On the Monotonicity of Interval Type-2 Fuzzy Logic Systems

Chengdong Li, *Member, IEEE*, Jianqiang Yi, *Senior Member, IEEE*, and Guiqing Zhang

***Abstract*—Qualitative knowledge is very useful for system modeling and control problems, especially when specific physical structure knowledge is unavailable and the number of training data points is small. This paper studies the incorporation of one common qualitative knowledge—monotonicity into interval type-2 (IT2) fuzzy logic systems (FLSs). Sufficient conditions on the antecedent and consequent parts of fuzzy rules are derived to guarantee the monotonicity between inputs and outputs. We take into account five type-reduction and defuzzification methods (the Karnik–Mendel method, the Du–Ying method, the Begian–Melek–Mendel method, the Wu–Tan method, and the Nie–Tan method). We show that IT2 FLSs are monotonic if the antecedent and consequents parts of their fuzzy rules are arranged according to the proposed monotonicity conditions. The derived monotonicity conditions are valid for the IT2 FLSs using any kind of IT2 fuzzy sets (FSs) (e.g., Trapezoidal IT2 FSs and Gaussian IT2 FSs) and stand for type-1 FLSs as well. Guidelines for applying the proposed conditions to modeling and control problems are also given. Our results will be useful in the design of monotonic IT2 FLSs for engineering applications when the monotonicity property is desired.**

***Index Terms*—Data-driven method, fuzzy logic system, modeling and control, monotonicity, type-2 fuzzy, type-reduction and defuzzification method.**

I. Introduction

RECENTLY, type-2 (T2) fuzzy logic systems (FLSs) [1]–[7] have attracted increasing interest, as T2 FLSs not only have the advantages of conventional FLSs (type-1 FLSs) but can provide the capability to model high levels of uncertainties and produce more complex input–output mappings and better results as well. To date, due to the computation complexity and theoretical analysis difficulty, the most widely studied and applied T2 FLSs are the interval ones, where interval type-2 (IT2) fuzzy sets (FSs)[1] [8]–[12] are adopted to reduce computational complexity. IT2 FLSs have found lots of applications in many areas, especially in the modeling and control fields [21]–[35].

For modeling, IT2 FLSs represent the input–output mappings of the systems to be identified and are usually constructed through data-driven methods. When constructing IT2 FLSs using data-driven methods, we often encounter that the data points are noisy and that the number of the data points is small. As discussed in [36]: *"in such cases, it is very important to fully exploit the additional nonquantitative knowledge about the system in order to obtain meaningful, interpretable models. Moreover, taking the qualitative knowledge about the system into account renders the model-identification process less vulnerable to noise and inconsistencies in the data and suppresses overfitting."* Monotonicity between the inputs and outputs is one of such qualitative knowledge in many modeling problems. Taking the identification of the water heating system [37] for example, the temperature of water will change with respect to the heat power monotonically. Therefore, the identified fuzzy model (type-2 or type-1) for the water heating system should be monotonic between the heat power and the temperature.

For control applications, IT2 FLSs are utilized to realize control laws to reduce control errors. In many cases, the control signal (output of IT2 FLS) should be monotonic with respect to the error and/or the change of error (inputs of IT2 FLS). One typical example is the control of a liquid level in a tank. An appropriate fuzzy controller (type-2 or type-1) for this system needs to open the valve larger as the liquid level deviates more from the required level. Another example is the temperature control of the refrigerator. The more the real temperature in the refrigerator deviates from the setpoint, the lager the control action is needed to be generated by IT2 FLS to increase the motor speed in the compressor.

From the previous discussion, we can see that it would be very helpful to find the conditions under which the FLSs can give monotonic input–output mappings. There are several meaningful papers on the monotonicity of type-1 (T1) FLSs [36]–[42]. Broekhoven *et al.* [36], [38] have studied the monotonicity issue on the Mamdani–Assilian models under the mean of maxima defuzzification and the center-of-gravity defuzzification. In [37], [39], and [40], sufficient parameter conditions are given to ensure a monotonic input–output mapping of the TSK T1 FLS. Kouikoglou *et al.* [41] have discussed how to ensure the monotonicity of the hierarchical sum-product T1 FLSs. Seki *et al.* [42] have derived the monotonicity conditions of the single input rule modules (SIRMs) connected T1 FLSs.

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[1]IT2 FSs are isomorphic to interval-valued FSs [12]. Both concepts are alternatively used by different researchers; for example, in [13]–[17], interval-valued FSs are utilized, while in most of the other references, IT2 FSs are adopted. As an extension of T1 FSs, IT2 FSs (interval-valued FSs) have close relationships with other extensions, e.g., intuitionistic FSs [18], interval-valued intuitionistic FSs [18], and L-FSs [19]. Deschrijver and Kerre [20] have made a comprehensive study on the relationships among such popular extensions of T1 FSs.

TABLE I
EXISTING RESULTS FOR MONOTONICITY OF IT2 FLSS

Reference	Number of input variables	Type-reduction and defuzzification method	Kinds of IT2 FSs	Other limitations
SIRMs connected IT2 FLS [43]	multi inputs	BMM	Gaussian and Trapzoidal	MFs should be differentiable
IT2 Fuzzy Neural Network [44]	multi inputs	BMM	Gaussian	MFs should be differentiable
Single input IT2 FLS [45], [46]	single input	KM	Trapezoidal	No more than two FSs can be fired simultaneously

However, for the monotonicity of IT2 FLSs, there have been only a few studies up until now because of the complexity of the input–output mappings of IT2 FLSs. Only for some special IT2 FLSs, e.g., SIRMs-connected IT2 FLS [43] which is essentially linear combination of the single input IT2 FLSs, the Begian–Melek–Mendel method-based IT2 FLS [44], and the single-input IT2 FLS [45], [46], the parameter conditions for the monotonicity have been given. Related results are shown in Table I. From this table, we can observe that the existing results have the following limitations.

1) Only two kinds of type-reduction and defuzzification methods have been considered. For IT2 FLSs, different type-reduction and defuzzification methods can give different input–output mappings. The popular type-reduction and defuzzification methods include the Karnik–Mendel (KM) method [1], [8], [47]–[50], the Du–Ying (DY) method [51], the Begian–Melek–Mendel (BMM) method [52], the Wu–Tan (WT) method [53], the Nie–Tan (NT) method [54], [55], and the uncertainty bound (UB) method [56]. There exist no monotonicity results for the other type-reduction and defuzzification methods.
2) The most widely used type-reduction and defuzzification method is the KM method. However, for this method, up until now, the results are only for single input IT2 FLSs.
3) There exist strict limitations on the membership functions (MFs) including both the kinds of IT2 FSs and the differentiability of the MFs of IT2 FSs.

Therefore, to be more practical, more work needs to be done to guarantee the monotonicity of multiinput IT2 FLSs with different kinds of MFs and out-processing methods. In this study, we present a unified framework for the monotonicity of IT2 FLSs. We derive the parameter conditions under which the IT2 FLSs can give monotonic input–output mappings. The derived conditions are composed of two parts: the conditions on the antecedent IT2 FSs and the conditions on the consequent weights. Particularly, we consider five kinds of type-reduction and defuzzification methods which cover most of practical IT2 FLSs. We also show how to use the proposed results to design appropriate monotonic IT2 FLSs for modeling and control applications. To the best of the authors' knowledge, this is the most comprehensive study on the monotonicity of IT2 FLSs. The main novelties of this paper are listed as follows.

1) The presented results are valid for all kinds of IT2 FSs including the Trapezoidal IT2 FSs, the Gaussian IT2 FSs, Triangular IT2 FSs, and even the general IT2 FSs.
2) The presented conditions are useful for the five widely used type-reduction and defuzzification methods which cover most of IT2 FLSs in practice.
3) The derived conditions are for multiinput IT2 FLSs, no matter which type-reduction and defuzzification method is adopted.
4) Except the proposed antecedent and consequent conditions, there exist no other limitations on the MFs of IT2 FSs. Hence, the monotonicity conditions can be more easily satisfied.
5) The presented results are also valid for T1 FLSs and general TSK FLSs and more loose than some existing results for T1 FLSs [37], [39], [42].

The rest of this paper is organized as follows. Section II studies the monotonicity of T1 FLS and the IT2 FLS using the KM method. Section III derives the monotonicity conditions on the antecedent T1 FSs and IT2 FSs. Section IV studies the monotonicity of the IT2 FLSs using the DY method, the BMM method, the WT method, the NT method, and the other out-processing methods. Section V extends the monotonicity conditions to general TSK IT2 FLSs, presents guidelines for applying the derived conditions to modeling and control problems and summarizes other fundamental properties of IT2 FLSs. Finally, conclusions are drawn in Section VI. The proofs of all lemmas and theorems are given in the Appendix.

II. MONOTONICITY OF FUZZY LOGIC SYSTEMS

In this study, we consider the general multiinput single-output FLS, whose input variables are supposed to be $\mathrm{x} = (x_1, \ldots, x_p) \in X_1 \times X_2 \times \cdots \times X_p$. However, our results can be readily extended to multiinput multioutput FLS, for the latter can be decomposed into several multiinput single-output FLSs [1], [8].

First, the structures of T1 FLSs and IT2 FLSs are introduced briefly.

A. T1 Fuzzy Logic Systems

By assigning the jth input variable N_j T1 FSs, we can obtain $\prod_{j=1}^{p} N_j$ fuzzy rules, each of which has the following form:

Rule$(i_1 \ldots i_p)$: If x_1 is $A_1^{i_1}$, x_2 is $A_2^{i_2}$, $\ldots$, x_p is $A_p^{i_p}$, Then $y_o(\mathrm{x})$ is $w^{i_1 \cdots i_p}$, where $i_j = 1, 2, \ldots, N_j$, $w^{i_1 \cdots i_p}$s are the crisp consequent weights, $A_j^{i_j}$s are antecedent T1 FSs for the input variables. This rule base can be seen as the simplest TSK model and the Mamdani model with height defuzzification, where $w^{i_1 \cdots i_p}$ represents the point with the maximum membership degree of the consequent T1 FS of Rule$(i_1 \ldots i_p)$ [57]. In fact, it represents the most frequently used T1 FLSs in engineering problems [57], [58].

Once a crisp input $\mathrm{x} = (x_1, x_2, \ldots, x_p)$ is applied to the T1 FLS, through the singleton fuzzifier, the firing strength of Rule $(i_1 \ldots i_p)$ can be calculated by the product operation as follows:

$$f^{i_1 \cdots i_p}(\mathrm{x}) = \prod_{j=1}^{p} \mu_{A_j^{i_j}}(x_j). \tag{1}$$

Then, the output of the T1 FLS is computed as

$$\begin{aligned} y_o(\mathrm{x}) &= \frac{\sum_{i_1=1}^{N_1} \cdots \sum_{i_p=1}^{N_p} f^{i_1 \cdots i_p}(\mathrm{x}) w^{i_1 \cdots i_p}}{\sum_{i_1=1}^{N_1} \cdots \sum_{i_p=1}^{N_p} f^{i_1 \cdots i_p}(\mathrm{x})} \\ &= \frac{\sum_{i_1=1}^{N_1} \cdots \sum_{i_p=1}^{N_p} w^{i_1 \cdots i_p} \prod_{j=1}^{p} \mu_{A_j^{i_j}}(x_j)}{\sum_{i_1=1}^{N_1} \cdots \sum_{i_p=1}^{N_p} \prod_{j=1}^{p} \mu_{A_j^{i_j}}(x_j)}. \end{aligned} \tag{2}$$

B. IT2 FLS

By blurring the T1 FSs and crisp weights, we can obtain the following type-2 fuzzy rule base:

Rule$(i_1 \ldots i_p)$: If x_1 is $\widetilde{A}_1^{i_1}$, x_2 is $\widetilde{A}_2^{i_2}$, $\ldots$, x_p is $\widetilde{A}_p^{i_p}$, Then, $y_o(\mathrm{x})$ is $[\underline{w}^{i_1 \cdots i_p}, \overline{w}^{i_1 \cdots i_p}]$, where $i_j = 1, 2, \ldots, N_j$, $\widetilde{A}_j^{i_j}$ s are the antecedent IT2 FSs for the input variables, and $[\underline{w}^{i_1 \cdots i_p}, \overline{w}^{i_1 \cdots i_p}]$s are the interval consequent weights. This rule base can be seen as the Mamdani model with KM method where $[\underline{w}^{i_1 \cdots i_p}, \overline{w}^{i_1 \cdots i_p}]$ can be viewed as the centroid of the consequent IT2 FS [57]. When $\underline{w}^{i_1 \cdots i_p} = \overline{w}^{i_1 \cdots i_p}$, this IT2 FLS can be viewed as the simplified TSK model [57]. Again, this rule base represents the most widely used IT2 FLSs in engineering applications, e.g., modeling problems [21] and control applications [30], [51], [58].

Once a crisp input $\mathrm{x} = (x_1, x_2, \ldots, x_p)$ is applied to the IT2 FLS, through the singleton fuzzifier and the type-2 inference process, the interval firing strength of Rule $(i_1, i_2, \ldots, i_p)$ can be calculated by the product operation as follows:

$$F^{i_1 \cdots i_p}(\mathrm{x}) = [\underline{f}^{i_1 \cdots i_p}(\mathrm{x}), \overline{f}^{i_1 \cdots i_p}(\mathrm{x})] \tag{3}$$

where

$$\underline{f}^{i_1 \cdots i_p}(\mathrm{x}) = \prod_{j=1}^{p} \underline{\mu}_{\widetilde{A}_j^{i_j}}(x_j) \tag{4}$$

$$\overline{f}^{i_1 \cdots i_p}(\mathrm{x}) = \prod_{j=1}^{p} \overline{\mu}_{\widetilde{A}_j^{i_j}}(x_j) \tag{5}$$

in which $\underline{\mu}_{\widetilde{A}_j^{i_j}}$ and $\overline{\mu}_{\widetilde{A}_j^{i_j}}$ denote the lower and upper MFs of the IT2 FS $\widetilde{A}_j^{i_j}$.

To generate crisp output, the output processing including type-reduction and defuzzification is needed. There exist several different type-reduction and defuzzification methods. The IT2 FLSs with different type-reduction and defuzzification methods have different input–output mappings. In this section, we only discuss the most widely used Karnik–Mendel type-reduction and Center-Of-Sets (COS) defuzzification method (KM method) [1], [47]–[50], while other output processing methods will be given in Section IV. The KM type-reducer which uses the Karnik–Mendel algorithms to realize the type-reduction are reported as the COS type-reducer in some early literatures on IT2 FLSs [8].

Let

$$y_l^n(\mathrm{x}) = \frac{\sum_{i_1=1}^{N_1} \cdots \sum_{i_p=1}^{N_p} f^{i_1 \cdots i_p}(\mathrm{x}) \underline{w}^{i_1 \cdots i_p}}{\sum_{i_1=1}^{N_1} \cdots \sum_{i_p=1}^{N_p} f^{i_1 \cdots i_p}(\mathrm{x})} \tag{6}$$

$$y_r^n(\mathrm{x}) = \frac{\sum_{i_1=1}^{N_1} \cdots \sum_{i_p=1}^{N_p} f^{i_1 \cdots i_p}(\mathrm{x}) \overline{w}^{i_1 \cdots i_p}}{\sum_{i_1=1}^{N_1} \cdots \sum_{i_p=1}^{N_p} f^{i_1 \cdots i_p}(\mathrm{x})} \tag{7}$$

where $f^{i_1 \cdots i_p}(\mathrm{x})$ means $\underline{f}^{i_1 \cdots i_p}(\mathrm{x})$ or $\overline{f}^{i_1 \cdots i_p}(\mathrm{x})$. Consequently, we have a total of $2^{\prod_{j=1}^{p} N_j}$ different combinations for both $y_l^n(\mathrm{x})$ and $y_r^n(\mathrm{x})$, i.e., $n = 1, \ldots, K$ where $K = 2^{\prod_{j=1}^{p} N_j}$.

