$$

for all $K=1,\ldots,M$; $i=1,\ldots,N$, i.e., (6). According to $\alpha\Psi=0$, one can further get

$$(\alpha^1,\ldots,\alpha^M)\begin{bmatrix}\Delta^1\otimes B^1 & & \\ & \ddots & \\ & & \Delta^M\otimes B^M\end{bmatrix}=0$$

which is equivalent to $\alpha^K(\Delta^K\otimes B^K)=0$, for all $K=1,\ldots,M$, that is $\delta_i^K\alpha_i^K B^K=0$, for all $K=1,\ldots,M$, $i=1,\ldots,N$. According to Theorem 1, span $(\Omega)\bigcap$ span $(\Xi)=\{0_{MNn}\}$, the system (Φ,Ψ,Ξ) is target controllable. ■

Remark 4: Corollary 1 presents a condition for verifying the target controllability when the system is complete state controllable, and also clarifies the relationship between complete state controllability and target controllability: if a multi-layer networked system is complete state controllable, the system is target controllable for all selected target subsets. For a linear system, the complete state controllability cannot guarantee the output controllability. Target controllability is different from output controllability. For target controllability, a subset of nodes is selected as the target node set, so only one element in each row of the output matrix is non-zero. However, for output controllability, there is no limit to the number of non-zero elements in each row of the system output matrix, so that the output controllability matrix may not be row full rank. Complete state controllability cannot guarantee output controllability, which is different from the target controllability. Therefore, a system is target controllable for all selected target sets if it is complete state controllable.

Many actual multi-layer networks only have edges that from the upper layers to the lower layers. The inter-layer coupling mode of such multi-layer networks is called the driven-response mode. There is the following conclusion for this type of multi-layer networks.

Corollary 2: For an M-layer network with the driven-response mode, the system is target controllable if and only if the following hold simultaneously:

1) For all $\eta=(\eta^1,0,\ldots,0)=\text{span}(\Xi)$, where $\eta^1\in\mathbb{C}^{1\times Nn}\backslash\{0\}$, if $\lambda_{ij}^1\in\sigma(\Phi^{11})$ and $\eta^1\in\Upsilon(\lambda_{ij}^1|\Phi^{11})$, then $\eta^1(\Delta^1\otimes B^1)\neq 0$, for all $i=1,\ldots,N$, $j=1,\ldots,r_i^1$.

2) For all $\eta=(\eta^1,\ldots,\eta^K,0,\ldots,0)=\text{span}(\Xi)$, where $\eta^K\in\mathbb{C}^{1\times Nn}\backslash\{0\}$, if $\lambda_{ij}^K\in\sigma(\Phi^{KK})$ and $\eta^K\in\Upsilon(\lambda_{ij}^K|\Phi^{KK})$, then $[\eta^1(\Delta^1\otimes B^1),\ldots,\eta^K(\Delta^K\otimes B^K)]\neq 0$, where $\eta^1(\lambda_{ij}^K I_{Nn}-\Phi^{11})=\eta^2\Phi^{21}+\cdots+\eta^K\Phi^{K1},\ldots,\eta^{K-1}(\lambda_{ij}^K I_{Nn}-\Phi^{K-1,K-1})=\eta^K\Phi^{K,K-1}$, for all $i=1,\ldots,N$, $j=1,\ldots,r_i^K$, $K=2,\ldots,M$.

Proof: Sufficiency: If the system (Φ,Ψ,Ξ) is not target controllable, then $\exists\ \lambda\in\sigma(\Phi)$ and $\eta=\text{span}(\Xi)$ satisfying $\eta(\lambda I_{MNn}-\Phi,\Psi)=0$. If $\lambda\in\sigma(\Phi^{KK})$, according to $(\eta^1,\ldots,\eta^K,0,\ldots,0)(\lambda I_{MNn}-\Phi)=0$, one can get $\eta^1(\lambda I_{Nn}-\Phi^{11})=\eta^2\Phi^{21}+\cdots+\eta^K\Phi^{K1}$, $\ldots,\eta^{K-1}(\lambda I_{Nn}-\Phi^{K-1,K-1})=\eta^K\Phi^{K,K-1}$, for all $K=2,\ldots,M$. Similarly, according to $(\eta^1,\ldots,\eta^K,0,\ldots,0)\Psi=0$, it is easy to get $[\eta^1(\Delta^1\otimes B^1),\ldots,\eta^K(\Delta^K\otimes B^K)]=0$. Condition 2) is not satisfied. If $\lambda\in\sigma(\Phi^{11})$ and $\lambda\notin\sigma(\Phi^{KK})$, one can get $\eta^1(\lambda I_{Nn}-\Phi^{11})=0$ according to $(\eta^1,0,\ldots,0)(\lambda I_{MNn}-\Phi)=0$, for all $K=2,\ldots,M$. Besides, one can get $\eta^1(\Delta^1\otimes B^1)=0$ according to $(\eta^1,0,\ldots,0)\Psi=[\eta^1(\Delta^1\otimes B^1),0,\ldots,0]=0$. Condition 1) is not satisfied.

