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Global stability analysis of fractional-order Hopfield neural networks with time delay

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ABSTRACT

In this paper, the global stability analysis of fractional-order Hopfield neural networks with time delay is investigated. A stability theorem for linear fractional-order systems with time delay is presented. And, a comparison theorem for a class of fractional-order systems with time delay is shown. The existence and uniqueness of the equilibrium point for fractional-order Hopfield neural networks with time delay are proved. Furthermore, the global asymptotic stability conditions of fractional-order neural networks with time delay are obtained. Finally, a numerical example is given to illustrate the effectiveness of the theoretical results.

1. Introduction

Fractional calculus, as a classical mathematical notion, is a generalization of the ordinary differentiation and integration to arbitrary non-integer order. Although fractional calculus has a long history, its applications to physics and engineering are just a recent focus of interest to many researchers. Compared with the classical integer-order systems, fractional-order systems provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. It would be far better if many practical problems are described by fractional-order dynamical systems rather than integer-order ones. In fact, real-world processes generally or most likely are fractional-order systems, such as phenomenological description of viscoelastic liquids [1], diffusion and wave propagation [2,3], colored noise [4], boundary layer effects in ducts [5], electromagnetic waves [6], fractional kinetics [7], electrode-electrolyte polarization [8], dielectric relaxation phenomena in polymeric materials [9] and fractional-order models of happiness [10]. All of the results demonstrate the importance of fractional calculus and motivate the development of new applications.

Since Hopfield neural network [11] was referred by Hopfield in 1984, it has attracted great attentions of many scientists and has been applied in various realms such as pattern recognition, associative memory and combinatorial optimization. In recent years, the researchers found that the fractional calculus could be well used in the study of neural networks. Kaslik et al. pointed out that the common capacitance from the continuous-time integer-order Hopfield neural network can be replaced by the fractance, giving birth to the so-called fractional-order Hopfield neural network model [13]. In fact, the fractional-order differentiation provides neurons with a fundamental and general computational ability that contributes to efficient information processing, stimulus anticipation, and frequency-independent phase shifts in oscillatory neuronal firings [12]. Now, a lot of results on fractional-order Hopfield neural networks have been obtained [13–19]. The stability and multi-stability of fractional-order Hopfield neural networks with ring or hub structures were investigated in [13,14]. The stability of fractional-order neural networks was fully investigated through an energy-like function analysis in [15]. A discrete time fractional-order Hopfield neural network was presented in [16]. The α-stability and α-synchronization for fractional-order Hopfield neural networks were investigated in [17]. It was pointed out that the stability in [17] was not α-stability but Mittag–Leffler stability in [18]. Some sufficient conditions are established to ensure the existence and uniqueness of the nontrivial solution for the fractional-order Hopfield neural networks in [19]. There were also several recent literatures discussing the topics including chaos and chaotic synchronization in fractional-order neural networks, which can be found in [20–23].

Note that most of the above results on the stability of fractional-order Hopfield neural networks did not consider time delay. In practice, because of finite switching speeds of the amplifiers, time delay is well-known to be unavoidable and it can cause oscillations or instabilities in dynamic systems. The stability as a very essential topic in fractional-order systems with time delay has attracted increasing interest in recent years [24–29]. However, there are just

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few results on the stability of fractional-order neural networks with time delay [30–32]. A sufficient condition was established for the uniform stability of fractional-order neural networks with time delay in [30]. The fractional-order neural networks of two and three neurons with time delay were discussed, and the stability conditions were derived in [31], Some sufficient conditions for stability of the fractional-order Hopfield neural networks with hub structure and time delays were obtained, and the stability conditions of two fractional-order Hopfield neural networks with different ring structures and time delays were derived in [32]. But, the above results of fractional-order Hopfield neural networks with time delay did not consider global stability.

Furthermore, the global stability is an important topic in the fractional-order neural networks. There are few results on the global analysis of fractional-order Hopfield neural networks. The authors consider the global stability of fractional-order neural networks without time delay in [17]. But, some conclusions in [17] are incorrect that lead to unqualified results of global stability which are given in [18]. The conditions on the global Mittag–Leffler stability are established by using Lyapunov method for memristor-based fractional-order neural networks without time delay [33]. However, to the best of our knowledge, there is no known result concerning a theoretical global stability analysis for fractional-order Hopfield neural networks with time delay.

Motivated by the above discussion, this paper is devoted to presenting a theoretical global stability analysis for fractional-order Hopfield neural networks with time delay. Firstly, a stability theorem for linear fractional-order systems with time delay is discussed. And a comparison theorem for a class of fractional-order systems with time delay is shown. Then, using the contraction mapping theorem, the existence and uniqueness of the equilibrium point for fractional-order Hopfield neural networks with time delay are proved. Finally, the global asymptotic stability of fractional-order neural networks with time delay is investigated, and the corresponding conditions for global asymptotic stability of fractional-order neural networks with time delay are also derived by using Lyapunov method.

The paper is structured as follows. In Section 2, the preliminaries concerning fractional-order differential systems with time delay are introduced. Some results for the stability analysis of fractional-order systems with time delay are given in Section 3. Then, the global asymptotic stability of fractional-order Hopfield neural networks with time delay is investigated in Section 4. And a numerical example is given in Section 5 to illustrate the effectiveness of the theoretical results. Some conclusions are included in Section 6.

2. Preliminaries

Some elementary notations are introduced for the Caputo fractional-order derivative and its properties. The main theoretical tools for the qualitative analysis of fractional-order dynamical systems are given in [34–36].

The Caputo fractional-order derivative is defined as

$$ {}_0D_t^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{q-n+1}}\, d\tau, $$

where n is an integer, $n-1<q\le n$ and $\Gamma(\cdot)$ is a Gamma function.

The Laplace transform of the Caputo fractional-order derivative is

$$ L\{{}_0D_t^q f(t); s\} = s^q F(s) - \sum_{k=0}^{n-1} s^{q-k-1} f^{(k)}(0), \quad n-1<q\le n. $$

When $f^{(k)}(0)=0, k=1,2,\dots,n$, then

$$ L\{{}_0D_t^q f(t); s\} = s^q F(s). $$

Some properties of the Caputo fractional-order derivative [37] are obtained as

(1) ${}_0D_t^q c = 0$, where c is any constant.
(2) If $x(t)\in C^m[0,T]$ for $T>0$ and $m-1<q<m\in Z^+$, then ${}_0D_t^q x(0)=0$.
(3) If $x(t)\in C^1[0,T]$ for some $T>0$, $q_1,q_2\in R^+$, $q_1+q_2\le 1$, then ${}_0D_t^{q_1}\,{}_0D_t^{q_2} x(t) = {}_0D_t^{q_1+q_2} x(t)$.

The following definitions will be used in this paper.

The Caputo derivative is employed in the paper, and the Adams–Bashforth–Moulton predictor–corrector scheme is applied to solve the fractional-order differential equations with time delay [38].