By the KM type-reducer, the left and right end points of the interval output of the IT2 FLS are computed as

$$y_l(\mathrm{x}) = \min_{n=1}^{K} y_l^n(\mathrm{x}) \tag{8}$$

$$y_r(\mathrm{x}) = \max_{n=1}^{K} y_r^n(\mathrm{x}). \tag{9}$$

The Karnik–Mendel algorithm [1], [8] or its enhanced ones [47]–[49] can be used to find $y_l(\mathrm{x})$ and $y_r(\mathrm{x})$ effectively. This is why, we usually call this COS type-reducer as the KM type-reducer. As the expressions in (8) and (9) are very convenient for our theoretical analysis, the details of Karnik–Mendel algorithm are omitted here. For more details, see [1], [8], and [47]–[49].

If the KM method is adopted, the crisp output of the IT2 FLS is calculated as

$$y_o(\mathrm{x}) = \frac{1}{2}[y_l(\mathrm{x}) + y_r(\mathrm{x})]. \tag{10}$$

When all sources of uncertainty disappear, IT2 FSs $\widetilde{A}_j^{i_j}$ s become T1FSs $A_j^{i_j}$ s, and interval weights $[\underline{w}^{i_1 \cdots i_p}, \overline{w}^{i_1 \cdots i_p}]$s become crisp weights $w^{i_1 \cdots i_p}$ s. Simultaneously, $y_l^n(\mathrm{x})$ and $y_r^n(\mathrm{x})$ all turn to the same function $y_o(\mathrm{x})$, which is the input–output mapping of a T1 FLS. Hence, the T1 FLS can be viewed as a special case of the IT2 FLS using the KM method.

C. Definition of Monotonicity

To present monotonicity conditions for FLSs, let us give the definition of monotonicity first.

Definition 1 [37], [39]: An FLS is said to be monotonically increasing with respect to (w.r.t) the kth input variable x_k, if $\forall x_k^1 \le x_k^2 \in X_k$ implies that $y_o(x_1, \ldots, x_k^1, \ldots, x_p) \le y_o(x_1, \ldots, x_k^2, \ldots, x_p)$ for all the combinations of $(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_p)$. And, an FLS is said to be monotonically decreasing w.r.t the kth input variable x_k, if $\forall x_k^1 \le x_k^2 \in X_k$ implies that $y_o(x_1, \ldots, x_k^1, \ldots, x_p) \ge y_o(x_1, \ldots, x_k^2, \ldots, x_p)$ for all combinations of $(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_p)$.

In this study, we just take the increasing monotonicity into account. Similar results can be obtained for decreasing monotonicity.

D. Monotonicity of IT2 Fuzzy Logic Systemss Using the Karnik–Mendel Method

The Karnik–Mendel type-reduction and COS defuzzification method (KM method) [1], [8], [47]–[50] are the most popular out-processing methods. The monotonicity of such IT2 FLS is studied in this section, while the monotonicity of the IT2 FLSs using other out-processing method is studied in Section IV.

For the increasing monotonicity, we have the following results for all $y_l^n(\mathrm{x})$ and $y_r^n(\mathrm{x})$.

Lemma 1: $y_l^n(\mathrm{x})$ and $y_r^n(\mathrm{x})$ $(n = 1, \ldots, K)$ are all monotonically increasing w.r.t x_k if we have the following.

1) For any $x_k^2 \geq x_k^1 \in X_k, 1 \leq l \leq m \leq N_k$, we have $\mu_{A_k^m}(x_k^2)\mu_{A_k^l}(x_k^1) \geq \mu_{A_k^m}(x_k^1)\mu_{A_k^l}(x_k^2)$, where $\mu_{A_k^m}$ means either $\underline{\mu}_{\widetilde{A}_k^m}$ or $\overline{\mu}_{\widetilde{A}_k^m}$, and $\mu_{A_k^l}$ means either $\underline{\mu}_{\widetilde{A}_k^l}$ or $\overline{\mu}_{\widetilde{A}_k^l}$.
2) $\underline{w}^{i_1\cdots i_{k-1}i_k i_{k+1}\cdots i_p} \leq \underline{w}^{i_1\cdots i_{k-1}(i_k+1)i_{k+1}\cdots i_p}$, $\overline{w}^{i_1\cdots i_{k-1}i_k i_{k+1}\cdots i_p} \leq \overline{w}^{i_1\cdots i_{k-1}(i_k+1)i_{k+1}\cdots i_p}$ for all the combinations of $(i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_p)$, where $i_k = 1, \cdots, N_k - 1$.

Proof: See Appendix A. □

The conclusion in this lemma is very important. Most of the proofs of the following theorems are based on the conclusion provided by this Lemma.

For the monotonicity of the IT2 FLS using the KM method, we have the following theorem.

Theorem 1: The IT2 FLS using the KM method is monotonically increasing w.r.t x_k if we have the following.

1) For any $x_k^2 \geq x_k^1 \in X_k, 1 \leq l \leq m \leq N_k$, we have $\mu_{A_k^m}(x_k^2)\mu_{A_k^l}(x_k^1) \geq \mu_{A_k^m}(x_k^1)\mu_{A_k^l}(x_k^2)$, where $\mu_{A_k^m}$ means either $\underline{\mu}_{\widetilde{A}_k^m}$ or $\overline{\mu}_{\widetilde{A}_k^m}$, and $\mu_{A_k^l}$ means either $\underline{\mu}_{\widetilde{A}_k^l}$ or $\overline{\mu}_{\widetilde{A}_k^l}$.[2]
2) $\underline{w}^{i_1\cdots i_{k-1}i_k i_{k+1}\cdots i_p} \leq \underline{w}^{i_1\cdots i_{k-1}(i_k+1)i_{k+1}\cdots i_p}$, $\overline{w}^{i_1\cdots i_{k-1}i_k i_{k+1}\cdots i_p} \leq \overline{w}^{i_1\cdots i_{k-1}(i_k+1)i_{k+1}\cdots i_p}$ for all the combinations of $(i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_p)$, where $i_k = 1, \cdots, N_k - 1$.

Proof: See Appendix B. □

E. Monotonicity of T1 Fuzzy Logic System

As discussed in Section II-B, T1 FLS can be viewed as a special case of the IT2 FLS using the KM method. Consequently, for the monotonicity of T1 FLS, we have the following result.

Theorem 2: The T1 FLS is monotonically increasing w.r.t x_k if we have the following.

1) For any $x_k^2 \geq x_k^1 \in X_k, 1 \leq l \leq m \leq N_k$, we have $\mu_{A_k^m}(x_k^2)\mu_{A_k^l}(x_k^1) \geq \mu_{A_k^m}(x_k^1)\mu_{A_k^l}(x_k^2)$.
2) $w^{i_1\cdots i_{k-1}i_k i_{k+1}\cdots i_p} \leq w^{i_1\cdots i_{k-1}(i_k+1)i_{k+1}\cdots i_p}$ for all the combinations of $(i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_p)$, where $i_k = 1, \ldots, N_k - 1$.

Note that our result on the monotonicity of both IT2 FLS and T1 FLS put no limitation on the shapes and types of IT2 FSs and T1 FSs. The second condition in Theorems 1 and 2 can be easily checked. On the other hand, we need to explore how the antecedent FSs in the rule base can meet the first condition in both theorems. In the next section, we will study this issue.

[2]In this study, we do not constrain the shape of the IT2 FSs, Hence, this theorem holds for any kind of IT2 FSs, as long as this condition can be satisfied. In the next section, we will show that this condition can easily be met when Trapezoidal and Gaussian IT2 FSs are adopted.

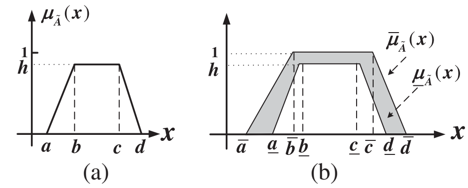

Fig. 1. Trapezoidal FSs: (a) Trapezoidal T1 FS. (b) Trapezoidal IT2 FS.

III. Monotonicity Conditions on the Antecedent Parts of Fuzzy Rules

In practical FLSs, the most frequently used FSs are the Gaussian FS and the Trapezoidal FS, a special case of which is the Triangular FS [1]–[3]. Next, we will show how the Trapezoidal and Gaussian FSs can satisfy the first condition in Theorems 1 and 2.

A. Monotonicity Conditions of the Trapezoidal Fuzzy Sets

Fig. 1(a) shows us a Trapezoidal T1 FS A, the MF of which can be expressed as

$$\mu_A(x) = \begin{cases} 0, & x \leq a \text{ or } x \geq d \\ h\frac{x-a}{b-a}, & a < x \leq b \\ h, & b < x \leq c \\ h\frac{x-d}{c-d}, & c < x < d \end{cases} \tag{11}$$

where $0 < h \leq 1$. We denote such a Trapezoidal T1 FS as $\mu_A(x) = \mu_A(x, a, b, c, d, h)$.

Note that Triangular T1 FSs are special cases of Trapezoidal T1 FSs when $b = c$. Generally, in many applications, normal Trapezoidal T1 FSs whose heights equal to 1 are adopted. However, in this study, we consider the general case where $0 < h \leq 1$.

By blurring the Trapezoidal T1 FS shown in Fig. 1(a), we can obtain the Trapezoidal IT2 FS $\widetilde{A}$ [see Fig. 1(b)], which can be described by its lower and upper MFs $\underline{\mu}_{\widetilde{A}}(x)$ and $\overline{\mu}_{\widetilde{A}}(x)$ as

$$\underline{\mu}_{\widetilde{A}}(x) = \underline{\mu}_{\widetilde{A}}(x, \underline{a}, \underline{b}, \underline{c}, \underline{d}, h) \tag{12}$$

$$\overline{\mu}_{\widetilde{A}}(x) = \overline{\mu}_{\widetilde{A}}(x, \overline{a}, \overline{b}, \overline{c}, \overline{d}, 1) \tag{13}$$

where $\overline{a} \leq \underline{a}$, $\overline{b} \leq \underline{b}$, $\underline{c} \leq \overline{c}$, $\underline{d} \leq \overline{d}$. We denote such a Trapezoidal IT2 FS as $\mu_{\widetilde{A}}(x) = \mu_{\widetilde{A}}(x, \overline{a}, \overline{b}, \overline{c}, \overline{d}, \underline{a}, \underline{b}, \underline{c}, \underline{d}, h)$. Again, when $\underline{b} = \underline{c}$ and $\overline{b} = \overline{c}$, the Trapezoidal IT2 FS becomes a Triangular IT2 FS.

Lemma 2: Consider two Trapezoidal T1 FSs $\mu_{A^l}(x) = \mu_{A^l}(x, a^l, b^l, c^l, d^l, h^l)$, $\mu_{A^r}(x) = \mu_{A^r}(x, a^r, b^r, c^r, d^r, h^r)$. If $a^l \leq a^r$, $b^l \leq b^r$, $c^l \leq c^r$, $d^l \leq d^r$ (as shown in Fig. 2), then, for any $x^2 \geq x^1 \in X$, we have $\mu_{A^r}(x^2)\mu_{A^l}(x^1) \geq \mu_{A^r}(x^1)\mu_{A^l}(x^2)$.

Proof: See Appendix C. □

From Lemma 2, we can make the following conclusion.

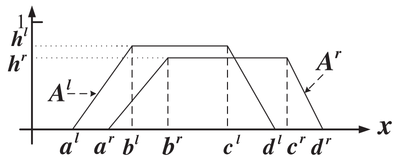

Fig. 2. Two Trapezoidal T1 FSs Used in Lemma 2.

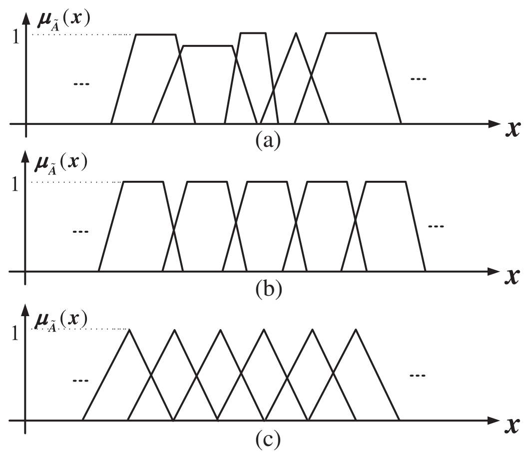

Fig. 3. Trapezoidal T1 FSs that satisfy the first condition in Theorem 2. (a) General case. (b) Normal case. (c) Triangular case.

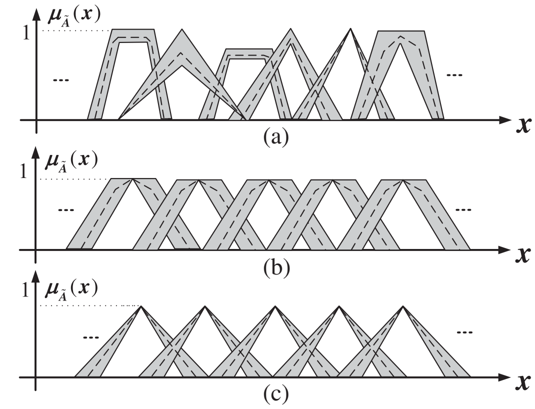

Fig. 4. Trapezoidal IT2 FSs that satisfy the first condition in Theorem 1. (a) General case. (b) Normal case. (c) Triangular case.

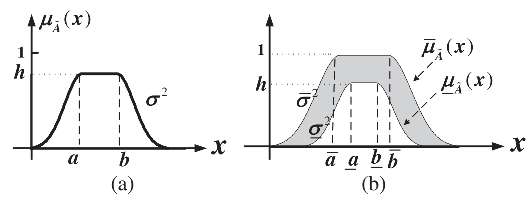

Fig. 5. Generalized Gaussian FS: (a) Generalized Gaussian T1 FS. (b) Generalized Gaussian IT2 FS.

Theorem 3: For Trapezoidal T1 FSs $A^1, A^2, \dots, A^N$, where $\mu_{A^j}(x) = \mu_{A^j}(x, a^j, b^j, c^j, d^j, h^j)$, if $a^l \le a^m$, $b^l \le b^m$, $c^l \le c^m$, $d^l \le d^m$ for any $1 \le l \le m \le N$, then the first condition in Theorem 2 can be met.

Fig. 3 demonstrates several examples that satisfy the first condition in Theorem 2. A general case is shown in Fig. 3(a), where the Trapezoidal T1 FSs have different widths and heights. In Fig. 3(b), the Trapezoidal T1 FSs are normal and have the same width. The triangular case is shown in Fig. 3(c). As well known, T1 FSs shown in Fig. 3(b) and (c) are widely used in fuzzy modeling and fuzzy control problems. See [1], [37], and [42] for examples.

Theorem 4: For the Trapezoidal IT2 FSs $\widetilde{A}^1, \widetilde{A}^2, \dots, \widetilde{A}^N$ in the input domain X, where $\mu_{\widetilde{A}^j}(x) = \mu_{\widetilde{A}^j}(x, \overline{a}^j, \overline{b}^j, \overline{c}^j, \overline{d}^j, \underline{a}^j, \underline{b}^j, \underline{c}^j, \underline{d}^j, h^j)$, if $\underline{a}^l \le \overline{a}^m$, $\underline{b}^l \le \overline{b}^m$, $\overline{c}^l \le \underline{c}^m$, $\overline{d}^l \le \underline{d}^m$ for any $1 \le l \le m \le N$, then the first condition in Theorem 1 can be met.