Necessity: If Condition 1) is not satisfied, then $\eta^1(\Delta^1\otimes B^1)=0$. There exists $\eta=\text{span}(\Xi)=(\eta^1,0,\ldots,0)$ satisfying

$$\eta(\lambda_{ij}^1 I_{MNn}-\Phi)=[\eta^1(\lambda_{ij}^1 I_{Nn}-\Phi^{11}),0,\ldots,0]=0$$

and

$$\eta\Psi=[\eta^1(\Delta^1\otimes B^1),0,\ldots,0]=0$$

that is, span $(\Omega)\bigcap$ span $(\Xi)=\eta\neq 0$. According to Theorem 1, the system (Φ,Ψ,Ξ) is target uncontrollable. If Condition 2) is not satisfied, then $[\eta^1(\Delta^1\otimes B^1),\ldots,\eta^K(\Delta^K\otimes B^K)]=0$. There exists $\eta=\text{span}\ (\Xi)=(\eta^1,\ldots,\eta^K,0,\ldots,0)$ satisfying

$$\begin{aligned}&\eta(\lambda_{ij}^K I_{MNn}-\Phi)\\ &\quad=[\eta^1(\lambda_{ij}^K I_{Nn}-\Phi^{11})-\cdots\\ &\qquad-\eta^K\Phi^{K1}\cdots\eta^{K-1}(\lambda_{ij}^K I_{Nn}-\Phi^{K-1,K-1})\\ &\qquad-\eta^K\Phi^{K,K-1},\eta^K(\lambda_{ij}^K I_{Nn}-\Phi^{KK}),0,\ldots,0]=0\end{aligned}$$

and

$$\eta\Psi=[\eta^1(\Delta^1\otimes B^1),\ldots,\eta^K(\Delta^K\otimes B^K),0,\ldots,0]=0.$$

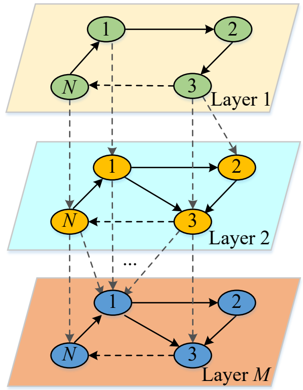
One can get span $(\Omega)\bigcap$ span $(\Xi)=\eta\neq 0$. According to Theorem 1, the system (Φ,Ψ) is target uncontrollable.

Remark 5: Compared to Theorem 1, the relationship between the uncontrollable subspace of layer K and the element η^K in the vector of linear combination of row vectors of the output matrix is considered in Corollary 2. Theorem 1 involves MNn dimension vector operations, and Corollary 2 only involves Nn dimension vectors and $Nn\times Nn$ dimension matrices, which means Corollary 2 is easier to calculate than Theorem 1 when verifying the target controllability of multi-layer networks with driven-response modes.

B. Target Controllability of Special Inter-Layer Structured Multi-Layer Networks

Some criteria have been provided for verifying the target controllability of multi-layer networks with general inter-layer structure in the previous section. In practical networks, there are many networks with special inter-layer structure, which are the basic units of large-scale networks. At the same time, it is easier to verify the target controllability of networked systems with special inter-layer structure.

1) Target Controllability of Multi-Layer Networks With Directed Chain Inter-Couplings: Consider M-layer networks, where the inter-layer couplings are directed chains and each layer has N identical nodes, as shown in Fig. 2(a). In the multi-layer networks, there are only coupling effects from the upper layers to the lower layers, that is, all the inter-layer topology matrices are zero except for $D^{K,K-1}$ $(K=2,\ldots,M)$. The system matrix Φ of the networked system (3) has the following form:



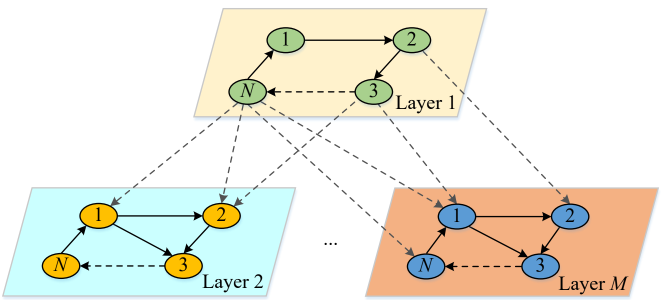


Fig. 2. Two special structures of multi-layer networks ((a) A directed chain multi-layer network; (b) A directed star multi-layer network).

$$\Phi=\begin{bmatrix}\Phi^{11} & & & & \\ \Phi^{21} & \Phi^{22} & & & \\ & \ddots & \ddots & & \\ & & \ddots & \Phi^{M-1,M-1} & \\ & & & \Phi^{M,M-1} & \Phi^{MM}\end{bmatrix} \tag{8}$$

where $\Phi^{KK}=I_N\otimes A^K+W^K\otimes H^K$, for all $K=1,\dots,M$; $\Phi^{K,K-1}=D^{K,K-1}\otimes H^{K,K-1}$, for all $K=2,\dots,M$.

The input matrix Ψ of the networked system (3) has the following form:

$$\Psi=\begin{bmatrix}\Psi^{1} & & & \\ & \Psi^{2} & & \\ & & \ddots & \\ & & & \ddots \\ & & & & \Psi^{M}\end{bmatrix} \tag{9}$$

where $\Psi^K=\Delta^K\otimes B^K$, for all $K=1,\dots,M$.

For multi-layer networks with directed chain inter-couplings, the following results are obtained.

Theorem 2: Consider the multi-layer networks described in networked system (3). Assume the inter-layer couplings are directed chains (i.e., the state matrix and input matrix are shown in (8) and (9), respectively). The topology matrix of each layer is diagonalizable. Let $\sigma(W^K)=\{\mu_1^K,\dots,\mu_N^K\}$, for all $K=1,\dots,M$. The networked system (Φ,Ψ,Ξ) is target controllable if the following conditions are satisfied simultaneously:

1) span $(T^K)\bigcap\langle W^K|\Delta^K\rangle^{\perp}=\{0_N\}$, $K=1,\dots,M$.