The l_1 norm is defined by $\|x(t)\| = \sum_{i=1}^n |x_i(t)|$, where $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T \in R^n$.

3. Some results on fractional-order systems with time delay

Some results on for fractional-order systems are given in this section, which will be used in global stability analysis of fractional-order Hopfield neural networks with time delay. The stability analysis of linear fractional-order systems with time delay is discussed. And a comparison theorem for a class of fractional-order systems with time delay is shown.

Consider the following linear fractional-order system with time delay:

$$ {}_0D_t^q X(t) = AX(t) + X(t_\tau), \tag{1} $$

where $A=(a_{ij})_{n\times n}$, $X(t)=(x_1(t),x_2(t),\dots,x_n(t))^T$, $X(t_\tau)=(\sum_{j=1}^n k_{1j}x_j(t-\tau_{1j}), \sum_{j=1}^n k_{2j}x_j(t-\tau_{2j}), \dots, \sum_{j=1}^n k_{nj}x_j(t-\tau_{nj}))^T$.

Especially, if $\tau_{ij}=\tau_j$, for $i=1,2,\dots,n$, $K=(k_{ij})_{n\times n}$, the system (1) can be written in the following vector form:

$$ {}_0D_t^q X(t) = AX(t) + KX(t-\tau), \tag{2} $$

where $X(t-\tau) = (x_1(t-\tau_1), x_2(t-\tau_2), \dots, x_n(t-\tau_n))^T$.

Taking Laplace transform [34,35] on both sides of (1), we have

$$ \begin{cases} s^qY_1(s)-s^{q-1}\phi_1(0) = k_{11}e^{-s\tau_{11}}(Y_1(s)+\int_{-\tau_{11}}^0 e^{-st}\phi_1(t)\,dt) + k_{12}e^{-s\tau_{12}}(Y_2(s)+\int_{-\tau_{12}}^0 e^{-st}\phi_2(t)\,dt) + a_{12}Y_2(s) \\ \qquad + \cdots + k_{1n}e^{-s\tau_{1n}}(Y_n(s)+\int_{-\tau_{1n}}^0 e^{-st}\phi_n(t)\,dt) + a_{1n}Y_n(s) + a_{11}Y_1(s) \\ s^qY_2(s)-s^{q-1}\phi_2(0) = k_{21}e^{-s\tau_{21}}(Y_1(s)+\int_{-\tau_{21}}^0 e^{-st}\phi_1(t)\,dt) + a_{21}Y_1(s) + k_{22}e^{-s\tau_{22}}(Y_2(s)+\int_{-\tau_{22}}^0 e^{-st}\phi_2(t)\,dt) \\ \qquad + \cdots + k_{2n}e^{-s\tau_{2n}}(Y_n(s)+\int_{-\tau_{2n}}^0 e^{-st}\phi_n(t)\,dt) + a_{2n}Y_n(s) + a_{22}Y_2(s) \\ \cdots \\ s^qY_n(s)-s^{q-1}\phi_n(0) = k_{n1}e^{-s\tau_{n1}}(Y_1(s)+\int_{-\tau_{n1}}^0 e^{-st}\phi_n(t)\,dt) + a_{n1}Y_1(s) + k_{n2}e^{-s\tau_{n2}}(Y_2(s)+\int_{-\tau_{n2}}^0 e^{-st}\phi_2(t)\,dt) \\ \qquad + a_{n2}Y_2(s) + \cdots + k_{nn}e^{-s\tau_{nn}}(Y_n(s)+\int_{-\tau_{nn}}^0 e^{-st}\phi_n(t)\,dt) + a_{nn}Y_n(s), \end{cases} \tag{3} $$

where $Y_i(s)$ is the Laplace transform of $x_i(t)$ with $Y_i(s)=L(x_i(t))$ and $\phi_i(t)$ $(1\le i\le n, t\in[-\tau,0])$ is an initial value.

We can rewrite (3) as follows:

$$\Delta(s)\begin{pmatrix} Y_1(s) \\ Y_2(s) \\ \vdots \\ Y_n(s) \end{pmatrix} = \begin{pmatrix} d_1(s) \\ d_2(s) \\ \vdots \\ d_n(s) \end{pmatrix},$$

in which

$$\begin{cases} d_1(s) = s^{q-1}\phi_1(0) + k_{11}e^{-s\tau_{11}}\int_{-\tau_{11}}^{0} e^{-st}\phi_1(t)\,dt + k_{12}e^{-s\tau_{12}}\int_{-\tau_{12}}^{0} e^{-st}\phi_2(t)\,dt \\ \qquad + \cdots + k_{1n}e^{-s\tau_{1n}}\int_{-\tau_{1n}}^{0} e^{-st}\phi_n(t)\,dt \\ d_2(s) = s^{q-1}\phi_2(0) + k_{21}e^{-s\tau_{21}}\int_{-\tau_{21}}^{0} e^{-st}\phi_1(t)\,dt + k_{22}e^{-s\tau_{22}}\int_{-\tau_{22}}^{0} e^{-st}\phi_2(t)\,dt \\ \qquad + \cdots + k_{2n}e^{-s\tau_{2n}}\int_{-\tau_{2n}}^{0} e^{-st}\phi_n(t)\,dt \\ \quad \cdots \\ d_n(s) = s^{q-1}\phi_n(0) + k_{n1}e^{-s\tau_{n1}}\int_{-\tau_{n1}}^{0} e^{-st}\phi_1(t)\,dt + k_{n2}e^{-s\tau_{n2}}\int_{-\tau_{n2}}^{0} e^{-st}\phi_2(t)\,dt \\ \qquad + \cdots + k_{nn}e^{-s\tau_{nn}}\int_{-\tau_{nn}}^{0} e^{-st}\phi_n(t)\,dt, \end{cases}$$

$$\Delta(s) = \begin{pmatrix} s^q - k_{11}e^{-s\tau_{11}} - a_{11} & -k_{12}e^{-s\tau_{12}} - a_{12} & \cdots & -k_{1n}e^{-s\tau_{1n}} - a_{1n} \\ -k_{21}e^{-s\tau_{21}} - a_{21} & s^q - k_{22}e^{-s\tau_{22}} - a_{22} & \cdots & -k_{2n}e^{-s\tau_{2n}} - a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -k_{n1}e^{-s\tau_{n1}} - a_{n1} & s^q - k_{n2}e^{-s\tau_{n2}} - a_{n2} & \cdots & -k_{nn}e^{-s\tau_{nn}} - a_{nn} \end{pmatrix}.$$

We call $\Delta(s)$ as the characteristic matrix of system (1) and $\det(\Delta(s))$ as the characteristic polynomial of $\Delta(s)$. The stability of system (1) is completely determined by the distribution of eigenvalues of $\det(\Delta(s))$.