Proof: See Appendix D. □

Fig. 4 demonstrates several examples that satisfy the first condition in Theorem 1. A general case is shown in Fig. 4(a), where the Trapezoidal IT2 FSs have different shapes. The Trapezoidal IT2 FSs in Fig. 4(b) and the Triangular IT2 FSs in Fig. 4(c) are widely used in fuzzy modeling and fuzzy control problems. See [1]–[3] for examples.

B. Monotonicity Conditions of the Gaussian Fuzzy Sets

In this section, we consider the monotonicity conditions of generalized Gaussian FSs as shown in Fig. 5, which include the most widely used standard Gaussian FSs.

The generalized Gaussian T1 FS A shown in Fig. 5(a) can be expressed as

$$\mu_A(x) = \mu_A(x, a, b, \sigma, h) = \begin{cases} h * e^{-\frac{(x-a)^2}{2\sigma^2}}, & x \le a \\ h, & a < x \le b \\ h * e^{-\frac{(x-b)^2}{2\sigma^2}}, & x \ge b. \end{cases} \tag{14}$$

When $a = b$ and $h = 1$, the generalized Gaussian T1 FS becomes a standard Gaussian T1 FS.

By blurring the generalized Gaussian T1 FS A, we can get the generalized Gaussian IT2 FS $\widetilde{A}$ [see Fig. 5(b)] which can be depicted by its lower and upper MFs $\underline{\mu}_{\widetilde{A}}(x)$ and $\overline{\mu}_{\widetilde{A}}(x)$ as

$$\underline{\mu}_{\widetilde{A}}(x) = \underline{\mu}_{\widetilde{A}}(x, \underline{a}, \underline{b}, \underline{\sigma}, h) \tag{15}$$

$$\overline{\mu}_{\widetilde{A}}(x) = \overline{\mu}_{\widetilde{A}}(x, \overline{a}, \overline{b}, \overline{\sigma}, 1) \tag{16}$$

where $\overline{a} \le \underline{a}$, $\underline{b} \le \overline{b}$, $\underline{\sigma} \le \overline{\sigma}$.

Next, we denote such a generalized Gaussian IT2 FS as $\mu_{\widetilde{A}}(x) = \mu_{\widetilde{A}}(x, \overline{a}, \overline{b}, \underline{a}, \underline{b}, \overline{\sigma}, \underline{\sigma}, h)$. Again, when $\underline{a} = \underline{b}$ and $\overline{a} = \overline{b}$, the generalized Gaussian IT2 FS turns to the widely used one.

Lemma 3: Consider two generalized Gaussian T1 FSs $\mu_{A^l}(x) = \mu_{A^l}(x, a^l, b^l, \sigma^l, h^l)$, $\mu_{A^r}(x) = \mu_{A^r}(x, a^r, b^r, \sigma^r, h^r)$. If $a^l \le a^r$, $b^l \le b^r$, and $(\sigma^r)^2 h^l = (\sigma^l)^2 h^r$ (as shown in Fig. 6), then, for any $x^2 \ge x^1 \in X$, we have $\mu_{A^r}(x^2)\mu_{A^l}(x^1) \ge \mu_{A^r}(x^1)\mu_{A^l}(x^2)$.

Proof: See Appendix E. □

Similarly, from Lemma 3, we can conclude that:

Theorem 5: For the generalized Gaussian T1 FSs $A^1, A^2, \dots, A^N$, where $\mu_{A^j}(x) = \mu_{A^j}(x, a^j, b^j, \sigma^j, h^j)$, if $a^l \le a^m$, $b^l \le b^m$, $\sigma^l = \sigma^m$, $h^l = h^m$ for any $1 \le l \le m \le N$, then the first condition in Theorem 2 can be met.

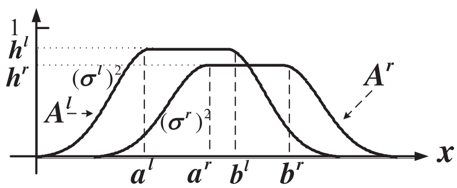

Fig. 6. Two generalized Gaussian T1 FSs used in Lemma 3.

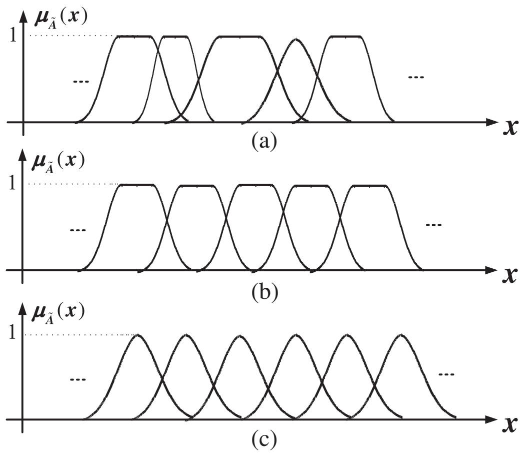

Fig. 7. Gaussian T1 FSs that satisfy the first condition in Theorem 2. (a) General case. (b) Normal case. (c) Standard case.

Fig. 7 shows several examples that satisfy the first condition in Theorem 2. A general case is shown in Fig. 7(a), where the generalized Gaussian T1 FSs have different shapes, but they have the same height and variation. In Fig. 7(b), the generalized Gaussian T1 FSs have the same shape. The standard Gaussian T1 FSs are shown in Fig. 7(c), which are also widely used in modeling and control.

Theorem 6: For the generalized Gaussian IT2 FSs $\widetilde{A}^1, \widetilde{A}^2, \ldots, \widetilde{A}^N$ in the input domain X, where $\mu_{\widetilde{A}^j}(x) = \mu_{\widetilde{A}^j}(x, \overline{a}^j, \overline{b}^j, \underline{a}^j, \underline{b}^j, \overline{\sigma}^j, \underline{\sigma}^j, h^j)$, if $\underline{a}^l \leq \overline{a}^m$, $\overline{b}^l \leq \underline{b}^m$, $\overline{\sigma}^l = \overline{\sigma}^m = \overline{\sigma}$, $\underline{\sigma}^l = \underline{\sigma}^m = \underline{\sigma}$, and $h^l = h^m = \frac{\underline{\sigma}^2}{\overline{\sigma}^2}$ for any $1 \leq l \leq m \leq N$, then the first condition in Theorem 1 can be met.

Proof: See Appendix F. □

Note that the generalized Gaussian IT2 FSs should have the same height determined by the uncertain variations. On the other hand, we do not put such constraints on Trapezoidal IT2 FSs.

Fig. 8 shows several cases that satisfy the first condition in Theorem 1. A generalized case is shown in Fig. 8(a), where the Gaussian IT2 FSs have different shapes, but they have the same height and uncertain variations. The generalized Gaussian IT2 FSs in Fig. 8(b) and (c) have the same shape, respectively. Note that the lower and upper MFs of each IT2 FS in Fig. 8(b) have different variations, while the lower and upper MFs of each IT2 FS in Fig. 8(c) have the same variation.

C. Monotonicity Conditions of the Embedded Fuzzy Sets

For IT2 FS $\widetilde{A}^j$, one of its embedded T1 FSs can be obtained by

$$\mu_{A^j}(x) = (1-\eta)\overline{\mu}_{\widetilde{A}^j}(x) + \eta\underline{\mu}_{\widetilde{A}^j}(x), \quad \eta \in [0,1]. \tag{17}$$

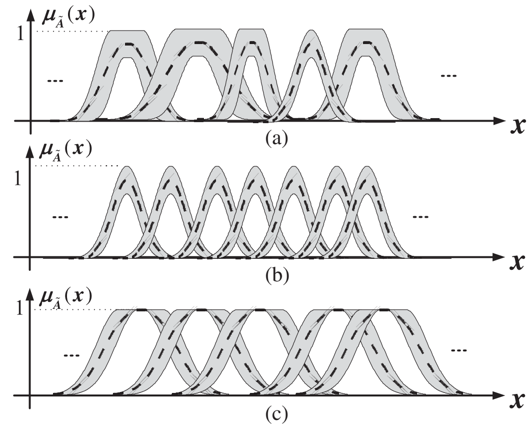

Fig. 8. Generalized Gaussian IT2 FSs that satisfy the first condition in Theorem 1. (a) General case. (b) The lower and upper MFs of each IT2 FS have different variations. (c) The lower and upper MFs of each IT2 FS have the same variation.

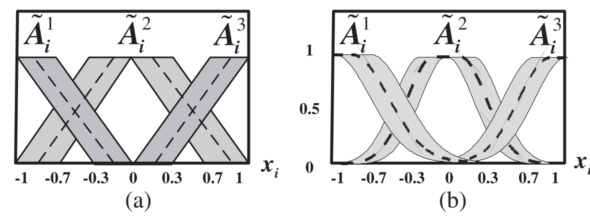

Fig. 9. Type-2 and Type-1 (dashed lines) fuzzy partitions. (a) Trapezoidal FSs. (b) Gaussian FSs.

The embedded T1 FSs of the IT2 FSs in Figs. 4 and 8 are depicted using the dashed lines. From Figs. 4 and 8, we can observe that the embedded T1 FSs may be neither the Trapezoidal type nor the generalized Gaussian type. Even though the embedded T1 FSs may be abnormal, we can still prove the following results.

Theorem 7: Suppose that IT2 FSs $\widetilde{A}^1, \widetilde{A}^2, \ldots, \widetilde{A}^N$ satisfy the first condition in Theorem 1. For their embedded T1 FSs A^1, A^2, $\cdots$, A^N obtained by (17), the first condition in Theorem 2 can be met.

Proof: See Appendix G. □

In [45] and [46], we have presented the monotonicity conditions for the single-input IT2 FLS using the KM method. From Theorems 4 and 6, we can conclude that the monotonicity conditions in [45] and [46] are special cases of the results in this study. In addition, from Theorems 3, 5 and 7, we can observe that in this study the shape or the height of the T1 FSs are not constrained. Therefore, our result for T1 FLS is more general and loose than that provided in [37], [39], and [42].

D. Examples

In this section, we provide two examples to verify Theorems 1–7.

1) Example 1: This example is presented to demonstrate Theorems 1, 4, and 6 using the single-input and double-input cases. In each case, two kinds of IT2 FSs are considered.

Fig. 9 shows the Trapezoidal and Gaussian IT2 FSs, which will be used in this example and the following examples.

TABLE II
SINGLE-INPUT FUZZY RULE BASE WITH INTERVAL CONSEQUENT WEIGHTS

x_1	$\widetilde{A}_1^1$	$\widetilde{A}_1^2$	$\widetilde{A}_1^3$
w	[0.7, 1.3]	[1.7, 2.3]	[2.7, 3.3]

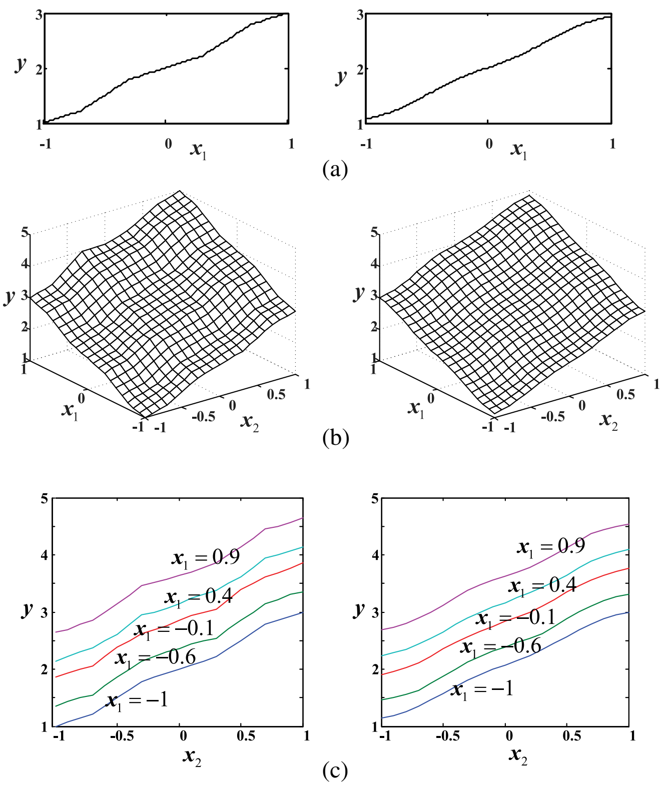

Fig. 10. Input–output mappings of the IT2 FLSs using the KM method. (a) Single-input IT2 FLSs. (b) Double-input IT2 FLSs. (c) Slices of the input-output mappings of the double-input IT2 FLSs.

TABLE III
DOUBLE-INPUT FUZZY RULE BASE WITH INTERVAL CONSEQUENT WEIGHTS

$x_1 \setminus x_2$	$\widetilde{A}_2^1$	$\widetilde{A}_2^2$	$\widetilde{A}_2^3$
$\widetilde{A}_1^1$	[0.7, 1.3]	[1.7, 2.3]	[2.7, 3.3]
$\widetilde{A}_1^2$	[1.7, 2.3]	[2.7, 3.3]	[3.7, 4.3]
$\widetilde{A}_1^3$	[2.7, 3.3]	[3.7, 4.3]	[4.7, 5.3]

For the single-input IT2 FLSs, their rule base is shown in Table II. And, the MFs of the IT2 FSs in Table II are depicted in Fig. 9(a) and (b). The input–output mappings of these single-input IT2 FLSs are shown in Fig. 10(a). The left one is for the IT2FLS with Trapezoidal IT2 FSs (Trapezoidal IT2 FLS), while the right one is for the IT2FLS with Gaussian IT2 FSs (Gaussian IT2 FLS).

For the double-input IT2 FLSs, their rule base is shown in Table III, and, the MFs of the IT2 FSs in Table III are also depicted in Fig. 9(a) and (b). The input–output mappings of these double-input IT2 FLSs are shown in Fig. 10(b). The left one is for the Trapezoidal IT2 FLS, while the right one is for the Gaussian IT2 FLS. To observe clearly, slices of the input–output mappings of the double-input IT2 FLSs shown in Fig. 10(b) are demonstrated in Fig. 10(c).

TABLE IV
SINGLE-INPUT FUZZY RULE BASE WITH CRISP CONSEQUENT WEIGHTS

x_1	$\widetilde{A}_1^1$	$\widetilde{A}_1^2$	$\widetilde{A}_1^3$
w	1	2	3

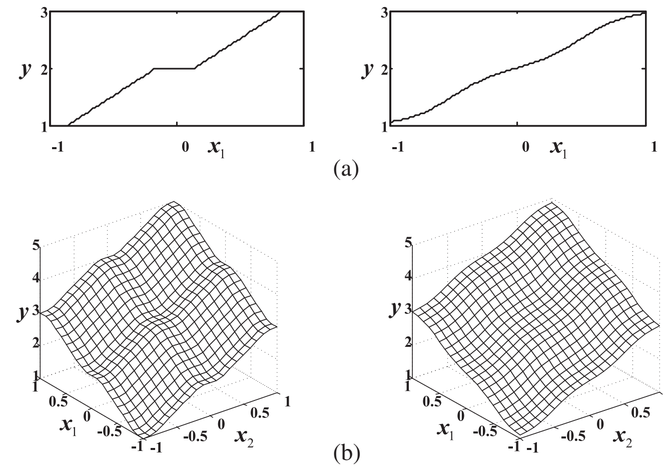

Fig. 11. Input–output mappings of the T1 FLSs. (a) single-input T1 FLSs. (b) Double-input T1 FLSs.