2) $(A^K+\mu_i^KH^K,B^K)$ is controllable, $K=1,\dots,M$, $i=1,\dots,N$.

3) The quantitaty of uncontrollable LEs of (W^K,Δ^K) are the same, $K=1,\dots,M$.

4) There exist $d_i^L\in\mathbb{C}$ and $h_{ij}^L\in\mathbb{C}$ satisfying both $p_i^LD^{L,L-1}=d_i^Lp_i^{L-1}$, $\phi_{ij}^L(k)H^{L,L-1}=h_{ij}^L\phi_{ij}^{L-1}(k)$ and $\lambda_{ij}^1+d_i^2h_{ij}^2=\lambda_{ij}^2+d_i^3h_{ij}^3=\cdots=\lambda_{ij}^{M-1}+d_i^Mh_{ij}^M=\lambda_{ij}^M=\lambda_{ij}$, where $\theta_{ij}^1=\cdots=\theta_{ij}^M=\theta_{ij}$ and $1\le k\le\theta_{ij}$, $K=1,\dots,M$, $L=2,\dots,M$, $i=1,\dots,N$, $j=1,\dots,r_i^K$.

Proof: p_i^K denotesthe LE corresponding to the eigenvalue μ_i^K of topology matrix W^K in layer K. λ_{ij}^K is the eigenvalue of $A^K+\mu_i^KH^K$, and $\phi_{ij}^K(k)$ is an element of LJC corresponding to the eigenvalue λ_{ij}^K of $A^K+\mu_i^KH^K$.

$$\eta_{ij}^1(1)=p_i^1\otimes\phi_{ij}^1(1),\dots,\eta_{ij}^1(\theta_{ij})=p_i^1\otimes\phi_{ij}^1(\theta_{ij})$$
$$\cdots$$
$$\eta_{ij}^M(1)=p_i^M\otimes\phi_{ij}^M(1),\dots,\eta_{ij}^M(\theta_{ij})=p_i^M\otimes\phi_{ij}^M(\theta_{ij})$$

represent the LJCs of eigenvalue from λ_{ij}^1 to λ_{ij}^M. Let

$$\xi_{ij}(1)=(\eta_{ij}^1(1),\eta_{ij}^2(1),\dots,\eta_{ij}^M(1))$$
$$\cdots$$
$$\xi_{ij}(\theta_{ij})=(\eta_{ij}^1(\theta_{ij}),\eta_{ij}^2(\theta_{ij}),\dots,\eta_{ij}^M(\theta_{ij})).$$

Since

$$\xi_{ij}(1)\Phi=\lambda_{ij}(p_i^1\otimes\phi_{ij}^1(1),\dots,p_i^M\otimes\phi_{ij}^M(1))=\lambda_{ij}\xi_{ij}(1)$$

one can get that $\xi_{ij}(1)$ is the left eigenvector of Φ.

Since $\xi_{ij}(k)\Phi=\lambda_{ij}\xi_{ij}(k)+\xi_{ij}(k-1)$, where $2\le k\le\theta_{ij}$, one can get $\xi_{ij}(1),\dots,\xi_{ij}(\theta_{ij})$ is the LJC about the eigenvalue λ_{ij} of Φ.

If λ_{ij} is a controllable eigenvalue, then the uncontrollable subspace about the eigenvalue λ_{ij} of the system (Φ,Ψ) is $\{0_{MNn}\}$. If λ_{ij} is an uncontrollable eigenvalue with geometric multiplicity 1, there exists $1\le l\le\theta_{ij}$ satisfying $[\xi_{ij}(1)^T\cdots\xi_{ij}(l)^T]^T\Psi=0$ and $\xi_{ij}(l+1)\Psi\neq0$. The uncontrollable subspace corresponding to the eigenvalue λ_{ij} of the system (Φ,Ψ) is $U_{ij}=\mathrm{span}\{\xi_{ij}(1),\dots,\xi_{ij}(l)\}$. If $\sigma=\lambda_{i_1j_1}=\cdots=\lambda_{i_qj_q}$ is an uncontrollable eigenvalue with geometric multiplicity more than 1, there exists non-zero vector $\beta_1\in\mathbb{C}^{1\times q}$ satisfying $\beta_1[\xi_{i_1j_1}(1)^T\cdots\xi_{i_qj_q}(1)^T]^T\Psi=0$.

$$v_1(\beta_1)=\beta_1[\xi_{i_1j_1}(1)^T\cdots\xi_{i_qj_q}(1)^T]^T,\dots$$
$$v_\theta(\beta_\theta)=\beta_1[\xi_{i_1j_1}(\theta)^T\cdots\xi_{i_qj_q}(\theta)^T]^T+\cdots$$
$$+\beta_\theta[\xi_{i_1j_1}(1)^T\cdots\xi_{i_qj_q}(1)^T]^T$$

denote the LJC of Φ with the top vector $v_1(\beta_1)$, where $\beta_2,\dots,\beta_\theta$ are arbitrary vectors. There exists $1\le l\le\theta$ satisfying both $[v_1(\beta_1)^T\cdots v_l(\beta_l)^T]^T\Psi=0$ and $v_{l+1}(\beta_{l+1})\Psi\neq0$. The uncontrollable subspace corresponding to the top vector $v_1(\beta_1)$ is $\mathrm{span}\{v_1(\beta_1),\dots,v_l(\beta_l)\}$. Then the uncontrollable subspace about the eigenvalue σ is $U(\sigma)=\mathrm{span}\{v_1,\dots,v_l\}$, where $U(\sigma)=U_{i_1j_1}=\cdots=U_{i_qj_q}$. In summary, the uncontrollable subspace of the system (Φ,Ψ) is $\langle\Phi|\Psi\rangle^{\perp}=\oplus_{i=1}^N\oplus_{j=1}^{r_i}U_{ij}$.