If $\tau_{ij} = 0$, system (1) can be written as

$$_0D_t^q X(t) = AX(t) + KX(t) = MX(t), \tag{4}$$

where

$$M = \begin{pmatrix} k_{11}+a_{11} & k_{12}+a_{12} & \cdots & k_{1n}+a_{1n} \\ k_{21}+a_{21} & k_{22}+a_{22} & \cdots & k_{2n}+a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ k_{n1}+a_{n1} & k_{n2}+a_{n2} & \cdots & k_{nn}+a_{nn} \end{pmatrix}.$$

Based on the characteristic polynomial $\det(\Delta(s))$ and the coefficient matrix M ($\tau_{ij} = 0$), we can obtain the following conclusions.

Theorem 3.1. *If all the roots of the characteristic equation* $\det(\Delta(s)) = 0$ *for* $q \in (0,1)$ *have negative real parts, then the zero solution of system (1) is Lyapunov asymptotically stable.*

Proof. If all the roots of the characteristic equation $\det(\Delta(s)) = 0$ have negative real parts for $q \in (0,1)$, then $\Delta(s)$ is an invertible matrix. From Eq. (3), one has

$$Y(s) = \Delta(s)^{-1}D(s),$$

where $Y(s) = (Y_1(s), Y_2(s), \ldots, Y_n(s))^T$, $D(s) = (d_1(s), d_2(s), \ldots, d_n(s))^T$. According to the final value theorem of the Laplace transform [39,40], one has

$$\lim_{t\to\infty} X(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \Delta(s)^{-1}D(s)s = 0.$$

Hence, the zero solution of system (1) is Lyapunov asymptotically stable. The proof is completed. □

Theorem 3.2. *If* $q \in (0,1)$, *all the eigenvalues of* M *satisfy* $|\arg(\lambda)| > \pi/2$ *and the characteristic equation* $\det(\Delta(s)) = 0$ *has no purely imaginary roots for any* $\tau_{ij} > 0, i,j = 1,\ldots,n$, *then the zero solution of system (1) is Lyapunov asymptotically stable.*

Proof. Because the proof of this theorem is almost as same as the one in Theorem 2.1 of Chapter 8 in [41] or in Theorem 2.2.6 of Chapter 2 in [42] for the classical different equations, the more details about proof can be found in [41,42]. Here, a brief illustration is taken. If $\tau_{ij} = 0$, all the eigenvalues of M satisfy $|\arg(\lambda)| > \pi/2$, that is to say that all the roots of the characteristic equation M have negative real parts. When $\tau_{ij} \neq 0$, the characteristics of $\Delta(s)$ are continuously changing with τ_{ij}. Also, the characteristic equation $\det(\Delta(s)) = 0$ has no purely imaginary roots for any $\tau_{ij} > 0, i,j = 1,\ldots,n$, so all the roots of the characteristic equation $\det(\Delta(s)) = 0$ have negative real parts. Then the zero solution of system (1) is Lyapunov asymptotically stable. The proof is completed. □

Remark 1. In this paper, the system $_0D_t^q x(t) = Ax(t) + Kx(t-\tau)$ ($A \neq 0$) is considered, and its stability is not guaranteed under conditions that the eigenvalues of M are satisfied $|\arg(\lambda)| > q\pi/2$ in [27]. In fact, when the eigenvalues of M are satisfied $q\pi/2 < |\arg(\lambda)| \le \pi/2$, and the characteristic equation $\det(\Delta(s)) = 0$ has no purely imaginary roots for any $\tau > 0$, the zero solution of $_0D_t^q x(t) = Ax(t) + Kx(t-\tau)(A \neq 0)$ has unstable situation [31].

Next, a comparison theorem for a class of fractional-order systems with time delay is shown.

Lemma 3.3. *Consider the following two fractional-order systems with time delay:*

$$\begin{cases} _0D_t^q x(t) = \overline{f}_1(t, x(t)) + \overline{g}_1(t, x(t-\tau)), \\ 0 < q \le 1, x(t) = h(t), \\ \quad t \in [-\tau, 0], \end{cases} \tag{5}$$

and

$$\begin{cases} _0D_t^q y(t) = \overline{f}_2(t, y(t)) + \overline{g}_2(t, y(t-\tau)), \quad 0 < q \le 1, \\ y(t) = h(t), t \in [-\tau, 0], \end{cases} \tag{6}$$

where $\overline{f}_1(t,x(t))$ *and* $\overline{f}_2(t,y(t))$ *are Lipschitz continuous in* $[0,+\infty) \times G(G \subset R)$. *Similarly,* $\overline{g}_1(t,x(t-\tau))$ *and* $\overline{g}_2(t,y(t-\tau))$ *are Lipschitz continuous in* $[-\tau,+\infty) \times G(G \subset R)$.

If

$$\overline{f}_1(t,x(t)) \le \overline{f}_2(t,y(t)), \quad \overline{g}_1(t,x(t-\tau)) \le \overline{g}_2(t,y(t-\tau)), \quad \forall t \in [0,+\infty),$$

then

$$x(t) \le y(t), \quad \forall t \in [0,+\infty).$$

Proof. The solutions of systems (5) and (6) can be expressed in the following form:

$$x(t) = h(0) + \frac{1}{\Gamma(q)}\int_0^t \left(t-s\right)^{q-1}[\overline{f}_1(s,x(s)) + \overline{g}_1(s,x(s-\tau))]\,ds, \tag{7}$$

and

$$y(t) = h(0) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[\overline{f}_2(s,y(s)) + \overline{g}_2(s,y(s-\tau))]\,ds. \tag{8}$$

Subtracting Eq. (8) from Eq. (7), one has

$$y(t) - x(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[\overline{f}_2(s,y(s)) - \overline{f}_1(s,x(s)) + \overline{g}_2(s,y(s-\tau)) - \overline{g}_1(s,x(s-\tau))]\,ds. \tag{9}$$

Take $m_1(t) = \overline{f}_2(t,y(t)) - \overline{f}_1(t,x(t))$, $m_2(t-\tau) = \overline{g}_2(t,y(t-\tau)) - \overline{g}_1(t,x(t-\tau))$. It is easy to know $m_1(t) \ge 0$, $m_2(t-\tau) \ge 0$, $t \in [0,+\infty)$.

Then Eq. (9) can be rewritten as

$$y(t) - x(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} m_1(t)\,ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} m_2(s-\tau)\,ds. \tag{10}$$

Since t^{q-1} is a nonnegative function, it then follows from Eq. (10) that $x(t) \le y(t)$, $\forall t \in [0,+\infty)$. The proof is completed. □

Remark 2. According to [43], if the functions $\overline{f}_j(j=1,2)$ and $\overline{g}_j(j=1,2)$ are Lipschitz continuous, there is the existence of solutions in systems (5) and (6) with Caputo fractional-order derivative.

Note that the conditions of Lemma 3.3 are very strong, hence the following conclusion is given, which will be used in this paper.