TABLE V
DOUBLE-INPUT FUZZY RULE BASE WITH CRISP CONSEQUENT WEIGHTS

$x_1 \setminus x_2$	$\widetilde{A}_2^1$	$\widetilde{A}_2^2$	$\widetilde{A}_2^3$
$\widetilde{A}_1^1$	1	2	3
$\widetilde{A}_1^2$	2	3	4
$\widetilde{A}_1^3$	3	4	5

From Fig. 10, we can observe that the input–output mappings of both the single-input and double-input IT2 FLSs with different kinds of IT2 FSs are consistent with the conclusions in Theorems 1, 4, and 6. Another thing needs to be mentioned is that the Gaussian IT2 FLSs perform smoother than the Trapezoidal IT2 FLSs.

2) Example 2: This example is done to demonstrate Theorems 2, 3, and 5 using the single-input and double-input cases. In each case, we consider two kinds of T1 FSs that are depicted in Fig. 9(a) and (b) using the dashed lines.

For the single-input T1 FLSs, their rule base is shown in Table IV. The input–output mappings of these single-input T1 FLSs are shown in Fig. 11(a). The left one is for the Trapezoidal T1 FLS, while the right one is for the Gaussian T1 FLS.

For the double-input T1 FLSs, their rule base is shown in Table V. The input–output mappings of these double-input T1 FLSs are shown in Fig. 11(b). The left one is for the Trapezoidal T1 FLS, while the right one is for the Gaussian T1 FLS.

From Fig. 11, we can see that the input–output mappings of both the single-input and double-input T1 FLSs with different kinds of T1 FSs are consistent with the conclusions in Theorems 2, 3, and 5. Again, the Gaussian T1 FLSs perform smoother than the Trapezoidal ones.

IV. Monotonicity of IT2 Fuzzy Logic Systems Using Other Out-processing Methods

In this section, the monotonicity of IT2 FLSs using other four different out-processing methods (the DY method [51], the BMM method [52], the WT method [53], and the NT method [54], [55]) is investigated. In these out-processing methods, all the consequent interval weights become crisp weights, i.e. $\underline{w}^{i_1\cdots i_p} = \overline{w}^{i_1\cdots i_p} = w^{i_1\cdots i_p}$. Therefore, the expressions of $y_l^n(\mathrm{x})$ and $y_r^n(\mathrm{x})$ in (6) and (7) become $y^n(\mathrm{x}) = y_l^n(\mathrm{x}) = y_r^n(\mathrm{x})$.

A. Monotonicity of IT2 Fuzzy Logic Systems Using the Du–Ying, Begian–Melek–Mendel, Wu–Tan and Nie–Tan methods

1) The Du–Ying Method: Du and Ying proposed an out-processing method in [51]. The final output of the IT2 FLS using the DY method is computed as the average of all the $K\, y^n(\mathrm{x})$, where $K = 2^{\prod_{j=1}^{p} N_j}$, i.e.,

$$y_o(\mathrm{x}) = \frac{1}{K}\sum_{n=1}^{K} y^n(\mathrm{x}). \tag{18}$$

2) The Begian–Melek–Mendel Method: In [52], Biglarbegian *et al.* proposed another closed-form out-processing method. The final output of the IT2 FLS using the BMM method is computed as the linear combination of $y^1(\mathrm{x})$ and $y^K(\mathrm{x})$, i.e.,

$$\begin{aligned} y_o(\mathrm{x}) &= \alpha y^1(\mathrm{x}) + \beta y^K(\mathrm{x}) \\ &= \alpha \frac{\sum_{i_1=1}^{N_1}\cdots\sum_{i_p=1}^{N_p} \underline{f}^{i_1\cdots i_p}(\mathrm{x}) w^{i_1\cdots i_p}}{\sum_{i_1=1}^{N_1}\cdots\sum_{i_p=1}^{N_p} \underline{f}^{i_1\cdots i_p}(\mathrm{x})} \\ &\quad + \beta \frac{\sum_{i_1=1}^{N_1}\cdots\sum_{i_p=1}^{N_p} \overline{f}^{i_1\cdots i_p}(\mathrm{x}) w^{i_1\cdots i_p}}{\sum_{i_1=1}^{N_1}\cdots\sum_{i_p=1}^{N_p} \overline{f}^{i_1\cdots i_p}(\mathrm{x})}. \end{aligned} \tag{19}$$

3) The Wu–Tan Method: The WT method proposed in [53] adopts the equivalent T1 FS to realize the type-reduction and defuzzification. The final output of the IT2 FLS using the WT method is computed as

$$y_o(\mathrm{x}) = \frac{\sum_{i_1=1}^{N_1}\cdots\sum_{i_p=1}^{N_p} f^{i_1\cdots i_p}(\mathrm{x}) w^{i_1\cdots i_p}}{\sum_{i_1=1}^{N_1}\cdots\sum_{i_p=1}^{N_p} f^{i_1\cdots i_p}(\mathrm{x})} \tag{20}$$

where

$$f^{i_1\cdots i_p}(\mathrm{x}) = \prod_{j=1}^{p} \mu_{A_j^{i_j}}(x_j) \tag{21}$$

in which

$$\mu_{A_j^{i_j}}(x_j) = (1-\eta_j^{i_j}(x_j))\overline{\mu}_{\widetilde{A}_j^{i_j}}(x_j) + \eta_j^{i_j}(x_j)\underline{\mu}_{\widetilde{A}_j^{i_j}}(x_j) \tag{22}$$

where $\eta_j^{i_j}(x_j)$ is a function of the inputs and is different for different IT2 FSs [53].

4) The Nie–Tan Method: When $\eta_j^{i_j}(x_j) = \frac{1}{2}$, the WT method turns to another out-processing method — the NT method proposed by Nie and Tan in [54]. Hence, the NT method is a special case of the WT method.

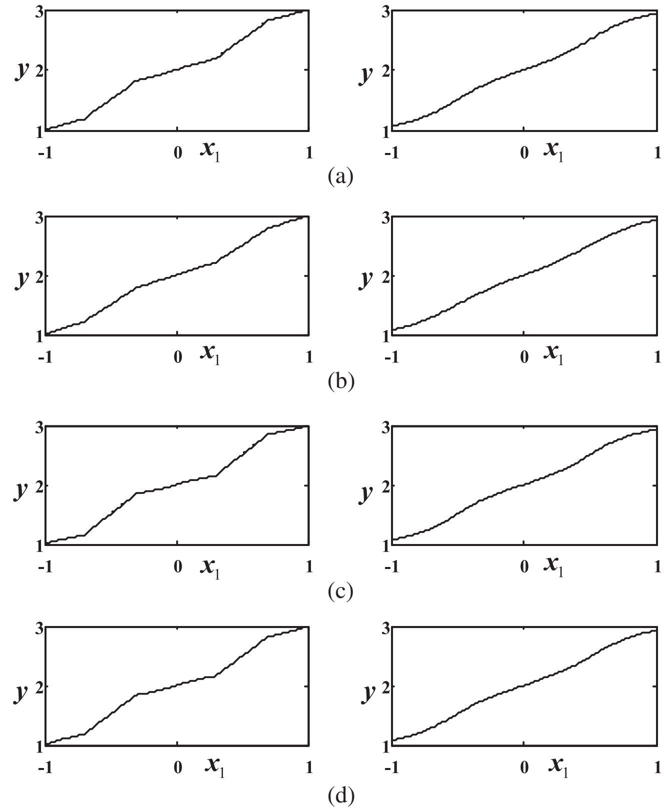

Fig. 12. Input–output mappings of the single-input IT2 FLSs using different type-reduction and defuzzification methods. (a) DY method. (b) BMM method. (c) WT method ($\eta(x) = 0.5$) and the NT method. (d) UB method.

For the IT2FLSs using these four type-reduction and defuzzification methods, we have the following results on the monotonicity:

Theorem 8: The IT2 FLSs using the DY, BMM, WT,[3] and NT methods are monotonically increasing w.r.t x_k if we have the following.

1) For any $x_k^2 \geq x_k^1 \in X_k, 1 \leq l \leq m \leq N_k$, we have $\mu_{A_k^m}(x_k^2)\mu_{A_k^l}(x_k^1) \geq \mu_{A_k^m}(x_k^1)\mu_{A_k^l}(x_k^2)$, where $\mu_{A_k^m}$ means either $\underline{\mu}_{\widetilde{A}_k^m}$ or $\overline{\mu}_{\widetilde{A}_k^m}$, and $\mu_{A_k^l}$ means either $\underline{\mu}_{\widetilde{A}_k^l}$ or $\overline{\mu}_{\widetilde{A}_k^l}$.
2) $w^{i_1\cdots i_{k-1} i_k i_{k+1}\cdots i_p} \leq w^{i_1\cdots i_{k-1}(i_k+1)i_{k+1}\cdots i_p}$ for all the combinations of $(i_1,\ldots,i_{k-1},\, i_{k+1},\ldots,i_p)$, where $i_k = 1,\cdots,\, N_k - 1$.

Proof: See Appendix H. □

B. Examples

Two examples illustrating Theorem 8 are presented in Figs. 12 and 13.

1) Single-Input Example: In this example, the fuzzy rule base is shown in Table IV, and the MFs of the IT2 FSs in

[3]As $\eta_j^{i_j}(x_j)$ is a function of the inputs and is different for different IT2 FSs, it is quite difficult to prove the monotonicity of the IT2 FLS using the WT method. Here, we assume that $\eta_j^1(x_j) = \eta_j^2(x_j) = \cdots = \eta_j^{N_j}(x_j) = \eta_j$ in the WT method. For the IT2 FLS using the general WT method, the proof of such a conclusion is still under investigation.

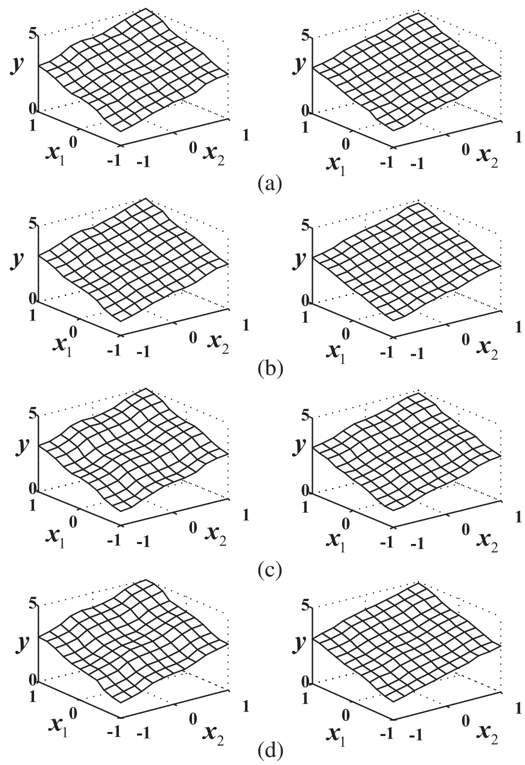

Fig. 13. Input–output mappings of the double-input IT2 FLSs using different type-reduction and defuzzification methods. (a) DY method. (b) BMM method. (c) WT method ($\eta(x) = 0.5$) and the NT method. (d) UB method.

Table IV are also depicted in Fig. 9(a) and (b). The input–output mappings of the single-input IT2 FLSs using the DY method, the BMM method, and the WT (NT) method are shown in Fig. 12(a)–(c), respectively. The left column is for the Trapezoidal IT2 FLSs, while the right column is for the Gaussian IT2 FLSs.

2) Double-Input Example: In this example, the fuzzy rule base is shown in Table V. And, the MFs of the IT2 FSs in Table V are also depicted in Fig. 9(a) and (b). The input–output mappings of the double-input IT2 FLSs using the DY method, the BMM method, and the WT (NT) method are shown in Fig. 13(a)–(c), respectively. The left column is for the Trapezoidal IT2 FLS, while the right column is for the Gaussian IT2 FLSs.

From Figs. 12 and 13, we can observe that the input–output mappings of both the single-input and double-input IT2 FLSs with different kinds of IT2 FSs are consistent with the conclusions in Theorem 8.

C. Other Type-reduction and Defuzzification Methods

There exist other two type-reduction and defuzzification methods which have been also widely used in fuzzy modeling and control.

The first one is the uncertainty bound type-reduction and COS defuzzification method (UB method) [56]. The formula of the input–output mapping of the IT2 FLS using the UB method is so complex that it is quite difficult for us to derive monotonicity conditions for such FLS. Taking the same rule base and IT2 FSs in the above examples, the input–output mappings of the single-input and double-input IT2 FLSs using the UB method are plotted in Figs. 12(d) and 13(d), respectively. In Figs. 12(d) and 13(d), IT2 FLSs using the UB method seems monotonic. However, we can not conclude that the derived conditions can assure the monotonicity of the IT2 FLS using the UB method. Such an example can be found in the first column of in [57], Fig. 5(c)]. Therefore, the monotonicity of the IT2 FLS using the UB method is still an open problem.

Another widely used type-reduction and defuzzification method is the Coupland–John (CJ) method proposed in [59]. The CJ method is a geometric type-reduction and defuzzification method which is used in the Mamdani IT2 FLSs. For analysis simplicity, only the monotonicity of the TSK IT2 FLS is considered in this study, therefore; the monotonicity of the IT2 FLSs using the CJ method is left for our future study.

V. Summary and Discussions

This section first presents the summarization and extension of our results. Then, guidelines for applying the proposed results to modeling and control problems are provided. Finally, relationships among monotonicity and other fundamental properties of IT2 FLSs are discussed.

A. Summarization and Extension to General TSK IT2 Fuzzy Logic System

In Sections II– IV, we have derived the monotonicity conditions for the IT2 FLSs using five kinds of type-reduction and defuzzification methods. From these results, we can observe that such conditions are almost the same and only differ in the consequent parts, which take interval values in the KM method but crisp values in the other four methods.

In the rule base of the IT2 FLS in Section II-B, if we replace the consequent interval of Rule$(i_1 \cdots i_p)$ by $y^{i_1 \cdots i_p}(\mathrm{x}) = c_1^{i_1 \cdots i_p} x_1 + c_2^{i_1 \cdots i_p} x_2 + \cdots + c_p^{i_1 \cdots i_p} x_p + c_{p+1}^{i_1 \cdots i_p}$, then the simplified TSK IT2 FLS becomes general TSK IT2 FLS. Similarly, for the general TSK IT2 FLS, we have the following conclusion:

Theorem 9: Given the KM method, the DY method, the BMM method, the WT method, and the NT method, no matter which one is adopted, the general TSK IT2 FLS is monotonically increasing w.r.t x_k if we have the following.

1) For any $x_k^2 \geq x_k^1 \in X_k, 1 \leq l \leq m \leq N_k$, we have $\mu_{A_k^m}(x_k^2)\mu_{A_k^l}(x_k^1) \geq \mu_{A_k^m}(x_k^1)\mu_{A_k^l}(x_k^2)$, where $\mu_{A_k^m}$ means either $\underline{\mu}_{\widetilde{A}_k^m}$ or $\overline{\mu}_{\widetilde{A}_k^m}$, and $\mu_{A_k^l}$ means either $\underline{\mu}_{\widetilde{A}_k^l}$ or $\overline{\mu}_{\widetilde{A}_k^l}$.
2) $\forall \mathrm{x} \in X$, $y^{i_1 \cdots i_{k-1} i_k i_{k+1} \cdots i_p}(\mathrm{x}) \leq y^{i_1 \cdots i_{k-1}(i_k+1) i_{k+1} \cdots i_p}(\mathrm{x})$ for all the combinations of $(i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_p)$, where $i_k = 1, \ldots, N_k - 1$.

Proof: This theorem can be proved in the similar way as Theorems 1 and 8. In the proof process, we just need to replace $w^{i_1 \cdots i_p}$ by $y^{i_1 \cdots i_p}(\mathrm{x})$. □

The derived monotonicity conditions are composed of two parts: the antecedent part and the consequent part. The antecedent monotonicity conditions are the constraints on the parameters of the antecedent IT2 FSs of fuzzy rules. The consequent monotonicity conditions are on the consequent parameters of the fuzzy rules and can be checked easily.