The uncontrollable eigenvectors of (W^K,Δ^K) are $p_{k_1}^K,\dots,$

$p_{k_s}^K$. The form of LE of Φ is $(p_i^1 \otimes \phi_{ij}^1(1), \ldots, p_i^M \otimes \phi_{ij}^M(1))$. Since $(A^K + \mu_i^K H^K, B^K)$ is controllableand $\phi_{ij}^K(1) B^K \neq 0$, the uncontrollable subspace of the system (Φ, Ψ) is

$$\text{span}\left\{[p_{k_1}^1 \otimes I_n, \ldots, p_{k_1}^M \otimes I_n], \ldots, [p_{k_S}^1 \otimes I_n, \ldots, p_{k_S}^M \otimes I_n]\right\}.$$

According to

$$\text{span } (T^K) \bigcap \text{span}\left\{p_{k_1}^K, \ldots, p_{k_S}^K\right\} = \{0_N\}$$

it is easy to get

$$\text{span}(\Xi) \bigcap \langle \Phi | \Psi \rangle^\perp = 0.$$

If the system is not target controllable, then

$$rank(\Xi\Psi, \Xi\Phi\Psi, \ldots, \Xi\Phi^{MNn-1}\Psi) < (m_1 + \cdots + m_M)n.$$

There exists non-zero vector α satisfying

$$\begin{aligned}&\alpha[\Xi\Psi, \Xi\Phi\Psi, \ldots, \Xi\Phi^{MNn-1}\Psi]\\ &\quad = \alpha\Xi[\Psi, \Phi\Psi, \ldots, \Phi^{MNn-1}\Psi] = 0.\end{aligned}$$

In other words, there exists vector $\beta = \alpha\Xi \in \langle \Phi | \Psi \rangle^\perp$ satisfying span $(\Xi) \bigcap \langle \Phi | \Psi \rangle^\perp \neq \{0_{MNn}\}$. ■

Remark 6: Condition 1) guarantees span $(T^K) \bigcap \langle W^K | \Delta^K \rangle^\perp = \{0_N\}$, that is the K-th layer networked system (W^K, Δ, T^K) is target controllable. According to Theorem 2, one can see that the target controllability of the entire networked systems is relevant to the target controllability of some lower-dimensional systems. In other words, Theorem 2 gives the target controllable conditions for the multi-layer networks. Compared to Lemma 3, the conditions in Theorem 2 involve to analyze a low dimensional matrix and are easier to apply in practical large-scale networks.

2) Target Controllability of Multi-Layer Networks With Directed Star Inter-Couplings: Consider M-layer networks, where the inter-layer couplings are directed stars and each layer has N identical nodes, as shown in Fig. 2(b). In the multi-layer networks, there are only coupling effects from the first layer to the other layers, i.e., all the inter-layer topology matrices are zero except for D^{K1} $(K = 2, \ldots, M)$. The system matrix Φ of the networked system (3) has the following form:

$$\Phi = \begin{bmatrix} \Phi^{11} & & & & \\ \Phi^{21} & \Phi^{22} & & & \\ \vdots & & \ddots & & \\ \vdots & & & \Phi^{M-1,M-1} & \\ \Phi^{M1} & & & & \Phi^{MM} \end{bmatrix} \tag{10}$$

where $\Phi^{KK} = I_N \otimes A^K + W^K \otimes H^K$, $K = 1, \ldots, M$; $\Phi^{L1} = D^{L1} \otimes H^{L1}$, $K = 2, \ldots, M$.

The input matrix Ψ of the networked system (3) has the following form:

$$\Psi = \begin{bmatrix} \Psi^1 & & & & \\ & \Psi^2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \Psi^M \end{bmatrix} \tag{11}$$

where $\Psi^K = \Delta^K \otimes B^K$, $K = 1, \ldots, M$.

For multi-layer networks that inter-layer couplings are directed stars, a sufficient condition is provided to verify the target controllability.

Theorem 3: Consider the multi-layer networks described in networked system (3). Assume the inter-layer couplings are directed stars (i.e., the state matrix and input matrix are shown in (10) and (11), respectively). The topology matrix of each layer is diagonalizable. Let $\sigma(W^K) = \{\mu_1^K, \ldots, \mu_N^K\}$, for all $K = 1, \ldots, M$. The networked system (Φ, Ψ, Ξ) is target controllable if the following conditions are satisfied simultaneously:

1) $\text{span}(T^K) \bigcap \langle W^K | \Delta^K \rangle^\perp = \{0_N\}$, $K = 1, \ldots, M$.

2) $(A^K + \mu_i^K H^K, B^K)$ is controllable, $K = 1, \ldots, M$, $i = 1, \ldots, N$.

3) The quantitaty of uncontrollable LEs of (W^K, Δ^K) are the same, $K = 1, \ldots, M$.

4) There exist $d_i^L \in \mathbb{C}$ and $h_{ij}^L \in \mathbb{C}$ satisfying both $p_i^L D^{L1} = d_i^L p_i^1$, $\phi_{ij}^L(k) H^{L1} = h_{ij}^L \phi_{ij}^1(k)$ and $\lambda_{ij}^1 + d_i^2 h_{ij}^2 + \cdots + d_i^M h_{ij}^M = \lambda_{ij}^2 = \cdots = \lambda_{ij}^M = \lambda_{ij}$, where $\theta_{ij}^1 = \cdots = \theta_{ij}^M = \theta_{ij}$ and $1 \leq k \leq \theta_{ij}$, $K = 1, \ldots, M$, $L = 2, \ldots, M$, $i = 1, \ldots, N$, $j = 1, \ldots, r_i^K$.