Lemma 3.4. *Consider the following the fractional-order differential inequality with time delay:*

$$\begin{cases} {}_0D_t^q x(t) \le -ax(t)+bx(t-\tau), & 0<q\le 1, \\ x(t)=h(t), & t\in[-\tau,0], \end{cases} \tag{11}$$

and the linear fractional-order differential systems with time delay

$$\begin{cases} {}_0D_t^q y(t) = -ay(t)+by(t-\tau), & 0<q\le 1, \\ y(t)=h(t), & t\in[-\tau,0], \end{cases} \tag{12}$$

where $x(t)$ and $y(t)$ are continuous and nonnegative in $(0,+\infty)$, and $h(t)\ge 0, t\in[-\tau,0]$.

If $a>0$ and $b>0$, then

$$x(t)\le y(t), \quad \forall t\in[0,+\infty).$$

Proof. From system (11), there exists a nonnegative function $m(t)$ satisfying

$$\begin{cases} {}_0D_t^q x(t) = -ay(t)+bx(t-\tau)-m(t), & 0<q\le 1, \\ x(t)=h(t), \end{cases}$$
$$t\in[-\tau,0]. \tag{13}$$

According to [44], the initial value problem (13) has, on the interval $[0,k\tau]$, a unique solution that can be represented by $x(t)=x_{i\tau}(t)$, and

$$x_{i\tau}(t)=\int_0^t (t-s)^{q-1}E_{q,q}(-a(t-s)^q)\phi_{i\tau}\, ds + c_{i\tau}E_{q,1}(-at^q),$$
$$0<q\le 1, \quad t\in[(i-1)\tau, i\tau], \tag{14}$$

where $c_{i\tau}$ is a constant, $i=1,2,\ldots,k$, and k is a greatest positive integer. $x_{0\tau}(t)=h(t)$ and $\phi_{i\tau}$ is expressed as

$$\phi_{i\tau}(t)=\begin{cases} bx_{0\tau}(t-\tau)-m(t), & 0<t\le\tau, \\ bx_{\tau}(t-\tau)-m(t), & \tau<t\le 2\tau, \\ \vdots & \\ bx_{(k-1)\tau}(t-\tau)-m(t), & (k-1)\tau<t\le k\tau. \end{cases} \tag{15}$$

Since both t^{q-1} and $E_{q,q}(-at^q)$ are nonnegative functions [45], due to $x(t)=x_{i\tau}(t)$ and $m(t)\ge 0$, Eq. (14) can be written as

$$x_{i\tau}(t)\le \int_0^t (t-s)^{q-1}E_{q,q}(-a(t-s)^q)bx_{i\tau}(s-\tau)\, ds + c_{i\tau}E_{q,1}(-at^q),$$
$$0<q\le 1, \quad t\in[(i-1)\tau, i\tau]. \tag{16}$$

Similarly, the solution of system (12) can be written as

$$y_{i\tau}(t)= \int_0^t (t-s)^{q-1}E_{q,q}(-a(t-s)^q)by_{i\tau}(s-\tau)\, ds + c_{i\tau}E_{q,1}(-at^q),$$
$$0<q\le 1, \quad t\in[(i-1)\tau, i\tau]. \tag{17}$$

Next, we will consider $x(t)\le y(t)$, $t\in[(i-1)\tau, i\tau]$, $i=1,2,\ldots,k$. We use the method of induction on k.

Let us first prove that $x(t)\le y(t)$ holds for $k=1$. If $t\in[0,\tau]$, then $t-\tau\in[-\tau,0]$ and $x(t-\tau)=y(t-\tau)=h(t-\tau)$.

According to the systems (16) and (17), we have

$$x_\tau(t)\le \int_0^t (t-s)^{q-1}E_{q,q}(-a(t-s)^q)bh(s-\tau)\, ds + c_\tau E_{q,1}(-at^q)=y_\tau(t).$$

Note that taking initial conditions into account in the systems (12) and (13), the solution is uniquely determined since we must have $c_\tau=h(0)$. So we prove that $x(t)\le y(t)$ holds for $k=1$.

Next suppose that $x(t)\le y(t)$ holds for k, that is, let us assume that for $t\in[(k-1)\tau, k\tau]$, then one has

$$x_{i\tau}(t)\le y_{i\tau}(t), \quad i=1,2,\ldots,k.$$

Let us prove that it will also be valid for $k+1$.

If the $t\in[k\tau,(k+1)\tau]$, the system (16) can be written as

$$\begin{aligned} x(t) &\le \int_0^t (t-s)^{q-1}E_{q,q}(-a(t-s)^q)bx(s-\tau)\, ds + c_{(k+1)\tau}E_{q,1}(-at^q), \\ &= \int_0^\tau (t-s)^{q-1}E_{q,q}(-a(t-s)^q)bx_\tau(s-\tau)\, ds \\ &\quad + \sum_{j=2}^{k}\int_{(j-1)\tau}^{j\tau} (t-s)^{q-1}E_{q,q}(-a(t-s)^q)bx_{j\tau}(s-\tau)\, ds \\ &\quad + \int_{k\tau}^t (t-s)^{q-1}E_{q,q}(-a(t-s)^q)bx_{(k+1)\tau}(s-\tau)\, ds + c_{(k+1)\tau}E_{q,1}(-at^q). \end{aligned} \tag{18}$$

When $s\in[k\tau,t]$, $s-\tau\in[(k-1)\tau, t-\tau]\subset[(k-1)\tau,k\tau]$. According to the assumed condition, we have $x(s-\tau)\le y(s-\tau)$.

From Eq. (18), we have $x(t)\le y(t)$, $t\in[k\tau,(k+1)\tau]$. This completes the proof. □

Remark 3. According to [44], if the function $\phi_{i\tau}$ is continuous, $x(t)=x_{i\tau}(t)$ and $x(t)$ can be represented by

$$x_{i\tau}(t)=\int_0^t (t-s)^{q-1}E_{q,q}(-a(t-s)^q)\phi_{i\tau}\, ds + c_{i\tau}E_{q,1}(-at^q).$$

In Lemma 3.4, if $\phi_{i\tau}$ is left continuous, it still has $x(t)=x_{i\tau}(t)$ except countable points. From Eqs. (13) and (17), the same constant $c_{i\tau}$ is used. According to [35] (see Theorem 5.15) and [44], the constant $c_{i\tau}$ just depends on the initial conditions.

Lemma 3.5 (*Zhang et al.* [46]). *If $h(t)\in C^1([0,+\infty),R)$ denotes a continuously differentiable function, then the following inequality holds almost everywhere:*

$${}_0D_t^q|h(t)|\le \operatorname{sgn}(h(t))\,{}_0D_t^q h(t), \quad 0<q\le 1. \tag{19}$$

4. Global stability analysis of fractional-order Hopfield neural network with time delay

Consider the following fractional-order Hopfield neural network with time delay:

$$\begin{aligned} {}_0D_t^q x_i(t) &= -a_ix_i(t)+\sum_{j=1}^n b_{ij}f_j(x_j(t))+\sum_{j=1}^n c_{ij}g_j(x_j(t-\tau)) \\ &\quad + d_i, \quad i=1,2,\ldots,n, \quad t>0, \end{aligned} \tag{20}$$

where $q\in(0,1)$, n corresponds to the number of units in a neural network, $x(t)=(x_1(t),x_2(t),\ldots,x_n(t))^T$ corresponds to the state vector at time t, $a_i>0$ is the self-regulating parameters of the neurons, $f_j(x_j(t))$ and $g_j(x_j(t-\tau))$ denote, respectively, the measures of response or activation to its incoming potentials of the unit j at time t and $t-\tau$; b_{ij}, c_{ij} are constants with b_{ij} denoting the synaptic connection weight of the unit j to the unit i at time t , and c_{ij} denoting the synaptic connection weight of the unit j to the unit i at time $t-\tau$, d_i is the constant control input vector.