B. Ordering of T1 and IT2 Fuzzy Sets in the Antecedent Monotonicity Conditions

From the derived results in Section III, we can observe that the antecedent parameter conditions in Theorems 3–7 arrange the antecedent T1 FSs and IT2 FSs in the monotonicity order. In this section, we explore the relationships between the antecedent monotonicity conditions and the ordering of T1 and IT2 FSs.

The ordering of T1 FSs defined in [42], [60], and [61] is considered. For two T1 FSs A and B in $\mathcal{F}(\mathbf{R})$ which is the set of all the convex T1 FSs, the α-level of A and B are denoted as $A_\alpha = [A_\alpha^L, A_\alpha^R]$ and $B_\alpha = [B_\alpha^L, B_\alpha^R]$. Then, a fuzzy ordering $A \preceq B$ exists if $A_\alpha^L \leq B_\alpha^L$ and $A_\alpha^R \leq B_\alpha^R$ are satisfied for any $\alpha \in [0, 1]$ [42], [60], [61]. From the results in [61], the defined order $\preceq$ on the set $\mathcal{F}(\mathbf{R})$ is a partial order, i.e., it is reflexive, antisymmetric, and transitive.

We can relate the antecedent conditions in Theorems 3 and 5 to the ordering of T1 FSs as follows: 1) If the Trapezoidal T1 FSs $A^1, A^2, \ldots, A^N$ in Theorem 3 have the same height, then $A^1 \preceq A^2 \preceq \ldots \preceq A^N$ (see Fig. 3(b) and (c) for examples); and 2) the generalized Gaussian T1 FSs $A^1, A^2, \ldots, A^N$ in Theorem 5 satisfy that $A^1 \preceq A^2 \preceq \ldots \preceq A^N$ (see Fig. 7 for examples). The proof of this conclusion is straightforward and is omitted here.

The ordering of IT2 FSs can be obtained by extending the ordering of T1 FSs. For two IT2 FSs $\widetilde{A}$ and $\tilde{B}$ in $\tilde{\mathcal{F}}(\mathbf{R})$ which is the set of all the convex IT2 FSs, a fuzzy ordering $\widetilde{A} \preceq \widetilde{B}$ exists if $\underline{\widetilde{A}} \preceq \underline{\widetilde{B}}$ and $\overline{\widetilde{A}} \preceq \overline{\widetilde{B}}$, where $\underline{\widetilde{A}}$ and $\underline{\widetilde{B}}$ denote the lower MFs of $\widetilde{A}$ and $\tilde{B}$, $\overline{\widetilde{A}}$ and $\overline{\widetilde{B}}$ denote the upper MFs of $\widetilde{A}$ and $\tilde{B}$. It is easy to prove that the defined order $\preceq$ on $\tilde{\mathcal{F}}(\mathbf{R})$ is still a partial order.

The relationship between the antecedent monotonicity conditions in Theorems 4 and 6 and the ordering of IT2 FSs can be derived as follows: 1) if the lower MFs of the Trapezoidal IT2 FSs $\widetilde{A}^1, \widetilde{A}^2, \ldots, \widetilde{A}^N$ in Theorem 4 have the same height, then $\widetilde{A}^1 \preceq \widetilde{A}^2 \preceq \ldots \preceq \widetilde{A}^N$ (see Fig. 4(b) and (c) for examples); and 2) the generalized Gaussian IT2 FSs $\widetilde{A}^1, \widetilde{A}^2, \ldots, \widetilde{A}^N$ in Theorem 6 satisfy that $\widetilde{A}^1 \preceq \widetilde{A}^2 \preceq \ldots \preceq \widetilde{A}^N$ (see Fig. 8 for examples). Again, the proof of this conclusion is omitted.

Besides the predefined ordering of FSs, there are other definitions, e.g., the linear orders in [15]. It is valuable to further explore whether the derived antecedent monotonicity conditions follow the other fuzzy orders. If so, it is necessary to define the lattice structure and study its relationship with other orders for IT2 FSs. This is not the focus of this study, but it will be one of our theoretical research directions.

C. Application to Modeling Problems: Data-Driven Method

As stated in Section I, monotonicity is meaningful for many modeling problems. This section will show how to incorporate monotonicity into IT2 FLSs for modeling applications.

For modeling, we usually use input–output data to construct IT2 FLSs to reflect the input–output mappings of real systems. Suppose that M input–output data points (x^1, y^1), $(\mathrm{x}^2, y^2), \ldots, (\mathrm{x}^M, y^M)$ are given. To obtain satisfactory IT2 FLS, we always minimize the following function:

$$E = \frac{1}{2}\sum_{i=1}^{M} |y_o(\mathrm{x}^i, \Theta) - y^i|^2 \tag{23}$$

TABLE VI
CONTROL RULE BASE FOR THE LIQUID LEVEL SYSTEM

$e \setminus \dot{e}$	$N_{\dot{e}}$	$ZR_{\dot{e}}$	$P_{\dot{e}}$
N_e	u_{11}	u_{12}	u_{13}
ZR_e	u_{21}	u_{22}	u_{23}
P_e	u_{31}	u_{32}	u_{33}

where $y_o(\mathrm{x}^i, \Theta)$ is the output of the IT2 FLS, and Θ is the parameter vector of all the antecedent and consequent parameters of the IT2 FLS.

When the monotonicity is required, the parameters of the IT2 FLS should be constrained. From previous theorems, the constraints on the parameters of the antecedent IT2 FSs and the consequent interval weights can be rewritten as the following linear-inequality:

$$P\Theta \geq 0. \tag{24}$$

Therefore, constructing monotonic IT2 FLSs for modeling problems can be realized by solving the following constrained optimization problem:

$$\begin{cases} \min_\Theta \frac{1}{2}\sum_{i=1}^{M} |y_o(\mathrm{x}^i, \Theta) - y^i|^2 \\ \text{subject to} \quad P\Theta \geq 0. \end{cases} \tag{25}$$

Based on the previous discussion, we provide the following guidelines for the design of monotonic IT2 FLS for modeling problems.

1) Partition the input domains using IT2 FSs. This can be done intuitively or by clustering algorithms under the conditions in Theorems 3–6.
2) Set initial consequent weights under the second condition in Theorems 1, 2, 8, and 9.
3) Optimize all or part of the parameters of the IT2 FLS under the first and second conditions in previous theorems. This can be done by classical constrained nonlinear optimization algorithms, e.g., the penalty function method, or through evolutionary computation algorithms, such as genetic algorithms, particle swarm algorithms, etc.

D. Application to Control Problems

As mentioned in Section I, in many control problems, the control signal should be monotonic with respect to the error and/or the change of error. This section will show how the proposed monotonicity results can be applied to control problems.

Take the control of a liquid level in a tank for example. Assume that the required liquid level is y_d and that the real liquid level is $y(t)$. The error is defined as $e = y_d - y(t)$, and the change of error is denoted as $\dot{e}$. We use the IT2 FLS whose inputs are e and $\dot{e}$ to generate the control signal u. The control rule base for this application is shown in Table VI.[4]

From our experience, the larger e and $\dot{e}$ are, the greater the control signal should be. Hence, the output u of the IT2 FLS should be monotonically increasing w.r.t. e and $\dot{e}$. To ensure that u is monotonically increasing w.r.t. e, we can set N_e, ZR_e, and P_e to meet the first condition in the previous theorems and

[4]In this table, N means Negative, ZR means Zero, and P means positive.

$u_{1j} \leq u_{2j} \leq u_{3j}$ $(j = 1, 2, 3)$. In a similar way, to ensure that u is monotonically increasing w.r.t. $\dot{e}$, we can set $N_{\dot{e}}$, $ZR_{\dot{e}}$, and $P_{\dot{e}}$ to meet the first condition in the previous theorems and $u_{i1} \leq u_{i2} \leq u_{i3}$ $(i = 1, 2, 3)$. To design sound fuzzy rule base for the fuzzy controller, the FSs and weights in the rule table need to comply with such constraints.

E. Other Fundamental Properties of IT2 Fuzzy Logic System

Besides the monotonicity, there are several other fundamental properties of IT2 FLSs investigated by researchers recently, such as the continuity, smoothness, adaptiveness, novelty, stability, and robustness [43], [52], [57], [58], [62]–[68].

In [57], Wu and Mendel have studied the continuity of the input–output mappings of FLSs, including T1 FLSs and IT2 FLSs. They showed that, to obtain a continuous input–output mapping, the lower MFs should cover every input domain. Another concept related to continuity is smoothness which requires that both the IT2 FLS and its derivative are continuous. Therefore, the issue of smoothness is more difficult than the study on continuity. So far, there are no results to guarantee the smoothness of IT2 FLSs. From Figs. 10–13, we can observe that the Gaussian FLSs are more smoother than the Trapezoidal FLSs. However, this fact still needs to be proved theoretically.

As studied in [58] by Wu, adaptiveness and novelty are two fundamental differences between IT2 FLSs and T1 FLSs. Wu has also demonstrated that only the IT2 FLSs using the KM method have the properties of adaptiveness and novelty.

Stability and robustness are another two important theoretical issues for IT2 FLSs, especially for IT2 fuzzy controllers. There have been many studies on the stability of IT2 FLSs [52], [62]–[66], whereas only a few researchers have investigated the robustness of IT2 FLSs [43], [67], [68]. More works need to be done on the robustness of IT2 FLSs.

These fundamental studies can not only deepen our theoretical understanding of IT2 FLSs but guide us to choose and design appropriate IT2 FLSs for modeling and control applications as well.

VI. Conclusion

In real-world applications, specific physical structure knowledge about systems may be difficult to obtain, but some qualitative knowledge (e.g., monotonicity) may be obvious for engineers. Hence, it is quite important for engineers to incorporate such qualitative knowledge into system design. We have, in this paper, addressed incorporating the monotonicity property into IT2 FLSs, and we have presented sufficient monotonicity conditions on the parameters of IT2 FLSs—the conditions on the antecedent IT2 FSs and the consequent weights—to ensure the monotonicity between their inputs and outputs. We have also provided the guidelines for applying the derived conditions to modeling and control problems. In our future work, we will study how to apply the proposed theoretical results to real-world modeling and control applications.

Appendix A

Proof of Lemma 1

We only prove that $y_l^n(\mathrm{x}^1) \leq y_l^n(\mathrm{x}^2)$, where $\mathrm{x}^1 = (x_1, \ldots, x_k^1, \ldots, x_p) \leq \mathrm{x}^2 = (x_1, \ldots, x_k^2, \ldots, x_p)$.

Note that

$$\begin{aligned} y_l^n(\mathrm{x}) &= \frac{\sum_{i_1=1}^{N_1} \cdots \sum_{i_p=1}^{N_p} f^{i_1\cdots i_p}(\mathrm{x}) \underline{w}^{i_1\cdots i_p}}{\sum_{i_1=1}^{N_1} \cdots \sum_{i_p=1}^{N_p} f^{i_1\cdots i_p}(\mathrm{x})} \\ &= \frac{\sum_{i_1=1}^{N_1} \cdots \sum_{i_p=1}^{N_p} \underline{w}^{i_1\cdots i_p} \prod_{j=1}^{p} \mu_{A_j^{i_j}}(x_j)}{\sum_{i_1=1}^{N_1} \cdots \sum_{i_p=1}^{N_p} \prod_{j=1}^{p} \mu_{A_j^{i_j}}(x_j)} \end{aligned} \tag{26}$$

where $\prod_{j=1}^{p} \mu_{A_j^{i_j}}(x_j)$ means either $\prod_{j=1}^{p} \underline{\mu}_{\tilde{A}_j^{i_j}}(x_j)$ or $\prod_{j=1}^{p} \overline{\mu}_{\tilde{A}_j^{i_j}}(x_j)$.

The expression of $y_l^n(\mathrm{x})$ can be rewritten, as (27), shown at the bottom of the page where $M = \prod_{j=1, j\neq k}^{p} N_k$, $v = v(i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_p)$ is a combination of $i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_p$, and $\alpha^v = \alpha^{v(i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_p)} = \prod_{j=1, j\neq k}^{p} \mu_{A_j^{i_j}}(x_j)$.

Then, we have

$$\begin{aligned} &y_l^n(\mathrm{x}^2) - y_l^n(\mathrm{x}^1) \\ &= \frac{\sum_{s=1}^{M} \alpha^s \sum_{l=1}^{N_k} \underline{w}^{sl} \mu_{A_k^l}(x_k^2)}{\sum_{s=1}^{M} \alpha^s \sum_{l=1}^{N_k} \mu_{A_k^l}(x_k^2)} - \frac{\sum_{t=1}^{M} \alpha^t \sum_{m=1}^{N_k} \underline{w}^{tm} \mu_{A_k^m}(x_k^1)}{\sum_{t=1}^{M} \alpha^t \sum_{m=1}^{N_k} \mu_{A_k^m}(x_k^1)} \\ &= \frac{1}{\varepsilon}\Bigg[\sum_{s=1}^{M}\sum_{t=1}^{M} \alpha^s\alpha^t \sum_{l=1}^{N_k}\sum_{m=1}^{N_k} \underline{w}^{sl} \mu_{A_k^l}(x_k^2)\mu_{A_k^m}(x_k^1) \\ &\quad - \sum_{s=1}^{M}\sum_{t=1}^{M} \alpha^s\alpha^t \sum_{l=1}^{N_k}\sum_{m=1}^{N_k} \underline{w}^{tm} \mu_{A_k^l}(x_k^2)\mu_{A_k^m}(x_k^1)\Bigg] \end{aligned} \tag{28}$$

$$\begin{aligned} y_l^n(\mathrm{x}) &= \frac{\sum_{i_1=1}^{N_1} \cdots \sum_{i_{k-1}=1}^{N_{k-1}} \sum_{i_{k+1}=1}^{N_{k+1}} \cdots \sum_{i_p=1}^{N_p} \left[\prod_{j=1, j\neq k}^{p} \mu_{A_j^{i_j}}(x_j) \sum_{i_k=1}^{N_k} \underline{w}^{i_1\cdots i_p} \mu_{A_k^{i_k}}(x_k)\right]}{\sum_{i_1=1}^{N_1} \cdots \sum_{i_{k-1}=1}^{N_{k-1}} \sum_{i_{k+1}=1}^{N_{k+1}} \cdots \sum_{i_p=1}^{N_p} \left[\prod_{j=1, j\neq k}^{p} \mu_{A_j^{i_j}}(x_j) \sum_{i_k=1}^{N_k} \mu_{A_k^{i_k}}(x_k)\right]} \\ &= \frac{\sum_{v=1}^{M} \alpha^v \sum_{i_k=1}^{N_k} \underline{w}^{v i_k} \mu_{A_k^{i_k}}(x_k)}{\sum_{v=1}^{M} \alpha^v \sum_{i_k=1}^{N_k} \mu_{A_k^{i_k}}(x_k)} \end{aligned} \tag{27}$$

where $\varepsilon = \left[\sum_{s=1}^{M} \alpha^s \sum_{l=1}^{N_k} \mu_{A_k^l}(x_k^2)\right]\left[\sum_{t=1}^{M} \alpha^t \sum_{m=1}^{N_k} \mu_{A_k^m}(x_k^1)\right] > 0$.

