Seeking Consensus in Networks of Linear Agents: Communication Noises and Markovian Switching Topologies

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***Abstract*—The stochastic consensus problem of linear multi-input multi-output (MIMO) multi-agent systems (MASs) with communication noises and Markovian switching topologies is studied in this technical note. The agent's full state is first estimated by the state observer, and then the estimated state is exchanged with neighbor agents through a noisy communication environment. The communication topology is randomly switching and the switching law is described by a continuous-time Markovian chain. Then a consensus protocol is proposed for this MAS, and some sufficient conditions are obtained for ensuring the mean square and almost sure consensus. In addition, if the communication topology is fixed, some necessary and sufficient conditions for the mean square consensus can be obtained according to whether or not each agent in the system has parents.**

***Index Terms*—Communication noise, linear multi-agent system, Markovian switching topology, stochastic consensus.**

I. Introduction

Recently, the communication noise becomes an attractive topic in the field of consensus of multi-agent systems (MASs) [1]–[9]. In order to attenuate the effect of communication noises, a time-varying gain, namely the stochastic-approximation type gain, was first introduced in the consensus protocol proposed in [1]. In [2], it is proven that the stochastic-approximation type gain is not only sufficient but also necessary for ensuring the mean square consensus of first-order integral MASs. In [3], two kinds of communication constraints (communication noises and delays) were considered simultaneously, and the stochastic-approximation type consensus protocol was still valid in this situation. Some interesting applications (e.g., the distributed parameter estimation) of the stochastic consensus of MASs were done in [4], [5]. In [6], a new protocol was presented for the mean square consensus of second-order integral MASs with additive communication noises. It is proved that the stochastic approximation type conditions are still the necessary and sufficient conditions. Extensions to the sampled-data based consensus protocol were made in [7]. Although some results on stochastic consensus were published, the study regarding linear MASs with communication noises is still very rare. Recently, few early attempts towards this challenge were made in [8], [9] where the mean square consensus problems of continuous-time/discrete-time linear single-input single-output (SISO) MASs with fixed topologies were solved. However, many interesting questions are still left open such as the almost sure consensus and the consensus under switching topologies, which become the motivation of the study presented in this technical note.

In this technical note, the MAS works under randomly switching communication topologies. The switching signal is modeled by a homogeneous ergodic Markovian chain with right continuous trajectories. The transition matrix of this Markovian chain is assumed to be a "doubly stochastic generator matrix". Moreover, it is assumed that only the agent's output is available. A state observer is constructed to estimate the agent's state. The estimated state is sent to its neighbor agents for the purpose of reaching consensus. A consensus protocol is then proposed by combining the agent's own estimated state and the relative estimated states between the agent and its neighbors. A time-varying gain matrix $a(t)K_2$ is applied to the relative estimated states to attenuate the noise's effect. It is proved that the mean square and almost sure consensus can be solved by the proposed protocol if the following conditions hold: 1) all possible communication topology digraphs are balanced and the union of them has a spanning tree; 2) $\int_0^\infty a(t)dt = \infty$ and $\int_0^\infty a^2(t)dt < \infty$; 3) all roots of "parameters polynomials" have negative real parts. In addition, the stochastic consensus of MASs with the fixed topology is further discussed. According to whether or not each agent in the system has parents, some necessary and sufficient conditions are obtained for the mean square consensus of MASs, respectively.

This technical note is a continuation and improvement of the previous papers [6]–[9]. In [6] and [7], only the mean square consensus problems of first-order/second-order integral MASs were studied. Although the linear MAS was studied in [8], [9], this technical note still has some distinguished features which are summarized as follows: 1) the agent is described by the stabilizable linear MIMO dynamics rather than the controllable SISO dynamics; 2) both the randomly switching topology and the fixed topology are investigated; 3) the proposed protocol is based on the agent's output rather than its full state; and 4) the proposed protocol is able to solve not only the mean square consensus problem but also the almost sure consensus problem.