The global stability of fractional-order neural networks with time delay will be discussed. In order to obtain the main results, the following assumptions are given.

Assumption 1. (A1). The neuron activation functions f_j, g_j are Lipschitz continuous. That is, there exist positive constants L_j, K_j, $j=1,2,\ldots,n$, such that

$$|f_j(u)-f_j(v)|<L_j|u-v|, \quad |g_j(u)-g_j(v)|<K_j|u-v|, \quad u,v\in R.$$

Assumption 2. (A2). a_i, b_{ij}, c_{ij}, L_j and K_j satisfy the following condition:

$$\widehat{K}<\lambda\sin\left(\frac{q\pi}{2}\right), \quad 0<q\le 1,$$

where $\widehat{K} = \max_{1 \le i \le n}(\sum_{j=1}^{n} |c_{ji}|K_i)$, $\lambda = \min_{1 \le i \le n}(a_i - \sum_{j=1}^{n} |b_{ji}|L_i)$.

Theorem 4.1. *If Assumptions (A1) and (A2) hold, then there exists a unique equilibrium point in system (20).*

Proof. Taking $a_i x_i^* = u_i^*$, construct a mapping $\Phi : R^n \to R^n$, as

$$\Phi_i u_i = \sum_{j=1}^{n} b_{ij} f_j\left(\frac{u_j^*}{a_j}\right) + \sum_{j=1}^{n} c_{ij} g_j\left(\frac{u_j^*}{a_j}\right) + \widehat{d}_i, \quad (21)$$

where $\Phi(u) = (\Phi_1(u), \Phi_2(u), \ldots, \Phi_n(u))^T$.

Now, we show that Φ is a contraction mapping on R^n endowed with the l_1 norm.

According to Assumption (A2), we have

$$\max_{1 \le i \le n}\left(\sum_{j=1}^{n} |c_{ji}|K_i\right) < \left(a_i - \sum_{j=1}^{n} |b_{ji}|L_i\right), \quad 1 \le i \le n.$$

Take

$$\theta = \max_{1 \le i \le n}\left(\frac{\max_{1 \le i \le n}\left(\sum_{j=1}^{n} |c_{ji}|K_i\right) + \sum_{j=1}^{n} |b_{ji}|L_i}{a_i}\right),$$

obviously drawn $\theta < 1$.

Consider two different vectors $u = (u_1, u_2, \ldots, u_n)^T$ and $v = (v_1, v_2, \ldots, v_n)^T$, and we obtain

$$\begin{aligned}
\|\Phi(u) - \Phi(v)\| &= \sum_{i=1}^{n} |\Phi_i(u) - \Phi_i(v)| \\
&= \sum_{i=1}^{n} \left| \sum_{j=1}^{n} b_{ij}\left[f_j\left(\frac{u_j}{a_j}\right) - f_j\left(\frac{v_j}{a_j}\right)\right] + \sum_{j=1}^{n} c_{ij}\left[g_j\left(\frac{u_j}{a_j}\right) - g_j\left(\frac{v_j}{a_j}\right)\right] \right| \\
&\le \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\frac{b_{ij}L_j + c_{ij}K_j}{a_i}\right)|u_j(t) - v_j|\right) \\
&\le \theta \sum_{i=1}^{n} |u_i(t) - v_i(t)| \\
&= \theta \|u(t) - v(t)\|. \quad (22)
\end{aligned}$$

So we can get

$$\|\Phi(u) - \Phi(v)\| < \theta \|u(t) - v(t)\|,$$

which implies that $\Phi(u)$ is a contraction mapping on R^n. Hence, there exists a unique fixed point such that $u^* \in R^n$ i.e. $\Phi(u^*) = u^*$,

$$u_i^* = \sum_{j=1}^{n} b_{ij} f_j\left(\frac{u_j^*}{a_j}\right) + \sum_{j=1}^{n} c_{ij} g_j\left(\frac{u_j^*}{a_j}\right) + \widehat{d}_i.$$

That is

$$-a_i x_i^* + \sum_{j=1}^{n} b_{ij} f_j(x_j^*) + \sum_{j=1}^{n} c_{ij} g_j(x_j^*) + d_i = 0,$$

which means that $u^* = x^*$ is an equilibrium point of system (20). This completes the proof. □

Theorem 4.2. *If Assumptions (A1) and (A2) hold, then system (20) is global asymptotically stable, and all the solutions of system (20) converge to the unique equilibrium point x^*.*

Proof. We first consider that all the solutions of system (20) will converge to the unique equilibrium point x^*.

Assume that $y(t) = (y_1(t), y_2(t), \ldots, y_n(t))$ and $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ are any two solutions of system (20) with the different initial conditions. Take $e_i(t) = y_i(t) - x_i(t)$, then $e_i(t-\tau) = y_i(t-\tau) - x_i(t-\tau), i = 1, 2, \ldots, n$.

According to system (20), one has

$$\begin{aligned}
{}_0D_t^q e_i(t) = &-a_i e_i(t) + \sum_{j=1}^{n} b_{ij}(f_j(y_j(t)) - f_j(x_j(t))) \\
&+ \sum_{j=1}^{n} c_{ij}(g_j(y_j(t-\tau)) - g_j(x_j(t-\tau))). \quad (23)
\end{aligned}$$

Based on Lemma 3.5, $e_i(t)$ satisfies

$${}_0D_t^q |e_i(t)| \le \text{sgn}(e_i(t))\, {}_0D_t^q e_i(t), \quad 0 < q \le 1.$$

Let $V(t) = \sum_{i=1}^{n} |e_i(t)|$, then $V(t-\tau) = \sum_{i=1}^{n} |e_i(t-\tau)|$.