Consider the following fact:

$$\sum_{s=1}^{M}\sum_{t=1}^{M}\alpha^s\alpha^t\sum_{l=1}^{N_k}\sum_{m=1}^{N_k}\underline{w}^{tm}\mu_{A_k^l}(x_k^2)\mu_{A_k^m}(x_k^1) = \sum_{s=1}^{M}\sum_{t=1}^{M}\alpha^s\alpha^t\sum_{l=1}^{N_k}\sum_{m=1}^{N_k}\underline{w}^{sm}\mu_{A_k^l}(x_k^2)\mu_{A_k^m}(x_k^1). \quad (29)$$

Substituting (29) into (28) leads to

$$\begin{aligned} & y_l^n(\mathrm{x}^2) - y_l^n(\mathrm{x}^1) \\ &= \frac{1}{\varepsilon}\left[\sum_{s=1}^{M}\sum_{t=1}^{M}\alpha^s\alpha^t\sum_{l=1}^{N_k}\sum_{m=1}^{N_k}(\underline{w}^{sl}-\underline{w}^{sm})\mu_{A_k^l}(x_k^2)\mu_{A_k^m}(x_k^1)\right] \\ &= \frac{1}{\varepsilon}\left[\sum_{s=1}^{M}\sum_{t=1}^{M}\alpha^s\alpha^t\sum_{l=1}^{N_k}\sum_{m=l+1}^{N_k}(\underline{w}^{sm}-\underline{w}^{sl})\Big(\mu_{A_k^m}(x_k^2)\mu_{A_k^l}(x_k^1) - \mu_{A_k^l}(x_k^2)\mu_{A_k^m}(x_k^1)\Big)\right]. \end{aligned} \quad (30)$$

From the first condition, we have

$$\mu_{A_k^m}(x_k^2)\mu_{A_k^l}(x_k^1) - \mu_{A_k^l}(x_k^2)\mu_{A_k^m}(x_k^1) \geq 0. \quad (31)$$

From the second condition, for $m > l$

$$\underline{w}^{sm} - \underline{w}^{sl} \geq 0. \quad (32)$$

From (30), (31), and (32), we have $y_l^n(\mathrm{x}^2) \geq y_l^n(\mathrm{x}^1)$.

In the similar way, we can also obtain that $y_r^n(\mathrm{x}^2) \geq y_r^n(\mathrm{x}^1)$. Therefore, this lemma holds.

APPENDIX B
PROOF OF THEOREM 1

To begin, let us consider the following fact: If $a_1 \leq b_1, a_2 \leq b_2, \ldots, a_n \leq b_n$, then $\min\{a_1, a_2, \ldots, a_n\} \leq \min\{b_1, b_2, \ldots, b_n\}$, and $\max\{a_1, a_2, \ldots, a_n\} \leq \max\{b_1, b_2, \ldots, b_n\}$.

From Lemma 1, $\forall \mathrm{x}^1 = (x_1, \ldots, x_k^1, \ldots, x_p) \leq \mathrm{x}^2 = (x_1, \ldots, x_k^2, \ldots, x_p)$, we have $y_l^n(\mathrm{x}^1) \leq y_l^n(\mathrm{x}^2)$ and $y_r^n(\mathrm{x}^1) \leq y_r^n(\mathrm{x}^2)$ $(n = 1, \ldots, K)$. Then, from the aforementioned fact, we can derive that

$$y_l(\mathrm{x}^1) = \min_{n=1}^{K} y_l^n(\mathrm{x}^1) \leq \min_{n=1}^{K} y_l^n(\mathrm{x}^2) = y_l(\mathrm{x}^2) \quad (33)$$

$$y_r(\mathrm{x}^1) = \max_{n=1}^{K} y_r^n(\mathrm{x}^1) \leq \max_{n=1}^{K} y_r^n(\mathrm{x}^2) = y_r(\mathrm{x}^2). \quad (34)$$

From (10), (33), and (34), we have

$$\begin{aligned} y_o(\mathrm{x}^1) &= \frac{1}{2}[y_l(\mathrm{x}^1) + y_r(\mathrm{x}^1)] \\ &\leq \frac{1}{2}[y_l(\mathrm{x}^2) + y_r(\mathrm{x})^2] = y_o(\mathrm{x}^2) \end{aligned} \quad (35)$$

which means this theorem holds.

APPENDIX C
PROOF OF LEMMA 2

For T1 FS A^l, let $S_1^l = \{x|x \leq a^l\}$, $S_2^l = \{x|a^l < x \leq b^l\}$, $S_3^l = \{x|b^l < x \leq c^l\}$, $S_4^l = \{x|c^l < x < d^l\}$, $S_5^l = \{x|x \geq d^l\}$.

For T1 FS A^r, let $S_1^r = \{x|x \leq a^r\}$, $S_2^r = \{x|a^r < x \leq b^r\}$, $S_3^r = \{x|b^r < x \leq c^r\}$, $S_4^r = \{x|c^r < x < d^r\}$, $S_5^r = \{x|x \geq d^r\}$.

Denote $R_1 = S_1^l \cup S_1^r$, $R_2 = S_5^l \cup S_5^r$, $R_3 = S_2^l \cap S_2^r$, $R_4 = S_3^l \cap S_2^r$, $R_5 = S_3^l \cap S_3^r$, $R_6 = S_4^l \cap S_2^r$, $R_7 = S_4^l \cap S_3^r$, and $R_8 = S_4^l \cap S_4^r$. Note that some R_i may be empty set.

As shown in Fig. 2, the input domain $X = R_1 \cup R_2 \cup R_3 \cup R_4 \cup R_5 \cup R_6 \cup R_7 \cup R_8$.

1) If x^1 or $x^2 \in R_1$, then $\mu_{A^r}(x^1) = 0$; as a result, $\mu_{A^r}(x^2)\mu_{A^l}(x^1) \geq \mu_{A^r}(x^1)\mu_{A^l}(x^2)$.

2) If x^1 or $x^2 \in R_2$, then $\mu_{A^l}(x^2) = 0$; as a result, $\mu_{A^r}(x^2)\mu_{A^l}(x^1) \geq \mu_{A^r}(x^1)\mu_{A^l}(x^2)$.

If x^1 and x^2 do not lie in $R_1 \cup R_2$, i.e. $x^1, x^2 \in [a^r, d^l] = R_3 \cup R_4 \cup R_5 \cup R_6 \cup R_7 \cup R_8$, then $\mu_{A^l}(x^2) > 0$ and $\mu_{A^l}(x^1) > 0$. Therefore, in order to prove that $\mu_{A^r}(x^2)\mu_{A^l}(x^1) \geq \mu_{A^r}(x^1)\mu_{A^l}(x^2)$, we just need to prove the following equation:

$$\frac{\mu_{A^r}(x^2)}{\mu_{A^l}(x^2)} - \frac{\mu_{A^r}(x^1)}{\mu_{A^l}(x^1)} \geq 0, \quad \forall x^2 \geq x^1 \quad (36)$$

which implies that $\frac{\mu_{A^r}(x)}{\mu_{A^l}(x)}$ is monotonically increasing w.r.t. x in $[a^r, d^l]$.

Because $\mu_{A^r}(x)$ and $\mu_{A^l}(x)$ are continuous, we only need to prove that $\frac{\mu_{A^r}(x)}{\mu_{A^l}(x)}$ is monotonically increasing w.r.t. x in R_3, R_4, R_5, R_6, R_7, and R_8, respectively. In any one of these six regions, $\mu_{A^l}(x)$ and $\mu_{A^r}(x)$ are differentiable. Therefore, in each region

$$\begin{aligned} &\frac{\mu_{A^r}(x)}{\mu_{A^l}(x)} \text{ is monotonically increasing} \\ &\Leftrightarrow \frac{d}{dx}\left(\frac{\mu_{A^r}(x)}{\mu_{A^l}(x)}\right) \geq 0 \\ &\Leftrightarrow \frac{d\mu_{A^r}(x)}{dx}\mu_{A^l}(x) - \frac{d\mu_{A^l}(x)}{dx}\mu_{A^r}(x) \geq 0. \end{aligned}$$

From the previous discussion, when $x \in [a^r, d^l]$, to prove that $\mu_{A^r}(x^2)\mu_{A^l}(x^1) \geq \mu_{A^r}(x^1)\mu_{A^l}(x^2)$, we just need to derive that $\frac{d\mu_{A^r}(x)}{dx}\mu_{A^l}(x) - \frac{d\mu_{A^l}(x)}{dx}\mu_{A^r}(x) \geq 0$ in each region R_i $(i = 3, 4, 5, 6, 7, 8)$.

3) In the region $R_3 = S_2^l \cap S_2^r$, we have

$$\begin{aligned} &\frac{d\mu_{A^r}(x)}{dx}\mu_{A^l}(x) - \frac{d\mu_{A^l}(x)}{dx}\mu_{A^r}(x) \\ &= \frac{h^r}{b^r - a^r}\frac{h^l(x - a^l)}{b^l - a^l} - \frac{h^l}{b^l - a^l}\frac{h^r(x - a^r)}{b^r - a^r} \\ &= \frac{h^r h^l[a^r - a^l]}{(b^r - a^r)(b^l - a^l)}. \end{aligned} \quad (37)$$

Since $a^l \leq a^r$, $a^r < b^r$ and $a^l < b^l$, we can conclude that $\frac{d\mu_{A^r}(x)}{dx}\mu_{A^l}(x) - \frac{d\mu_{A^l}(x)}{dx}\mu_{A^r}(x) \geq 0$.

4) In the regions R_4, R_5, R_6, R_7, we have

In each region, $\frac{d\mu_{A^r}(x)}{dx} \geq 0$ and $\frac{d\mu_{A^l}(x)}{dx} \leq 0$. Consequently, $\frac{d\mu_{A^r}(x)}{dx}\mu_{A^l}(x) - \frac{d\mu_{A^l}(x)}{dx}\mu_{A^r}(x) \geq 0$.

5) In the region $R_8 = S_4^l \cap S_4^r$, we have

$$\begin{aligned} &\frac{d\mu_{A^r}(x)}{dx}\mu_{A^l}(x) - \frac{d\mu_{A^l}(x)}{dx}\mu_{A^r}(x) \\ &= \frac{h^r}{c^r - d^r}\frac{h^l(x-d^l)}{c^l - d^l} - \frac{h^l}{c^l - d^l}\frac{h^r(x-d^r)}{c^r - d^r} \\ &= \frac{h^r h^l[d^r - d^l]}{(c^r - d^r)(c^l - d^l)}. \end{aligned} \tag{38}$$

Since $d^l \leq d^r$, $c^r < d^r$ and $c^l < d^l$, we can conclude that $\frac{d\mu_{A^r}(x)}{dx}\mu_{A^l}(x) - \frac{d\mu_{A^l}(x)}{dx}\mu_{A^r}(x) \geq 0$.

From the previous discussions, this lemma holds.

Appendix D
Proof of Theorem 4

To begin, consider the following fact existing in the definition: For the Trapezoidal IT2 FS $\mu_{\widetilde{A}}(x) = \mu_{\widetilde{A}}(x, \overline{a}, \overline{b}, \overline{c}, \overline{d}, \underline{a}, \underline{b}, \underline{c}, \underline{d}, h)$, its parameters satisfy that $\overline{a} \leq \underline{a}$, $\overline{b} \leq \underline{b}$, $\underline{c} \leq \overline{c}$, $\underline{d} \leq \overline{d}$.

To prove this theorem, we just need to consider the following four cases.

Case 1: $\mu_{A^m} = \underline{\mu}_{\widetilde{A}^m}$ and $\mu_{A^l} = \underline{\mu}_{\widetilde{A}^l}$.

From the parameter constraints and the condition that $\underline{a}^l \leq \overline{a}^m$, $\underline{b}^l \leq \overline{b}^m$, $\overline{c}^l \leq \underline{c}^m$, $\overline{d}^l \leq \underline{d}^m$, we obtain $\underline{a}^l \leq \underline{a}^m$, $\underline{b}^l \leq \underline{b}^m$, $\underline{c}^l \leq \underline{c}^m$, $\underline{d}^l \leq \underline{d}^m$. Then, from Lemma 2, we have $\underline{\mu}_{A^m}(x^2)\underline{\mu}_{A^l}(x^1) \geq \underline{\mu}_{A^m}(x^1)\underline{\mu}_{A^l}(x^2)$ for any $x^2 \geq x^1 \in X$.

Case 2: $\mu_{A^m} = \underline{\mu}_{\widetilde{A}^m}$ and $\mu_{A^l} = \overline{\mu}_{\widetilde{A}^l}$.

From the parameter constraint and the condition that $\underline{a}^l \leq \overline{a}^m$, $\underline{b}^l \leq \overline{b}^m$, $\overline{c}^l \leq \underline{c}^m$, $\overline{d}^l \leq \underline{d}^m$, we obtain $\overline{a}^l \leq \underline{a}^m$, $\overline{b}^l \leq \underline{b}^m$, $\overline{c}^l \leq \underline{c}^m$, $\overline{d}^l \leq \underline{d}^m$. Then, from Lemma 2, we have $\underline{\mu}_{A^m}(x^2)\overline{\mu}_{A^l}(x^1) \geq \underline{\mu}_{A^m}(x^1)\overline{\mu}_{A^l}(x^2)$ for any $x^2 \geq x^1 \in X$.

Case 3: $\mu_{A^m} = \overline{\mu}_{\widetilde{A}^m}$ and $\mu_{A^l} = \underline{\mu}_{\widetilde{A}^l}$.

From the parameter constraint and the condition that $\underline{a}^l \leq \overline{a}^m$, $\underline{b}^l \leq \overline{b}^m$, $\overline{c}^l \leq \underline{c}^m$, $\overline{d}^l \leq \underline{d}^m$, we obtain $\underline{a}^l \leq \overline{a}^m$, $\underline{b}^l \leq \overline{b}^m$, $\underline{c}^l \leq \overline{c}^m$, $\underline{d}^l \leq \overline{d}^m$. Then, from Lemma 2, we have $\overline{\mu}_{A^m}(x^2)\underline{\mu}_{A^l}(x^1) \geq \overline{\mu}_{A^m}(x^1)\underline{\mu}_{A^l}(x^2)$ for any $x^2 \geq x^1 \in X$.

Case 4: $\mu_{A^m} = \overline{\mu}_{\widetilde{A}^m}$ and $\mu_{A^l} = \overline{\mu}_{\widetilde{A}^l}$

From the parameter constraint and the condition that $\underline{a}^l \leq \overline{a}^m$, $\underline{b}^l \leq \overline{b}^m$, $\overline{c}^l \leq \underline{c}^m$, $\overline{d}^l \leq \underline{d}^m$, we obtain $\overline{a}^l \leq \overline{a}^m$, $\overline{b}^l \leq \overline{b}^m$, $\overline{c}^l \leq \overline{c}^m$, $\overline{d}^l \leq \overline{d}^m$. Then, from Lemma 2, we have $\overline{\mu}_{A^m}(x^2)\overline{\mu}_{A^l}(x^1) \geq \overline{\mu}_{A^m}(x^1)\overline{\mu}_{A^l}(x^2)$ for any $x^2 \geq x^1 \in X$.

Appendix E
Proof of Lemma 3

For T1 FS A^l, let $S_1^l = \{x | x \leq a^l\}$, $S_2^l = \{x | a^l < x \leq b^l\}$, $S_3^l = \{x | x > b^l\}$.

For T1 FS A^r, let $S_1^r = \{x | x \leq a^r\}$, $S_2^r = \{x | a^r < x \leq b^r\}$, and $S_3^r = \{x | x > b^r\}$.

Denote $R_1 = S_1^l \cap S_1^r$, $R_2 = S_2^l \cap S_1^r$, $R_3 = S_2^l \cap S_2^r$, $R_4 = S_3^l \cap S_1^r$, $R_5 = S_3^l \cap S_2^r$, and $R_6 = S_3^l \cap S_3^r$. For some cases, some R_is may also be empty sets.