Proof: p_i^K denotes LE corresponding to eigenvalue μ_i^K of the topology matrix W^K in layer K. λ_{ij}^K denotes the eigenvalue of $A^K + \mu_i^K H^K$, and $\phi_{ij}^K(k)$ is the element of the LJC corresponding to the eigenvalue λ_{ij}^K of $A^K + \mu_i^K H^K$. Due to the fact that the state matrix of the multi-layer network with directed star inner-couplings is a lower triangular matrix, the eigenvalues of the state matrix of the entire networked system are equal to the union of the eigenvalues of each diagonal block matrix, i.e.,

$$\begin{aligned}&\sigma(\Phi) = \sigma(\Phi^{11}) \bigcup \sigma(\Phi^{22}) \bigcup \cdots \bigcup \sigma(\Phi^{MM})\\ &\eta_{ij}^1(1) = p_i^1 \otimes \phi_{ij}^1(1), \ldots, \eta_{ij}^1(\theta_{ij}) = p_i^1 \otimes \phi_{ij}^1(\theta_{ij})\\ &\qquad \cdots\\ &\eta_{ij}^M(1) = p_i^M \otimes \phi_{ij}^M(1), \ldots, \eta_{ij}^M(\theta_{ij}) = p_i^M \otimes \phi_{ij}^M(\theta_{ij})\end{aligned}$$

represent the LJCs corresponding to the eigenvalues from λ_{ij}^1 to λ_{ij}^M. Let

$$\begin{aligned}&\xi_{ij}(1) = (\eta_{ij}^1(1), \eta_{ij}^2(1), \ldots, \eta_{ij}^M(1))\\ &\qquad \vdots\\ &\xi_{ij}(\theta_{ij}) = (\eta_{ij}^1(\theta_{ij}), \eta_{ij}^2(\theta_{ij}), \ldots, \eta_{ij}^M(\theta_{ij})).\end{aligned}$$

Since $\xi_{ij}(1)\Phi = \lambda_{ij}(p_i^1 \otimes \phi_{ij}^1(1), \ldots, p_i^M \otimes \phi_{ij}^M(1)) = \lambda_{ij}\xi_{ij}(1)$, $\xi_{ij}(1)$ is the LE of Φ. Since $\xi_{ij}(k)\Phi = \lambda_{ij}\xi_{ij}(k) + \xi_{ij}(k-1)$, for $2 \leq k \leq \theta_{ij}$, $\xi_{ij}(1), \ldots, \xi_{ij}(\theta_{ij})$ is the LJC about the eigenvalue λ_{ij} of Φ.

If λ_{ij} is a controllable eigenvalue, then the uncontrollable subspace corresponding to the eigenvalue λ_{ij} of (Φ, Ψ) is $\{0_{MNn}\}$. If λ_{ij} is an uncontrollable eigenvalue with geometric multiplicity 1, then there exists $1 \leq l \leq \theta_{ij}$ satisfying both $[\xi_{ij}(1)^T \cdots \xi_{ij}(l)^T]^T \Psi = 0$ and $\xi_{ij}(l+1)\Psi \neq 0$. The uncontrollable subspace corresponding to the eigenvalue λ_{ij} of (Φ, Ψ) is $U_{ij} = \text{span}\{\xi_{ij}(1), \ldots, \xi_{ij}(l)\}$. If $\sigma = \lambda_{i_1 j_1} = \cdots = \lambda_{i_q j_q}$ is an uncontrollable eigenvalue with geometric multiplicity more than 1, then there exists non-zero vector $\beta_1 \in \mathbb{C}^{1\times q}$ satisfying

$\beta_1[\xi_{i_1j_1}(1)^T \cdots \xi_{i_qj_q}(1)^T]^T\Psi = 0$.

Let

$$v_1(\beta_1) = \beta_1[\xi_{i_1j_1}(1)^T \cdots \xi_{i_qj_q}(1)^T]^T$$

$$\vdots$$

$$v_\theta(\beta_\theta) = \beta_1[\xi_{i_1j_1}(\theta)^T \cdots \xi_{i_qj_q}(\theta)^T]^T + \cdots$$
$$+\beta_\theta[\xi_{i_1j_1}(1)^T \cdots \xi_{i_qj_q}(1)^T]^T$$

denote the LJC with the top vector is $v_1(\beta_1)$, where $\beta_2, \ldots, \beta_\theta$ are arbitrary vectors. There exists $1 \le l \le \theta$ satisfying both $[v_1(\beta_1)^T \cdots v_l(\beta_l)^T]^T\Psi = 0$ and $v_{l+1}(\beta_{l+1})\Psi \ne 0$. So the uncontrollable subspace corresponding to the top vector $v_1(\beta_1)$ is $\text{span}\{v_1(\beta_1), \ldots, v_l(\beta_l)\}$. Then the uncontrollable subspace of the system (Φ, Ψ) about the eigenvalue σ is $U(\sigma) = \text{span}\{v_1, \ldots, v_l\}$, where $U(\sigma) = U_{i_1j_1} = \cdots = U_{i_qj_q}$. In summary, the uncontrollable subspace of the system (Φ, Ψ) is $\langle\Phi|\Psi\rangle^\perp = \oplus_{i=1}^N \oplus_{j=1}^{r_i} U_{ij}$.