Calculating the fractional-order derivatives of $V(t)$ along the solutions of system (20), and using Lemma 3.5, one can get

$$\begin{aligned}
{}_0D_t^q V(t) &= \sum_{i=1}^{n} ({}_0D_t^q |e_i(t)|) \\
&\le \sum_{i=1}^{n} \text{sgn}(e_i(t))\, {}_0D_t^q e_i(t) \\
&= \sum_{i=1}^{n} \text{sgn}(e_i(t))\Bigg\{-a_i e_i(t) + \sum_{j=1}^{n} b_{ij}(f_j(y_j(t)) - f_j(x_j(t))) \\
&\quad + \sum_{j=1}^{n} c_{ij}(g_j(y_j(t-\tau)) - g_j(x_j(t-\tau)))\Bigg\} \\
&\le \sum_{i=1}^{n}\left(-a_i|e_i(t)| + \sum_{j=1}^{n} |b_{ij}L_j||e_j(t)| + \sum_{j=1}^{n} |c_{ij}K_j||e_j(t-\tau)|\right) \\
&= \sum_{i=1}^{n}\left(-a_i|e_i(t)| + \sum_{j=1}^{n} |b_{ji}L_i||e_i(t)|\right) + \sum_{i=1}^{n}\sum_{j=1}^{n} |c_{ji}K_i||e_i(t-\tau)| \\
&= \sum_{i=1}^{n}\left(-a_i + \sum_{j=1}^{n} |b_{ji}L_i|\right)|e_i(t)| + \sum_{i=1}^{n}\sum_{j=1}^{n} |c_{ji}K_i||e_i(t-\tau)| \\
&\le -\lambda V(t) + \widehat{K} V(t-\tau),
\end{aligned}$$

where $\widehat{K} = \max_{1 \le i \le n}(\sum_{j=1}^{n} |c_{ji}|K_i)$, $\lambda = \min_{1 \le i \le n}(a_i - \sum_{j=1}^{n} |b_{ji}|L_i)$.

Consider the following system:

$${}_0D_t^q W(t) = -\lambda W(t) + \widehat{K} W(t-\tau), \quad (24)$$

where $W(t) \ge 0 (W(t) \in R)$, and take the same initial conditions with $V(t)$.

Using Lemma 3.4, we have

$$0 < V(t) \le W(t) \quad (\forall t \in [0, +\infty)).$$

Note that there exists a unique zero equilibrium point in system (24).

When

$$\widehat{K} < \lambda \sin\left(\frac{q\pi}{2}\right), \quad 0 < q \le 1,$$

the characteristic equation $\det(\Delta(s)) = 0$ of system (24) has no purely imaginary roots for any τ. When $\tau = 0$, we obtain

$$\widehat{K} < \lambda \sin\left(\frac{q\pi}{2}\right) \le \lambda, \quad 0 < q \le 1,$$

then $\widehat{K} < \lambda$, $0 < q \le 1$. According to Theorem 3.2, the zero solution of system (24) is global Lyapunov asymptotically stable.

Because $0 < V(t) \le W(t)$, $V(t)$ is global Lyapunov asymptotically stable, i.e., $V(t) \to 0 (t \to +\infty)$. Then $V(t) = \sum_{i=1}^{n} |e_i(t)| \to 0$, and $|e_i(t)| \to 0$, which means that all the solutions of system (20) converge to the some one.

According to Theorem 4.1, the equilibrium point $x^*(t)$ is unique equilibrium point in system (20). That is to say, $x^*(t)$ also is a solution of system (20). Take $x(t) = x^*(t)$, then one has

$$\|y(t) - x^*(t)\| \to 0 \quad (t \to +\infty).$$

That is, $x^*(t)$ is uniformly attractive. Any different solution of system (20) converges to the $x^*(t)$.

Next, consider boundedness of all the solutions of system (20). Without loss of generality, assume that a solution of system (20) is $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$.

Let $\tau_{ij} = \tau$ and $\widehat{V}(t) = \sum_{i=1}^{n} |x_i(t)|$, then $\widehat{V}(t-\tau) = \sum_{i=1}^{n} |x_i(t-\tau)|$. Similarly, calculating the fractional-order derivatives of $\widehat{V}(t)$ along the solutions of system (20), and using Lemma 3.5, one has

$$\begin{aligned}
{}_0D_t^q\widehat{V}(t) &= \sum_{i=1}^{n}({}_0D_t^q|x_i(t)|)\\
&\le \sum_{i=1}^{n}\operatorname{sgn}(x_i(t))\,{}_0D_t^q x_i(t)\\
&= \sum_{i=1}^{n}\operatorname{sgn}(x_i(t))\left\{-a_ix_i(t)+\sum_{j=1}^{n}b_{ij}f_j(x_j(t)+\sum_{j=1}^{n}c_{ij}g_j(x_j(t-\tau))+d_i\right\}\\
&\le \sum_{i=1}^{n}\left(-a_i|x_i(t)|+\sum_{j=1}^{n}|b_{ij}L_j||x_j(t)|+\sum_{j=1}^{n}|c_{ij}K_j||x_j(t-\tau)|+d\right)\\
&= \sum_{i=1}^{n}\left(-a_i|x_i(t)|+\sum_{j=1}^{n}|b_{ji}L_i||x_i(t)|\right)+\sum_{i=1}^{n}\sum_{j=1}^{n}|c_{ji}K_i||x_i(t-\tau)|+d\\
&= \sum_{i=1}^{n}\left(-a_i+\sum_{j=1}^{n}|b_{ji}L_i|\right)|x_i(t)|+\sum_{i=1}^{n}\sum_{j=1}^{n}|c_{ji}K_i||x_i(t-\tau)|+d\\
&\le -\lambda\widehat{V}(t)+\widehat{K}V(t-\tau)+d, \qquad (25)
\end{aligned}$$

where $\widehat{K}=\max_{1\le i\le n}(\sum_{j=1}^{n}|c_{ji}|K_i)$, $\lambda=\min_{1\le i\le n}(a_i-\sum_{j=1}^{n}|b_{ji}|L_i)$ and $d=\max_{1\le i\le n}|d_i|$.

Consider the following system:

$${}_0D_t^q\overline{W}(t)=-\lambda\overline{W}(t)+\widehat{K}\overline{W}(t-\tau)+d$$

where $\overline{W}(t)\ge 0(\overline{W}(t)\in R)$, and take the same initial conditions with $\widehat{V}(t)$.

Due to Lemma 3.4, we have

$$0<\widehat{V}(t)\le\overline{W}(t)\quad(\forall t\in[0,+\infty)).$$

According to the property of Caputo fractional-order derivatives, we obtain

$${}_0D_t^q(\overline{W}(t)-\tilde{d})=-\lambda(\overline{W}(t)-\tilde{d})+\widehat{K}(\overline{W}(t-\tau)-\tilde{d}), \qquad (26)$$

where $\tilde{d}=d/(\lambda-\widehat{K})$.

Take $\hat{W}(t)=\overline{W}(t)-\tilde{d}$, then system (26) becomes

$${}_0D_t^q\hat{W}(t)=-\lambda\hat{W}(t)+\widehat{K}\hat{W}(t-\tau). \qquad (27)$$

When

$$\widehat{K}<\lambda\sin\left(\frac{q\pi}{2}\right),\quad 0<q\le 1,$$

the characteristic equation $\det(\Delta(s))=0$ of system (27) has no purely imaginary roots for any τ. When $\tau=0$, we have

$$\widehat{K}<\lambda\sin\left(\frac{q\pi}{2}\right)\le\lambda,\quad 0<q\le 1,$$

then $\widehat{K}<\lambda, 0<q\le 1$. According to Theorem 3.2, the zero solution of system (27) is global Lyapunov asymptotically stable.