As shown in Fig. 6, the input domain $X = R_1 \cup R_2 \cup R_3 \cup R_4 \cup R_5 \cup R_6$. In any one of these six regions, $\mu_{A^l}(x)$ and $\mu_{A^r}(x)$ are continuous and differentiable. Consequently, from the discussion in the proof of Lemma 2 in Appendix C, to prove that $\mu_{A^r}(x^2)\mu_{A^l}(x^1) \geq \mu_{A^r}(x^1)\mu_{A^l}(x^2)$ for any $x^2 \geq x^1$ is equivalent to showing that $\frac{d\mu_{A^r}(x)}{dx}\mu_{A^l}(x) - \frac{d\mu_{A^l}(x)}{dx}\mu_{A^r}(x) \geq 0$ in each region.

1) $x \in R_1 = S_1^l \cap S_1^r$.

$$\begin{aligned} &\frac{d\mu_{A^r}(x)}{dx}\mu_{A^l}(x) - \frac{d\mu_{A^l}(x)}{dx}\mu_{A^r}(x) \\ &= \frac{-h^r(x-a^r)\mu_{A^r}(x)}{(\sigma^r)^2}\mu_{A^l}(x) - \frac{-h^l(x-a^l)\mu_{A^l}(x)}{(\sigma^l)^2}\mu_{A^r}(x) \\ &= \frac{\mu_{A^r}(x)\mu_{A^l}(x)[(\sigma^r)^2 h^l - (\sigma^l)^2 h^r]x}{(\sigma^r)^2 * (\sigma^l)^2} \\ &+ \frac{\mu_{A^r}(x)\mu_{A^l}(x)[(\sigma^l)^2 h^r a^r - (\sigma^r)^2 h^l a^l]}{(\sigma^r)^2 * (\sigma^l)^2}. \end{aligned} \tag{39}$$

Since $a^l \leq a^r$ and $(\sigma^r)^2 h^l = (\sigma^l)^2 h^r$, we can conclude that $\frac{d\mu_{A^r}(x)}{dx}\mu_{A^l}(x) - \frac{d\mu_{A^l}(x)}{dx}\mu_{A^r}(x) \geq 0$.

2) In the regions R_2, R_3, R_4, and R_5, we have

In each region, $\frac{d\mu_{A^r}(x)}{dx} \geq 0$ and $\frac{d\mu_{A^l}(x)}{dx} \leq 0$. Consequently, $\frac{d\mu_{A^r}(x)}{dx}\mu_{A^l}(x) - \frac{d\mu_{A^l}(x)}{dx}\mu_{A^r}(x) \geq 0$.

3) $x \in R_6 = S_3^l \cap S_3^r$.

$$\begin{aligned} &\frac{d\mu_{A^r}(x)}{dx}\mu_{A^l}(x) - \frac{d\mu_{A^l}(x)}{dx}\mu_{A^r}(x) \\ &= \frac{-h^r(x-b^r)\mu_{A^r}(x)}{(\sigma^r)^2}\mu_{A^l}(x) - \frac{-h^l(x-b^l)\mu_{A^l}(x)}{(\sigma^l)^2}\mu_{A^r}(x) \\ &= \frac{\mu_{A^r}(x)\mu_{A^l}(x)[(\sigma^r)^2 h^l - (\sigma^l)^2 h^r]x}{(\sigma^r)^2 * (\sigma^l)^2} \\ &+ \frac{\mu_{A^r}(x)\mu_{A^l}(x)[(\sigma^l)^2 h^r b^r - (\sigma^r)^2 h^l b^l]}{(\sigma^r)^2 * (\sigma^l)^2}. \end{aligned} \tag{40}$$

Since $b^l \leq b^r$ and $(\sigma^r)^2 h^l = (\sigma^l)^2 h^r$, we can conclude that $\frac{d\mu_{A^r}(x)}{dx}\mu_{A^l}(x) - \frac{d\mu_{A^l}(x)}{dx}\mu_{A^r}(x) \geq 0$.

From the previous discussions, this lemma holds.

Appendix F
Proof of Theorem 6

To begin, consider the following fact in the definition: for the generalized Gaussian IT2 FS $\mu_{\widetilde{A}}(x) = \mu_{\widetilde{A}}(x, \overline{a}, \overline{b}, \underline{a}, \underline{b}, \overline{\sigma}, \underline{\sigma}, h)$, its parameters satisfy that $\overline{a} \leq \underline{a}$, $\underline{b} \leq \overline{b}$, $\underline{\sigma} \leq \overline{\sigma}$.

Again, to prove this theorem, we just need to prove the following four cases.

Case 1: $\mu_{A^m} = \underline{\mu}_{\widetilde{A}^m}$ and $\mu_{A^l} = \underline{\mu}_{\widetilde{A}^l}$

From the parameter constraints and the condition that $\underline{a}^l \leq \overline{a}^m$, $\overline{b}^l \leq \underline{b}^m$, $\underline{\sigma}^l = \underline{\sigma}^m = \underline{\sigma}$, $h^l = h^m$, we obtain $\underline{a}^l \leq \underline{a}^m$, $\underline{b}^l \leq \underline{b}^m$, $(\underline{\sigma}^m)^2 h^l = (\underline{\sigma}^l)^2 h^m$. Then, from Lemma 3, we have $\underline{\mu}_{A^m}(x^2)\underline{\mu}_{A^l}(x^1) \geq \underline{\mu}_{A^m}(x^1)\underline{\mu}_{A^l}(x^2)$ for any $x^2 \geq x^1 \in X$.

Case 2: $\mu_{A^m} = \underline{\mu}_{\widetilde{A}^m}$ and $\mu_{A^l} = \overline{\mu}_{\widetilde{A}^l}$.

From the condition that $\underline{a}^l \leq \overline{a}^m$, $\overline{b}^l \leq \underline{b}^m$, $\overline{\sigma}^l = \overline{\sigma}^m = \overline{\sigma}$, $\underline{\sigma}^l = \underline{\sigma}^m = \underline{\sigma}$, $h^m = \frac{\underline{\sigma}^2}{\overline{\sigma}^2}$ and the fact that the height of $\overline{\mu}_{\widetilde{A}^l}$

is 1, we obtain $\overline{a}^l \leq \underline{a}^m$, $\overline{b}^l \leq \underline{b}^m$, $(\underline{\sigma}^m)^2 * 1 = (\overline{\sigma}^l)^2 h^m$. Then, from Lemma 3, we have $\underline{\mu}_{A^m}(x^2)\overline{\mu}_{A^l}(x^1) \geq \underline{\mu}_{A^m}(x^1)\overline{\mu}_{A^l}(x^2)$ for any $x^2 \geq x^1 \in X$.

Case 3: $\mu_{A^m} = \overline{\mu}_{\widetilde{A}^m}$ and $\mu_{A^l} = \underline{\mu}_{\widetilde{A}^l}$.

From the condition that $\underline{a}^l \leq \overline{a}^m$, $\overline{b}^l \leq \underline{b}^m$, $\overline{\sigma}^l = \overline{\sigma}^m = \overline{\sigma}$, $\underline{\sigma}^l = \underline{\sigma}^m = \underline{\sigma}$, $h^l = \frac{\underline{\sigma}^2}{\overline{\sigma}^2}$ and the fact that the height of $\overline{\mu}_{\widetilde{A}^m}$ is 1, we obtain $\underline{a}^l \leq \overline{a}^m$, $\underline{b}^l \leq \overline{b}^m$, $(\overline{\sigma}^m)^2 h^l = (\underline{\sigma}^l)^2 * 1$. Then, from Lemma 3, we have $\overline{\mu}_{A^m}(x^2)\underline{\mu}_{A^l}(x^1) \geq \overline{\mu}_{A^m}(x^1)\underline{\mu}_{A^l}(x^2)$ for any $x^2 \geq x^1 \in X$.

Case 4: $\mu_{A^m} = \overline{\mu}_{\widetilde{A}^m}$ and $\mu_{A^l} = \overline{\mu}_{\widetilde{A}^l}$.

From the condition that $\underline{a}^l \leq \overline{a}^m$, $\overline{b}^l \leq \underline{b}^m$, $\overline{\sigma}^l = \overline{\sigma}^m$ and the fact that the heights of $\overline{\mu}_{\widetilde{A}^m}$ and $\overline{\mu}_{\widetilde{A}^l}$ are 1, we obtain $\overline{a}^l \leq \overline{a}^m$, $\overline{b}^l \leq \overline{b}^m$, $(\overline{\sigma}^m)^2 * 1 = (\overline{\sigma}^l)^2 * 1$. Then, from Lemma 3, we have $\overline{\mu}_{A^m}(x^2)\overline{\mu}_{A^l}(x^1) \geq \overline{\mu}_{A^m}(x^1)\overline{\mu}_{A^l}(x^2)$ for any $x^2 \geq x^1 \in X$.

Appendix G
Proof of Theorem 7

As $\widetilde{A}^1, \widetilde{A}^2, \cdots, \widetilde{A}^N$ satisfy the first condition in Theorem 1, then, $\forall x^2 \geq x^1 \in X$, $m \geq l$, we have

$$\underline{\mu}_{\widetilde{A}^m}(x^2)\underline{\mu}_{\widetilde{A}^l}(x^1) \geq \underline{\mu}_{\widetilde{A}^m}(x^1)\underline{\mu}_{\widetilde{A}^l}(x^2) \tag{41}$$

$$\underline{\mu}_{\widetilde{A}^m}(x^2)\overline{\mu}_{\widetilde{A}^l}(x^1) \geq \underline{\mu}_{\widetilde{A}^m}(x^1)\overline{\mu}_{\widetilde{A}^l}(x^2) \tag{42}$$

$$\overline{\mu}_{\widetilde{A}^m}(x^2)\underline{\mu}_{\widetilde{A}^l}(x^1) \geq \overline{\mu}_{\widetilde{A}^m}(x^1)\underline{\mu}_{\widetilde{A}^l}(x^2) \tag{43}$$

$$\overline{\mu}_{\widetilde{A}^m}(x^2)\overline{\mu}_{\widetilde{A}^l}(x^1) \geq \overline{\mu}_{\widetilde{A}^m}(x^1)\overline{\mu}_{\widetilde{A}^l}(x^2). \tag{44}$$

Then

$$\begin{aligned}
&\mu_{A^m}(x^2)\mu_{A^l}(x^1) \\
&= [(1-\eta)\overline{\mu}_{\widetilde{A}^m}(x^2) + \eta\underline{\mu}_{\widetilde{A}^m}(x^2)] * [(1-\eta)\overline{\mu}_{\widetilde{A}^l}(x^1) + \eta\underline{\mu}_{\widetilde{A}^l}(x^1)] \\
&= (1-\eta)^2\overline{\mu}_{\widetilde{A}^m}(x^2)\overline{\mu}_{\widetilde{A}^l}(x^1) + \eta(1-\eta)\overline{\mu}_{\widetilde{A}^m}(x^2)\underline{\mu}_{\widetilde{A}^l}(x^1) \\
&\quad + \eta(1-\eta)\underline{\mu}_{\widetilde{A}^m}(x^2)\overline{\mu}_{\widetilde{A}^l}(x^1) + \eta^2\underline{\mu}_{\widetilde{A}^m}(x^2)\underline{\mu}_{\widetilde{A}^l}(x^1) \\
&\geq (1-\eta)^2\overline{\mu}_{\widetilde{A}^l}(x^2)\overline{\mu}_{\widetilde{A}^m}(x^1) + \eta(1-\eta)\overline{\mu}_{\widetilde{A}^l}(x^2)\underline{\mu}_{\widetilde{A}^m}(x^1) \\
&\quad + \eta(1-\eta)\underline{\mu}_{\widetilde{A}^l}(x^2)\overline{\mu}_{\widetilde{A}^m}(x^1) + \eta^2\underline{\mu}_{\widetilde{A}^l}(x^2)\underline{\mu}_{\widetilde{A}^m}(x^1) \\
&= [(1-\eta)\overline{\mu}_{\widetilde{A}^m}(x^1) + \eta\underline{\mu}_{\widetilde{A}^m}(x^1)] * [(1-\eta)\overline{\mu}_{\widetilde{A}^l}(x^2) + \eta\underline{\mu}_{\widetilde{A}^l}(x^2)] \\
&= \mu_{A^m}(x^1)\mu_{A^l}(x^2).
\end{aligned} \tag{45}$$

Therefore, the first condition in Theorem 2 can be met. Hence, this theorem holds.

Appendix H
Proof of Theorem 8

A. Proof of the Monotonicity of IT2 Fuzzy Logic System Using the Du-Ying Method

From Lemma 1, $\forall \mathrm{x}^1 = (x_1, \ldots, x_k^1, \ldots, x_p) \leq \mathrm{x}^2 = (x_1, \ldots, x_k^2, \ldots, x_p)$, we have $y^n(\mathrm{x}^1) \leq y^n(\mathrm{x}^2)$ $(n = 1, \cdots, K)$. Then, we can derive that

$$y_o(\mathrm{x}^1) = \frac{1}{K}\sum_{n=1}^{K} y^n(\mathrm{x}^1) \leq \frac{1}{K}\sum_{n=1}^{K} y^n(\mathrm{x}^2) = y_o(\mathrm{x}^2). \tag{46}$$

B. Proof of the Monotonicity of IT2 Fuzzy Logic System Using the Begian–Melek–Mendel Method

From Lemma 1, $\forall \mathrm{x}^1 = (x_1, \ldots, x_k^1, \ldots, x_p) \leq \mathrm{x}^2 = (x_1, \ldots, x_k^2, \ldots, x_p)$, we have $y^1(\mathrm{x}^1) \leq y^1(\mathrm{x}^2)$ and $y^K(\mathrm{x}^1) \leq y^K(\mathrm{x}^2)$. Then, we can derive that

$$\begin{aligned}
y_o(\mathrm{x}^1) &= \alpha y^1(\mathrm{x}^1) + \beta y^K(\mathrm{x}^1) \\
&\leq \alpha y^1(\mathrm{x}^2) + \beta y^K(\mathrm{x}^2) = y_o(\mathrm{x}^2).
\end{aligned} \tag{47}$$

C. Proof of the Monotonicity of IT2 Fuzzy Logic System Using the Wu–Tan Method

From the assumption that $\eta_j^1(x_j) = \eta_j^2(x_j) = \cdots = \eta_j^{N_j}(x_j) = \eta_j$, we can conclude that the input–output mapping of the IT2 FLS using the WT method is equal to input–output mapping of the T1 FLS whose antecedent T1 FSs are obtained as

$$\mu_{A_j^{i_j}}(x_j) = (1-\eta_j)\overline{\mu}_{\widetilde{A}_j^{i_j}}(x_j) + \eta_j\underline{\mu}_{\widetilde{A}_j^{i_j}}(x_j), \quad j = 1, \ldots, p. \tag{48}$$

As the antecedent IT2 FSs for the kth input variable satisfy the first condition in Theorem 1 and $\eta_k^1(x_k) = \eta_k^2(x_k) = \cdots = \eta_k^{N_k}(x_k) = \eta_k$, from Theorem 7, the embedded T1 FSs satisfy the first condition in Theorem 2.

From Theorem 2, the equivalent T1 FLS is monotonically increasing w.r.t x_k, which means that the IT2 FLS using the WT method is also monotonically increasing w.r.t x_k.