Suppose the uncontrollable eigenvectors of the system (W^K, Δ^K) are $p_{k_1}^K, \ldots, p_{k_S}^K$. The form of the LE of Φ is $(p_i^1 \otimes \phi_{ij}^1(1), \ldots, p_i^M \otimes \phi_{ij}^M(1))$. Since $(A^K + \mu_i^K H^K, B^K)$ is controllable and $\phi_{ij}^K(1)B^K \ne 0$, the uncontrollable subspace of (Φ, Ψ) is $\text{span}\{[p_{k_1}^1 \otimes I_n, \ldots, p_{k_1}^M \otimes I_n], \ldots, [p_{k_S}^1 \otimes I_n, \ldots, p_{k_S}^M \otimes I_n]\}$. $\text{span}(T^K) \bigcap \langle W^K|\Delta^K\rangle^\perp = \{0_N\}$, so $\text{span}(\Xi) \bigcap \langle\Phi|\Psi\rangle^\perp = \{0_{MNn}\}$. If the system is not target controllable, then

In other words, there exists non-zero vector α satisfying that

$$\begin{aligned} &\alpha[\Xi\Psi, \Xi\Phi\Psi, \ldots, \Xi\Phi^{MNn-1}\Psi] \\ &= \alpha\Xi[\Psi, \Phi\Psi, \ldots, \Phi^{MNn-1}\Psi] \\ &= 0 \end{aligned}$$

that is, there exists vector $\beta = \alpha\Xi \in \langle\Phi|\Psi\rangle^\perp$ satisfying span (Ξ) $\bigcap \langle\Phi|\Psi\rangle^\perp \ne \{0_{MNn}\}$. ■

Remark 7: The Condition 1) in Theorems 2 and 3 verify the target controllability of the system (W^K, Δ^K, T^K). The matrices W^K, Δ^K, T^K describe the topology matrix, input and output channels, respectively in layer K of the networks, which are independent of the inter-couplings. Conditions 2) and 3) in Theorems 2 and 3 ensure the target controllability of the whole multi-layer network through the verification of the low-dimensional local matrix. These conditions verify the controllability of subsystems, and consider the situations in each single-layer network, which are independent of the inter-coupling modes. Condition 4) in Theorems 2 and 3 are prerequisites for verifying the target controllability of the entire systems by lower-dimensional matrices. Since Condition 4) involves the eigenvalues and LJCs of the state matrix Φ, it is necessary to consider the impact of differences between the directed chains and the directed stars.

Remark 8: Theorem 4 in [32] provides a sufficient condition for verifying the target controllability of single-layer networks. For multi-layer networks, each layer has different node dynamics, but Condition 2) of Theorem 4 in [32] is only suitable for the networked systems with the same node dynamics, but not for multi-layer networks. Theorems 2 and 3 derived here extend the condition of Theorem 4 in [32] to two typical multi-layer networks.

V. Simulated Examples

A. A Numerical Example

In this subsection, an example is presented for illustrating Theorems 2 and 3.

Example 2: Consider a two-layer networked system, where each layer has $N = 3$ nodes, as shown in Fig. 3.

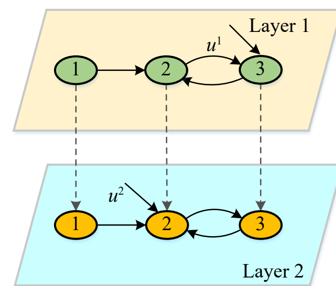


Fig. 3. A two-layer network with three identical nodes in each layer.

In Layer 1, the topology matrix is $W^1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Node 3 is selected as the controlled node (i.e., $\Delta^1 = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix}$). The state matrix is $A^1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, the input matrix is $B^1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, and the inner-coupling matrix is $H^1 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$. Node 2 is selected as the target node (i.e., $T^1 = [0\ \ 1\ \ 0\ ]$). In Layer 2, the topology matrix is $W^1 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$. Node 2 is selected as the-controlled node (i.e., $\Delta^2 = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 0 \end{bmatrix}$). The state matrix is $A^2 = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$, the input matrix is $B^2 = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ and the inner-coupling matrix is $H^2 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$. Node 3 is selected as the target node (i.e., $T^2 = [0\ \ 0\ \ 1]$). The topology matrix from Layer 1 to Layer 2 is $D^{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and the inter-coupling matrix is $H^{21} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