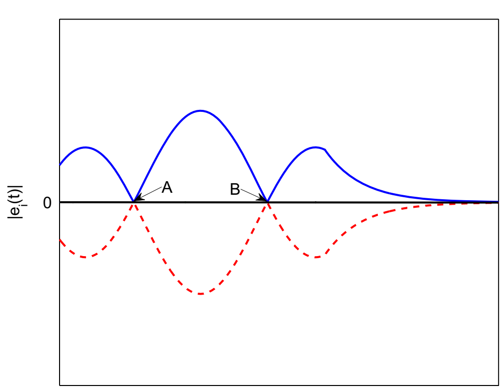

Fig. 1. A simple example of $|e_i(s)|$'s trajectory.

Hence,

$$\overline{W}(t)-\tilde{d}\to 0\quad(t\to+\infty).$$

For $0<\overline{W}(t)$ and $\forall\epsilon>0$, where ϵ is a sufficiently small number, we get

$$\overline{W}(t)<\tilde{d}+\epsilon.$$

Because

$$0<\widehat{V}(t)\le\overline{W}(t)<\tilde{d}+\epsilon,$$

one has

$$0<\widehat{V}(t)\le\tilde{d}. \qquad (28)$$

By the preceding condition $\widehat{V}(t)=\sum_{i=1}^{n}|x_i(t)|\le\tilde{d}$, with $t\to+\infty$, one has $\|x(t)\|\le\tilde{d}$, where $\|\bullet\|\in l_1$ norm. The boundedness of the solution $x(t)$ is given.

To sum up, all solutions of system (20) are bounded, and they converge to the unique equilibrium point x^*. So the system (20) is global asymptotically stable. This completes the proof. □

Remark 4. For Eq. (23), according to Caputo fractional-order derivative, if $|e_i(t)|$ is differentiable function almost everywhere, then ${}_0D_t^q|e_i(t)|\le\operatorname{sgn}(e_i(t))\,{}_0D_t^q e_i(t)$ is established almost everywhere in Lemma 3.5. Next we will discuss if $|e_i(t)|$ is indifferentiable function almost everywhere, then Lemma 3.5 is also right.

If the $|e_i(t)|$ is indifferentiable function almost everywhere, the indifferentiable points are at $|e_i(s)|=0$, $s\in\Omega$, where $\Omega\subset[0,+\infty)$ is not differentiable set. The indifferentiable points like points A and B are shown in Fig. 1. That is to say, $|e_i(s)|=0$ almost everywhere at Ω. Based on the Lusin Theorem [47], the $r(s)\equiv 0$ is obtained which $|e_i(s)|=r(s)$ is almost everywhere at Ω. Then

$$\begin{aligned}
{}_0D_t^q|e_i(t)| &= \frac{1}{\Gamma(1-q)}\int_0^t\frac{|e_i(t)|'}{(t-\tau)^q}\,d\tau\\
&= \frac{1}{\Gamma(1-q)}\int_{[0,t]/\Omega}\frac{|e_i(t)|'}{(t-\tau)^q}\,d\tau+\frac{1}{\Gamma(1-q)}\int_{\Omega}\frac{r'(t)}{(t-\tau)^q}\,d\tau\\
&= \frac{1}{\Gamma(1-q)}\int_{[0,t]/\Omega}\frac{|e_i(t)|'}{(t-\tau)^q}\,d\tau.
\end{aligned}$$

Here, the indifferentiable almost everywhere function $|e_i(t)|$ also satisfies Lemma 3.5.

Remark 5. Smoothing the $|e_i(s)|(s\in\Omega)$ is taken in Theorem 4.2, in fact, this is a generality method for indifferentiable functions. In addition, $e_i(t)$ is continuously differentiable function and asymptotically stable in Eq. (23), then $|e_i(t)|$ is always differentiable almost everywhere. And the indifferentiable points of $|e_i(s)|$ can be computed as $\sum_{i=1}^{n-2}C_n^i(n>2)$ at most, where C_n^i is a combinatorial number.

Remark 6. The boundedness of all the solutions of system (20) need to be considered. According to Eqs. (21) and (23), if the d_i is unbounded external input, the unique equilibrium point exists $x^*=\infty$, and all the solutions of system (20) still converge to the unique equilibrium point x^*. However, the unique equilibrium point $x^*=\infty$ is meaningless for the neural network.

Remark 7. According to Assumptions (A2), it can conclude the following condition which is given in [30]:

$$\max_{1\le i\le n}\left(\sum_{j=1}^{n}|c_{ji}|K_i\right)<\min_{1\le i\le n}(a_i)-\max_{1\le i\le n}\left(\sum_{j=1}^{n}|b_{ji}|L_i\right)$$
$$<\min_{1\le i\le n}\left(a_i-\sum_{j=1}^{n}|b_{ji}|L_i\right).$$

So the conditions of Theorem 4.2 are more general than their results.

Remark 8. Due to Assumptions (A1) and (A2), system (20) is global asymptotically stability, and the stability conditions do not contain delay τ. So stability conditions of system (20) are independent on the initial conditions and time delay.

5. Simulation

In this section, a numerical simulation is given by using MATLAB to illustrate the theoretical results of the paper. Without loss of generality, the initial conditions with random and periodic functions in the example are used. In addition, to solve the fractional-order differential equations with time delay, step-length $h=0.01$ in the Adams–Bashforth–Moulton predictor–corrector scheme is taken.

A fractional-order Hopfield neural network of four neurons with time delay is given as follows:

$$\begin{cases} {}_0^tD x_1(t) = -a_1x_1(t)+b_{11}\sin(x_1(t))+b_{12}\sin(x_2(t))+b_{13}\sin(x_3(t))+b_{14}\sin(x_4(t))+c_{11}\tanh(x_1(t-\tau)) \\ \quad +c_{12}\tanh(x_2(t-\tau))+c_{13}\tanh(x_3(t-\tau))+c_{14}\tanh(x_4(t-\tau))+d_1 \\ {}_0^tD x_2(t) = -a_2x_2(t)+b_{21}\sin(x_1(t))+b_{22}\sin(x_2(t))+b_{23}\sin(x_3(t))+b_{24}\sin(x_4(t))+c_{21}\tanh(x_1(t-\tau)) \\ \quad +c_{22}\tanh(x_2(t-\tau))+c_{23}\tanh(x_3(t-\tau))+c_{24}\tanh(x_4(t-\tau))+d_2 \\ {}_0^tD x_3(t) = -a_3x_3(t)+b_{31}\sin(x_1(t))+b_{32}\sin(x_2(t))+b_{33}\sin(x_3(t))+b_{34}\sin(x_4(t))+c_{31}\tanh(x_1(t-\tau)) \\ \quad +c_{32}\tanh(x_2(t-\tau))+c_{33}\tanh(x_3(t-\tau))+c_{34}\tanh(x_4(t-\tau))+d_3 \\ {}_0^tD x_4(t) = -a_4x_4(t)+b_{41}\sin(x_1(t))+b_{42}\sin(x_2(t))+b_{43}\sin(x_3(t))+b_{44}\sin(x_4(t))+c_{41}\tanh(x_1(t-\tau)) \\ \quad +c_{42}\tanh(x_2(t-\tau))+c_{43}\tanh(x_3(t-\tau))+c_{44}\tanh(x_4(t-\tau))+d_4 \end{cases} \tag{29}$$