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References

[1] J. M. Mendel, *Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions*. Upper Saddle River, NJ, USA: Prentice-Hall, 2001.
[2] J. M. Mendel and D. Wu, *Perceptual Computing: Aiding People in Making Subjective Judgments*. Hoboken, NJ, USA: Wiley-IEEE Press, 2010.
[3] O. Castillo and P. Melin, *Type-2 Fuzzy Logic Theory and Applications*. Berlin, Germany: Springer-Verlag, 2008.
[4] N. N. Karnik, J. M. Mendel, and Q. Liang, "Type-2 fuzzy logic systems," *IEEE Trans. Fuzzy Syst.*, vol. 7, no. 6, pp. 643–658, 1999.
[5] J. T. Starczewski, "Efficient triangular type-2 fuzzy logic systems," *Int. J. Approx. Reas.*, vol. 50, pp. 799–811, 2009.
[6] C. Wagner and H. Hagras, "Toward general type-2 fuzzy logic systems based on zSlices," *IEEE Trans. Fuzzy Syst.*, vol. 18, no. 4, pp. 637–660, Aug. 2010.
[7] S. Coupland and R. John, "A fast geometric method for defuzzification of type-2 fuzzy sets," *IEEE Trans. Fuzzy Syst.*, vol. 16, no. 4, pp. 929–941, Aug. 2008.
[8] Q. Liang and J. M. Mendel, "Interval type-2 fuzzy logic systems: Theory and design," *IEEE Trans. Fuzzy Syst.*, vol. 8, no. 5, pp. 535–550, Oct. 2000.
[9] J. M. Mendel, R. John, and F. Liu, "Interval type-2 fuzzy logic systems made simple," *IEEE Trans. Fuzzy Syst.*, vol. 14, no. 6, pp. 808–821, Dec. 2006.

[10] S. Greenfield, F. Chiclana, S. Coupland, and R. John, “The collapsing method of defuzzification for discretised interval type-2 fuzzy sets,” *Inf. Sci.*, vol. 179, no. 13, pp. 2055–2069, 2009.

[11] D. Wu, J. M. Mendel, and S. Coupland, “Enhanced interval approach for encoding words into interval type-2 fuzzy sets and its convergence analysis,” *IEEE Trans. Fuzzy Syst.*, vol. 20, no. 3, pp. 499–513, Jun. 2012.

[12] M. Pagola, C. Lopez-Molina, J. Fernandez, E. Barrenechea, and H. Bustince, “Interval type-2 fuzzy sets constructed from several membership functions: Application to the fuzzy thresholding algorithm,” *IEEE Trans. Fuzzy Syst.*, vol. 21, no. 2, pp. 230–244, Apr. 2013.

[13] H. Bustince, E. Barrenechea, M. Pagola, and J. Fernandez, “Interval-valued fuzzy sets constructed from matrices: Application to edge detection,” *Fuzzy Sets Syst.*, vol. 160, pp. 1819–1840, 2009.

[14] H. Bustince, “Indicator of inclusion grade for interval-valued fuzzy sets: Application to approximate reasoning based on interval-valued fuzzy sets,” *Int. J. Approx. Reas.*, vol. 23, no. 3, pp. 137–209, 2000.

[15] H. Bustince, J. Fernandez, A. Kolesarova, and R. Mesiar, “Generation of linear orders for intervals by means of aggregation functions,” *Fuzzy Sets Syst.*, vol. 220, pp. 69–77, 2013.

[16] E. Barrenechea, H. Bustince, B. De Baets, and C. Lopez-Molina, “Construction of interval-valued fuzzy relations with application to the generation of fuzzy edge images,” *IEEE Trans. Fuzzy Syst.*, vol. 19, no. 5, pp. 819–830, Oct. 2011.

[17] C. Cornelis, G. Deschrijver, and E. E. Kerre, “Advances and challenges in interval-valued fuzzy logic,” *Fuzzy Sets Syst.*, vol. 157, pp. 622–627, 2006.

[18] K. Atanassov, *Intuitionistic Fuzzy Sets: Theory and Applications*. Heidelberg, Germany: Physica-Verlag, 1999.

[19] J. Goguen, “L-fuzzy sets,” *J. Math. Anal. Appl.*, vol. 18, pp. 145–174, 1967.

[20] G. Deschrijver and E. E. Kerre, “On the relationship between some extensions of fuzzy set theory,” *Fuzzy Sets Syst.*, vol. 133, no. 2, pp. 227–235, 2003.

[21] G. M. Mendez and M. de los Angeles Hernandez, “Hybrid learning for interval type-2 fuzzy logic systems based on orthogonal least-squares and back-propagation methods,” *Inf. Sci.*, vol. 179, no. 13, pp. 2146–2157, 2009.

[22] S. Miller and R. John, “An interval type-2 fuzzy multiple echelon supply chain model,” *Knowl. Based Syst.*, vol. 23, no. 4, pp. 363–368, 2010.

[23] Y.-Y. Lin, J.-Y. Chang, N. R. Pal, and C.-T. Lin, “A mutually recurrent interval type-2 neural fuzzy system (MRIT2NFS) with self-evolving structure and parameters,” *IEEE Trans. Fuzzy Syst.*, vol. 21, no. 3, pp. 492–509, Jun. 2013.

[24] R. Hosseini, S. D. Qanadli, S. Barman, M. Mazinani, T. Ellis, and J. Dehmeshki, “An automatic approach for learning and tuning Gaussian interval type-2 fuzzy membership functions applied to lung CAD classification system,” *IEEE Trans. Fuzzy Syst.*, vol. 20, no. 2, pp. 224–234, Apr. 2012.

[25] D. Wu and J. M. Mendel, “Linguistic summarization using IF-THEN rules and interval type-2 fuzzy sets,” *IEEE Trans. Fuzzy Syst.*, vol. 19, no. 1, pp. 136–151, Feb. 2011.

[26] H. Hagras, “Type-2 FLCs: A new generation of fuzzy controllers,” *IEEE Comput. Intell. Mag.*, vol. 2, no. 1, pp. 30–43, Feb. 2007.

[27] H. Hagras and C. Wagner, “Towards the wide spread use of type-2 fuzzy logic systems in real world applications,” *IEEE Comput. Intell. Mag.*, vol. 7, no. 3, pp. 14–24, Aug. 2012.

[28] E. A. Jammeh, M. Fleury, C. Wagner, H. Hagras, and M. Ghanbari, “Interval type-2 fuzzy logic congestion control for video streaming across IP networks,” *IEEE Trans. Fuzzy Syst.*, vol. 17, no. 5, pp. 1123–1142, Oct. 2009.

[29] M. Manceur, N. Essounbouli, and A. Hamzaoui, “Second-order sliding fuzzy interval type-2 control for an uncertain system with real application,” *IEEE Trans. Fuzzy Syst.*, vol. 20, no. 2, pp. 262–275, Apr. 2012.

[30] S. Barkat, A. Tlemcani, and H. Nouri, “Noninteracting adaptive control of PMSM using interval type-2 fuzzy logic systems,” *IEEE Trans. Fuzzy Syst.*, vol. 19, no. 5, pp. 925–936, Oct. 2011.

[31] O. Castillo and P. Melin, “A review on the design and optimization of interval type-2 fuzzy controllers,” *Appl. Soft Comput.*, vol. 12, no. 4, pp. 1267–1278, 2012.

[32] O. Castillo, R. Martinez-Marroquin, P. Melin *et al.*, “Comparative study of bio-inspired algorithms applied to the optimization of type-1 and type-2 fuzzy controllers for an autonomous mobile robot,” *Inf. Sci.*, vol. 192, pp. 19–38, 2012.

[33] N. R. Cazarez-Castro, L. T. Aguilar, and O. Castillo, “Designing type-1 and type-2 fuzzy logic controllers via fuzzy Lyapunov synthesis for nonsmooth mechanical systems,” *Eng. Appl. AI*, vol. 25, no. 5, pp. 971–979, 2012.

[34] O. Linda and M. Manic, “Uncertainty-robust design of interval type-2 fuzzy logic controller for delta parallel robot,” *IEEE Trans. Ind. Informat.*, vol. 7, no. 4, pp. 661–670, Nov. 2011.

[35] C. Hsu and C. Juang, “Evolutionary robot wall-following control using type-2 fuzzy controller with species-DE-activated continuous ACO,” *IEEE Trans. Fuzzy Syst.*, vol. 21, no. 1, pp. 100–112, Feb. 2013.

[36] E. V. Broekhoven and B. D. Baets, “Only smooth rule bases can generate monotone Mamdani–Assilian models under center-of-gravity defuzzification,” *IEEE Trans. Fuzzy Syst.*, vol. 17, no. 5, pp. 1157–1174, Oct. 2009.

[37] P. Lindskog and L. Ljung, “Ensuring monotonic gain characteristics in estimated models by fuzzy model structures,” *Automatica*, vol. 36, pp. 311–317, 2000.

[38] E. V. Broekhoven and B. D. Baets, “Monotone Mamdani-Assilian models under mean of maxima defuzzification,” *Fuzzy Sets Syst.*, vol. 159, pp. 2819–2844, 2008.

[39] J. M. Won, S. Y. Park, and J. S. Lee, “Parameter conditions for monotonic Takagi-Sugeno-Kang fuzzy system,” *Fuzzy Sets Syst.*, vol. 132, pp. 135–146, 2002.

[40] K. M. Tay and C. P. Lim, “Optimization of Gaussian fuzzy membership functions and evaluation of the monotonicity property of fuzzy inference systems,” in *Proc. IEEE Int. Conf. Fuzzy Syst.*, Taipei, Taiwan, Jun. 2011, pp. 1219–1224.

[41] V. S. Kouikoglou and Y. A. Phillis, “On the monotonicity of hierarchical sumproduct fuzzy systems,” *Fuzzy Sets Syst.*, vol. 160, pp. 3530–3538, 2009.

[42] H. Seki, H. Ishii, and M. Mizumoto, “On the monotonicity of fuzzy inference methods related to T-S inference method,” *IEEE Trans. Fuzzy Syst.*, vol. 18, no. 3, pp. 629–634, Jun. 2010.

[43] C. Li, G. Zhang, J. Yi, and T. Wang, “On the properties of SIRMs connected type-1 and type-2 fuzzy inference systems,” in *Proc. IEEE Int. Conf. Fuzzy Syst.*, Taipei, Taiwan, Jun. 2011, pp. 1982–1988.

[44] C. Li, J. Yi, M. Wang, and G. Zhang, “Monotonic type-2 fuzzy neural network its application to thermal comfort prediction,” in *Neural Comput. Appl.* New York, NY, USA: Springer, 2012.

[45] C. Li, J. Yi, and D. Zhao, “Analysis and design of monotonic type-2 fuzzy inference systems,” in *Proc. IEEE Int. Conf. Fuzzy Syst.*, Jeju Island, Korea, Aug. 2009, pp. 1193–1198.

[46] C. Li, J. Yi, and T. Wang, “Encoding prior knowledge into data driven design of interval type-2 fuzzy logic systems,” *Int. J. Innov. Comput.*, vol. 7, no. 3, pp. 1133–1144, 2011.

[47] D. Wu, “Approaches for reducing the computational cost of interval type-2 fuzzy logic systems: Overview and comparisons,” *IEEE Trans. Fuzzy Syst.*, vol. 21, no. 1, pp. 80–89, Feb. 2013.

[48] C.-Y. Yeh, W.-H. Jeng, and S.-J. Lee, “An enhanced type-reduction algorithm for type-2 fuzzy sets,” *IEEE Trans. Fuzzy Syst.*, vol. 19, no. 2, pp. 227–240, Apr. 2011.

[49] X. Liu and J. M. Mendel, “Connect Karnik-Mendel algorithms to root-finding for computing the centroid of an interval type-2 fuzzy set,” *IEEE Trans. Fuzzy Syst.*, vol. 19, no. 4, pp. 652–665, Aug. 2011.

[50] J. M. Mendel, “On KM algorithms for solving type-2 fuzzy set problems,” *IEEE Trans. Fuzzy Syst.*, vol. 21, no. 3, pp. 426–446, Jun. 2013.

[51] X. Du and H. Ying, “Derivation and analysis of the analytical structures of the interval type-2 fuzzy-PI and PD controllers,” *IEEE Trans. Fuzzy Syst.*, vol. 18, no. 4, pp. 802–814, Aug. 2010.

[52] M. Biglarbegian, W. Melek, and J. M. Mendel, “On the stability of interval type-2 TSK fuzzy logic control systems,” *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 41, no. 5, pp. 798–818, Jun. 2010.

[53] D. Wu and W. W. Tan, “Computationally efficient type-reduction strategies for a type-2 fuzzy logic controller,” in *Proc. IEEE Int. Conf. Fuzzy Syst.*, Reno, NV, USA, May 2005, pp. 353–358.

[54] M. Nie and W. W. Tan, “Towards an efficient type-reduction method for interval type-2 fuzzy logic systems,” in *Proc. IEEE Int. Conf. Fuzzy Syst.*, Hong Kong, Jun. 2008, pp. 1425–1432.

[55] M. Nie and W. W. Tan, “Analytical structure and characteristics of symmetric centroid type-reduced interval type-2 fuzzy PI and PD controllers,” *IEEE Trans. Fuzzy Syst.*, vol. 20, no. 3, pp. 416–430, Jun. 2012.

[56] H. Wu and J. M. Mendel, “Uncertainty bounds and their use in the design of interval type-2 fuzzy logic systems,” *IEEE Trans. Fuzzy Syst.*, vol. 10, no. 5, pp. 622–639, Oct. 2002.

[57] D. Wu and J. M. Mendel, “On the continuity of type-1 and interval type-2 fuzzy logic systems,” *IEEE Trans. Fuzzy Syst.*, vol. 19, no. 1, pp. 179–192, Feb. 2011.

[58] D. Wu, "On the fundamental differences between interval type-2 and type-1 fuzzy logic controllers," *IEEE Trans. Fuzzy Syst.*, vol. 20, no. 5, pp. 832–848, Oct. 2012.

[59] S. Coupland and R. John, "Geometric type-1 and type-2 fuzzy logic systems," *IEEE Trans. Fuzzy Syst.*, vol. 15, no. 1, pp. 3–15, Feb. 2007.

[60] J. Ramik and J. Rimanek, "Inequality relation between fuzzy numbers and its use in fuzzy optimization," *Fuzzy Sets Syst.*, vol. 16, no. 2, pp. 123–138, 1985.

[61] E. Valvis, "A new linear ordering of fuzzy numbers on subsets of F(R)," *Fuzzy Optim. Decis. Making*, vol. 8, pp. 141–163, 2009.

[62] S. Jafarzadeh, S. Fadali, and A. Sonbol, "Stability analysis and control of discrete type-1 and type-2 TSK fuzzy systems—Part I: Stability analysis," *IEEE Trans. Fuzzy Syst.*, vol. 19, no. 6, pp. 989–1000, Dec. 2011.

[63] S. Jafarzadeh, S. Fadali, and A. Sonbol, "Stability analysis and control of discrete type-1 and type-2 TSK fuzzy systems: Part II control design," *IEEE Trans. Fuzzy Syst.*, vol. 19, no. 6, pp. 1001–1013, Dec. 2011.

[64] H. K. Lam, M. Narimani, and L. D. Seneviratne, "LMI-based stability conditions for interval type-2 fuzzy-model-based control systems," in *Proc. IEEE Int. Conf. Fuzzy Syst.*, Taipei, Taiwan, Jun. 2011, pp. 298–303.

[65] H. K. Lam and L. Seneviratne, "Stability analysis of interval type-2 fuzzy model- based control systems," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 38, no. 3, pp. 617–628, Jun. 2008.

[66] C. Li, J. Yi, and T. Wang, "Stability analysis of SIRMs based type- 2 fuzzy logic control systems," in *Proc. IEEE Int. Conf. Fuzzy Syst.*, Taipei, Taiwan, Jun. 2011, pp. 1–7.

[67] M. Biglarbegian, W. Melek, and J. M. Mendel, "On the robustness of type-1 and interval type-2 fuzzy logic systems in modeling," *Inf. Sci.*, vol. 181, no. 7, pp. 1325–1347, 2011.

[68] D. Wu and W. W. Tan, "Interval type-2 fuzzy PI controllers: Why they are more robust," in *Proc. IEEE Int. Conf. Granul. Comput.*, San Jose, CA, Aug. 2010, pp. 802–807.

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