It is easy to calculate that the eigenvalues of W^1 are $\mu_1^1 = 1$, $\mu_2^1 = -1$, $\mu_3^1 = 0$, and the corresponding LEs are $p_1^1 = (1\ 1\ 1)$, $p_2^1 = (1\ \ -1\ \ 1)$, $p_3^1 = (1\ 0\ 0)$, respectively. Since $p_1^1\Delta^1 \ne 0$, $p_2^1\Delta^1 \ne 0$ and $p_3^1\Delta^1 = 0$, p_3^1 is a uncontrollable eigenvector of (W^1, Δ^1), i.e., $p_3^1 \in \langle W^1|\Delta^1\rangle^\perp$. One can get $\text{span}(T^1 \bigcap \langle W^1|\Delta^1\rangle^\perp = \{0_N\}$. Similarly, one can easily calculate that the eigenvalues of W^2 are $\mu_1^2 = 2$, $\mu_2^2 = -2$, $\mu_3^2 = 0$, and the corresponding LEs are $p_1^2 = (1\ 1\ 1)$, $p_2^2 = (1\ \ -1\ \ 1)$, $p_3^2 = (1\ 0\ 0)$, respectively. Since $p_1^2\Delta^1 \ne 0$, $p_2^2\Delta^1 \ne 0$ and $p_3^2\Delta^1 = 0$, p_3^2 is a uncontrollable LE of (W^2,Δ^2), i.e., $p_3^2 \in \langle W^2|\Delta^2\rangle^{\perp}$. Then, $\mathrm{span}(T^2) \bigcap \langle W^2|\Delta^2\rangle^{\perp} = \{0_N\}$. The quantitaty of the uncontrollable LEs of (W^1,Δ^1) and (W^2,Δ^2) are the same, so Condition 3) of Theorem 3 is satisfied. $\mathrm{span}(T^1) \bigcap \langle W^1|\Delta^1\rangle^{\perp} = \{0_N\}$ and $\mathrm{span}(T^2) \bigcap \langle W^2|\Delta^2\rangle^{\perp} = \{0_N\}$, so Condition 1) of Theorem 3 is satisfied. $(A^1+\mu_1^1 H^1, B^1)$, $(A^1+\mu_2^1 H^1, B^1)$, $(A^1+\mu_3^1 H^1, B^1)$, $(A^2+\mu_1^2 H^2, B^2)$, $(A^2+\mu_2^2 H^2, B^2)$ and $(A^2+\mu_3^2 H^2, B^2)$ are all controllable, so Condition 2) of Theorem 3 is satisfied. There exist $d_1^2=1$, $h_{11}^2=5$, $h_{12}^2=4$, $d_2^2=1$, $h_{21}^2=-1$, $h_{22}^2=-2$, $d_3^2=1$, $h_{31}^2=2$ and $h_{32}^2=1$ satisfying $p_i^2 D^{21} = d_i^2 p_i^1$, $\phi_{ij}^2(1) H^{21} = h_{ij}^2 \phi_{ij}^1(1)$ and $\lambda_{ij}^1 + d_i^2 h_{ij}^2 = \lambda_{ij}^2 = \lambda_{ij}$, for all $i=1,2,3$, $j=1,2$, Condition 4) of Theorem 3 is satisfied. According to Theorem 3, the networked system is target controllable.

Example 2 is a two-layer network, which can be seen as a network with directed chain inter-layer couplings and verified by Theorem 2, or as a network with directed star inter-layer couplings and verified by Theorem 3. For two-layer networks, the conditions in Theorems 2 and 3 have the same description. Therefore, Theorem 2 can also verify the target controllability of the network in Example 2.

Remark 9: Example 2 verifies the effectiveness of Theorems 2 and 3 and also verifies that less computation is required to test controllability than Lemma 3. In the example, only 2×2 or 3×3 matrices are involved using Theorems 2 or 3. While in the calculation of Lemma 3, the matrix sizes of Φ, Ψ, and Ξ are 12×12, 12×12 and 4×12, respectively, whose matrices size are obviously larger.

B. A Practical Example

In this subsection, a practical example is presented to demonstrate the applicability and validity of the Corollary 2.

Example 3: Consider a traffic networked system consisting of a two-layer network of four unmanned vehicles to perform adaptive cruise and following tasks in two-lane traffic [25], as shown in Fig. 4. In Layer 1, the two cruise vehicles maintain consistency according to the relative position of each other. In Layer 2, the two following vehicles maintain the minimum safe distance according to the position of the vehicle in front.

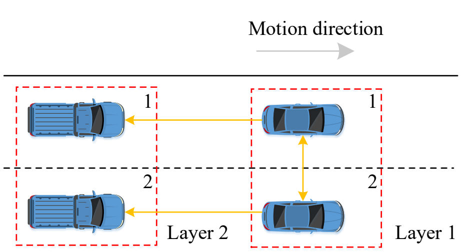


Fig. 4. An unmanned vehicle system in a two-lane adaptive cruise and following task.

Vehicle 2 in Layer 1 and Vehicles 1 and 2 in Layer 2 are controlled, thus $\Delta^1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\Delta^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The topology matrix of Layer 1 is $W^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, the topology matrix of Layer 2 is $W^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and the topology matrix from Layer 1 to Layer 2 is $D^{21} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

The system can be described as the form of (2) and (3), where $A^1 = \begin{bmatrix} -\frac{1}{\epsilon_u} & 0 & -\frac{1}{\epsilon_i} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $H^1 = \begin{bmatrix} 0 & 0 & \frac{1}{\epsilon_i} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B^1 = \begin{bmatrix} \frac{1}{\epsilon_u} \\ 0 \\ 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} -\frac{1}{\epsilon_u} + \frac{\epsilon_m^2}{2\epsilon_o} & \frac{\epsilon_m}{\epsilon_o} & -\frac{1}{\epsilon_o} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $B^2 = \begin{bmatrix} \frac{1}{\epsilon_u} \\ 0 \\ 0 \end{bmatrix}$, and $H^{21} = \begin{bmatrix} -\frac{\epsilon_m^2}{2\epsilon_o} & -\frac{\epsilon_m}{\epsilon_o} & \frac{1}{\epsilon_o} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, in which $\epsilon_u, \epsilon_i, \epsilon_o$ and ϵ_m are all time constants. Assume Vehicles 1 and 2 in Layer 1 and Vehicle 1 in Layer 2 are selected as target nodes, i.e., $T^1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $T^2 = [1\ \ 0]$.

In this model, the state is $x = [(x_1^1)^T, (x_2^1)^T, (x_1^2)^T, (x_2^2)^T]^T$, where $x_1^1 = [a_1^1, v_1^1, z_1^1]^T$, $x_2^1 = [a_2^1, v_2^1, z_2^1]^T$, $x_1^2 = [a_1^2, v_1^2, z_1^2]^T$, $x_2^2 = [a_2^2, v_2^2, z_2^2]^T$. a, v, z represent acceleration, velocity, and displacement, respectively.