The neural network parameters of system (29) are chosen as $q=0.96$, $\tau=3$, $d_1=0.3$, $d_2=-0.2$, $d_3=-0.1$, $d_4=0.4$,

$$A=\begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -2.5 & 0 \\ 0 & 0 & 0 & -3.8 \end{pmatrix}, B=\begin{pmatrix} 1 & -1.2 & 0.5 & 0.3 \\ -0.4 & 0.8 & -0.4 & -1 \\ 0.4 & -0.1 & -0.1 & 1.1 \\ -0.2 & 0.4 & -5.8 & 0.4 \end{pmatrix}$$

and

$$C=\begin{pmatrix} 0.1 & -0.5 & 0.15 & -0.2 \\ 0.3 & 0.1 & -0.25 & -0.5 \\ -0.1 & 0.15 & 0.1 & 0.1 \\ -0.4 & 0.2 & -0.4 & -0.15 \end{pmatrix}.$$

The initial values of $x_1(t)$, $x_2(t)$, $x_3(t)$ and $x_4(t)$ are chosen as $x_1(t)=\tilde{h}_1(t)$, $x_2(t)=\tilde{h}_2(t)$, $x_3(t)=\tilde{h}_3(t)$ and $x_4(t)=\tilde{h}_4(t)$, $t\in[-\tau,0]$, where $\tilde{h}_i(t)$ $(i=1,2,3,4)$ is a random function. Here, $\tilde{h}_i(t)$ $(i=1,2,3,4)$ is chosen as white Gaussian noise. Furthermore, the other initial values of $\hat{x}_1(t), \hat{x}_2(t)$ and $\hat{x}_3(t)$ are chosen as $\hat{x}_1(t)=\hat{h}_1(t)$, $\hat{x}_2(t)=\hat{h}_2(t)$, $\hat{x}_3(t)=\hat{h}_3(t)$ and $\hat{x}_4(t)=\hat{h}_4(t)$, $t\in[-\tau,0]$, where $\hat{h}_i(t)$ $(i=1,2,3,4)$ is a periodic function. Take $\hat{h}_1(t)=\hat{h}_3(t)=|\sin(t)|$ and $\hat{h}_2(t)=\hat{h}_4(t)=|\cos(t)|$. The unique equilibrium point of system (29) is $x^*=(0.2582,-0.0871,0.0561,0.0978)$. The convergent behaviors of system (29) are shown in Fig. 2.

6. Conclusion

In this paper, the global stability analysis for fractional-order Hopfield neural networks with time delay has been investigated. A stability theorem for linear fractional-order systems with time delay has been discussed. Furthermore, a comparison theorem for a class of fractional-order systems with time delay has been given. And the existence and uniqueness of the equilibrium point for the fractional-

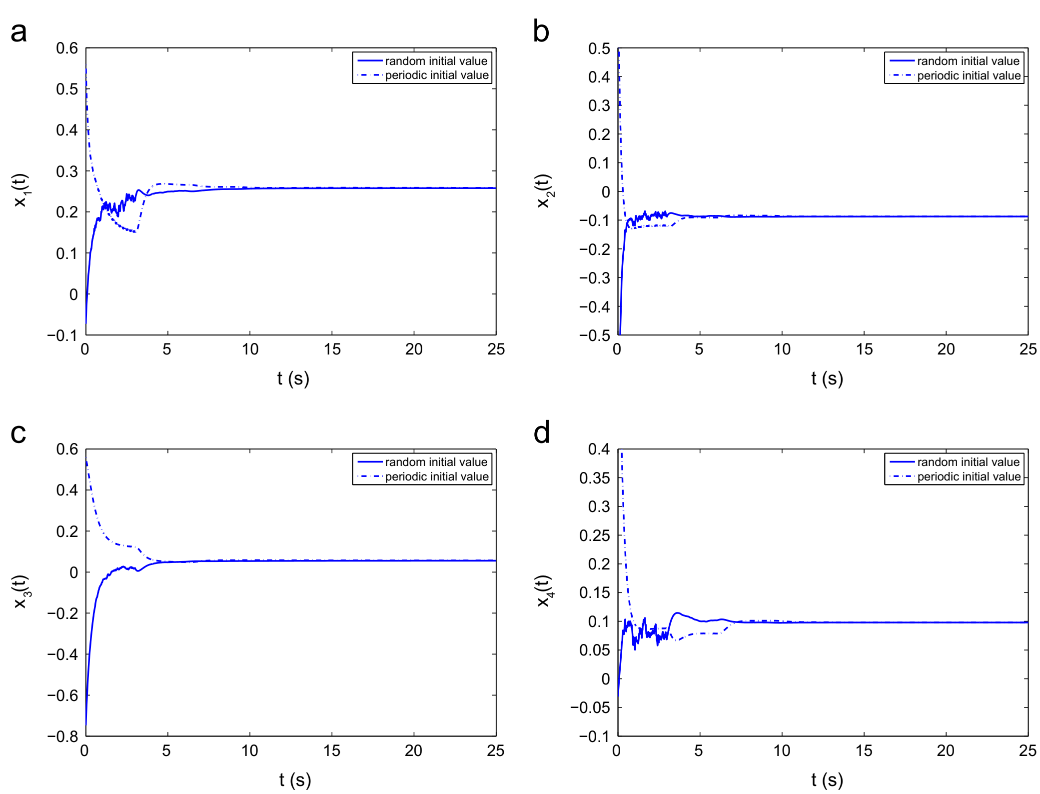

Fig. 2. The convergent behaviors of system (29). (a) The convergent behaviors of $x_1(t)$. (b) The convergent behaviors of $x_2(t)$. (c) The convergent behaviors of $x_3(t)$. (d) The convergent behaviors of $x_4(t)$.

order Hopfield neural networks with time delay have been proved by using the contraction mapping theorem. Finally, based on the above the results on the fractional-order systems with time delay, global asymptotic stability of fractional-order neural networks with time delay have been investigated, and the corresponding conditions for global asymptotic stability of fractional-order neural networks with time delay have been derived by using Lyapunov method.

There are some potential research directions that could be considered for the future works. Note that many stability conditions about fractional-order neural networks with time delay in the previously works and this paper are sufficient. So the sufficient and necessary stability condition of fractional-order neural networks with time delay will be studied in the future.

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