Let $\epsilon_u = 1$, $\epsilon_i = 1$, $\epsilon_o = 0.5$ and $\epsilon_m = 2$, then $H^1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $H^{21} = \begin{bmatrix} -4 & -4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A^1 = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 3 & 4 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, and $B^1 = B^2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Since this two-layer networked system is drive-response mode, the target controllability can be verified by Corollary 2. For $\eta = (\eta^1, 0) = \mathrm{span}(\Xi)$, where $\eta^1 \in \mathbb{C}^{1\times Nn} \backslash \{0\}$, it is easy to find there exist eigenvalue $\lambda_{11}^1 = -\frac{2362}{1393}$ and the corresponding LE $\eta_1^1 = [\frac{1213}{2079}\ -\frac{437}{1270}\ \frac{180}{887}\ -\frac{1213}{2079}\ \frac{437}{1270}\ -\frac{180}{887}]$, where $\eta_1^1 \in \Upsilon(\lambda_{11}^1|\Phi^{11})$, and $\lambda_{21}^1 = -1$ and the corresponding LE $\eta_2^1 = [-1\ \ 1\ -1\ -1\ \ 1\ -1]$, where $\eta_2^1 \in \Upsilon(\lambda_{21}^1|\Phi^{11})$. It is easy to get that $\eta_1^1(\Delta^1 \otimes B^1) \neq 0$ and $\eta_2^1(\Delta^1 \otimes B^1) \neq 0$, i.e., $\eta^1(\Delta^1 \otimes B^1) \neq 0$. Condition 1) in Corollary 2 is satisfied. For $\eta = (\eta^1, \eta^2) = \mathrm{span}(\Xi)$, where $\eta^1, \eta^2 \in \mathbb{C}^{1\times Nn} \backslash \{0\}$, it is easy to find there exist eigenvalue $\lambda_{11}^2 = \frac{470}{1183}$ of Φ^{22} and the corresponding LE $\eta_1^2 \in \Upsilon(\lambda_{11}^2|\Phi^{22})$, where $\eta_1^2 = [\frac{291}{2005}\ \frac{2456}{6723}\ \frac{811}{882}\ 0\ 0\ 0]$ satisfying $\eta_1^1(\lambda_{11}^2 I_{Nn} - \Phi^{11}) = \eta_1^2 \Phi^{21}$, eigenvalue $\lambda_{12}^2 = \frac{557}{143}$ of Φ^{22} and the corresponding LE $\eta_2^2 \in \Upsilon(\lambda_{12}^2|\Phi^{22})$, where $\eta_2^2 = [-\frac{695}{719}\ -\frac{439}{1769}\ -\frac{378}{5933}\ 0\ 0\ 0]$ satisfying $\eta_2^1(\lambda_{12}^2 I_{Nn} - \Phi^{11}) = \eta_2^2 \Phi^{21}$ and eigenvalue $\lambda_{13}^2 = -\frac{1905}{1474}$ of Φ^{22} and the corresponding LE $\eta_3^2 \in \Upsilon(\lambda_{13}^2|\Phi^{22})$, where $\eta_3^2 = [\frac{975}{1364}\ -\frac{224}{405}\ \frac{395}{923}\ 0\ 0\ 0]$ satisfying $\eta_3^1(\lambda_{13}^2 I_{Nn} - \Phi^{11}) = \eta_3^2 \Phi^{21}$. Since $[\eta^1(\Delta^1 \otimes B^1), \eta^2(\Delta^2 \otimes B^2)] \neq 0$, Condition 2) in Corollary 2 is satisfied. Therefore, it is concluded that the networked system is target controllable.

Remark 10: The example indicates that Corollary 2 is applicable in practical systems. If the system is modeled as a single-layer network with four nodes, attempts that verify the target controllability of the system by the methods of [32] can be carried out. The conditions of Theorems 3 and 4 in [32] require that the node dynamics matrix A and the input matrix B are the same for each vehicle. However, the node dynamics matrix A is different in each layer. For these reasons, the methods in [32] are not suitable for this practical example. In addition, applying Corollary 2 to verify the target controllability of the system, the matrix sizes involved are 6×6, 6×2 and 3×6, respectively. If the system is regarded as a single-layer network and the target controllability is verified by Lemma 3, the matrix sizes involved are 12×12, 12×4 and 9×12, respectively. Obviously, applying Corollary 2 to verify target controllability involves matrix computations more efficiently.

VI. Conclusion

This paper investigates the target controllability of multi-layer networks with high-dimensional nodes. It is found that the inter-layer couplings play an important role in the target controllability of multi-layer networks: even if there exists a layer of the multi-layer network which is not target controllable, the entire multi-layer network can still be target controllable due to the influence of inter-layer couplings. A necessary and sufficient condition for verifying the target controllability of multi-layer networks with general structure is given by establishing the relationship between the uncontrollable subspace and the output matrix. It is easy to find that even if the multi-layer network is not state controllable, it may still be target controllable. Then, two corollaries are provided, and the relationship between target controllability, complete state controllability, and output controllability has been clarified (target controllability can be seen as a special case of output controllability; if the system is complete state controllable, it must be target controllable, but not necessarily output controllable). For multi-layer networks with directed chain and directed star inter-layer couplings, sufficient conditions for verifying target controllability are given. These conditions are valid and easy to calculate. The results can provide some reference for the control of target nodes in real networks.

This paper considers the problem of target controllability, and nodes in each layer have the same dynamic characteristics. The impact of node heterogeneity in each layer on target controllability of multi-layer networks can be further explored.

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