Notations: $1_n = (1, \ldots, 1)^T \in \mathbb{R}^n$; $0_n = (0, \ldots, 0)^T \in \mathbb{R}^n$; I_n denotes the $n \times n$ dimensional identity matrix; Θ denotes zero matrix with proper dimension; $\otimes$ and $\oplus$ denote the Kronecker product and Kronecker sum, respectively. For a given matrix X, X^T denotes its transpose; $\|X\|_1$, $\|X\|_2$ and $\|X\|_F$ denote its 1-norm, 2-norm and Frobenius norm, respectively; $\mathrm{tr}(X)$ denotes the trace of X; $\mathrm{null}(X)$ denotes the null space of X. For a linear space $\mathcal{H}$, its orthogonal complement space is denoted by $\mathcal{H}^\perp$. $\mathrm{diag}(\cdot)$ denotes a block diagonal matrix formed by its inputs. For a random variable/vector x, $E\{x\}$ denotes its mathematical expectation, $D\{x\}$ denotes its variance. This technical note is based on the complete probability space $(\Omega, \mathcal{F}, P)$ which is equipped with a filtration $\{\mathcal{F}_t, t \geq 0\}$. Throughout this technical note, "m.s." and "a.s." are abbreviations

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for mean square and almost sure, respectively. For a random process $x(t)$, $x(t) \xrightarrow{m.s.} x^*$ and $x(t) \xrightarrow{a.s.} x^*$ mean that $x(t)$ is m.s. and a.s. convergent to a random variable x^* as time goes to infinity, respectively. Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$ denote the field of complex number, the field of real number and the set of positive real number, respectively. For any $x \in \mathbb{C}$, $\Re(x)$ denotes its real part. Define an operator φ in the following way: for a matrix $V = [V_1, V_2, \ldots, V_n] \in \mathbb{C}^{l\times n}$ with $V_i \in \mathbb{C}^l$, $\varphi(V) = [V_1^T, \ldots, V_n^T]^T \in \mathbb{C}^{nl\times 1}$. For a function $f(x)$, $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = -\min\{f(x), 0\}$. For a real function $\theta(t) : \mathbb{R} \to \mathbb{R}$, its indicator function is defined by

$$1_{[\theta(t)=c]} = \begin{cases} 1, & \text{if } \theta(t) = c \\ 0, & \text{if } \theta(t) \neq c. \end{cases}$$

II. Preliminary Results, Problem Formulation and Consensus Protocol

This technical note considers a MAS composed of N agents. The ith agent is described by the following continuous-time linear MIMO time-invariant system

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t),\ y_i(t) = Cx_i(t) \tag{1}$$

where $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^m$ and $y_i(t) \in \mathbb{R}^r$ are the state, input and output of the ith agent, respectively

$$A = \begin{bmatrix} A_a & A_c \\ A_{ve} & A_v \end{bmatrix} \in \mathbb{R}^{n\times n} \tag{2}$$

where $A_a^T = [A_e^T, \Theta] \in \mathbb{R}^{l_e\times(n-l_v)}$, $A_e \in \mathbb{R}^{l_e\times l_e}$, $A_c = \mathrm{diag}(\Theta, A_{1,2}, A_{2,3}, \ldots, A_{v-1,v}) \in \mathbb{R}^{(n-l_v)\times(n-l_e)}$, $A_{i,i+1} = [I_{l_i}, \Theta] \in \mathbb{R}^{l_i\times l_{i+1}}$ $(i = 1, \ldots, v-1)$, $A_{v,e} \in \mathbb{R}^{l_v\times l_e}$, $A_v = [A_{v,1}, \ldots, A_{v,v}] \in \mathbb{R}^{l_v\times(n-l_e)}$, $A_{v,i} \in \mathbb{R}^{l_v\times l_i}$ $(i = 1, \ldots, v)$, $l_1 \le l_2 \le \cdots \le l_v = m$ and $\sum_{i=1}^{v} l_v + l_e = n$; $B = [\Theta, \ldots, \Theta, I_m]^T \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{r\times n}$. We assume that B is of full column rank. If it is not, one can refer to the "γ-equivalence" in [10] to deal with this challenge. It is worth noting that any linear MIMO system can be transformed into the Yokoyama canonical form defined by (1) in [10]. Through this technical note, it is assumed that (A, B) is stabilizable and (A, C) is detectable. Since (A, B) is stabilizable, all eigenvalues of A_e must have negative real parts.

In the literature, the communication among agents is modelled by a digraph $\mathcal{G} = \{\mathcal{V}_\mathcal{G}, \mathcal{E}_\mathcal{G}, \mathcal{A}_\mathcal{G}\}$, where $\mathcal{V} = \{v_1, \ldots, v_N\}$, $\mathcal{E}_\mathcal{G} \subseteq \mathcal{V} \times \mathcal{V} = \{e_{ij} | i, j = 1, \ldots, N\}$ and $\mathcal{A}_\mathcal{G} = [\alpha_{ij}] \in \mathbb{R}^{N\times N}$ are the node set, edge set, and adjacency matrix, respectively. Agent i is denoted by node v_i. The edge $e_{ij} \in \mathcal{E}_\mathcal{G}$ denotes the information flow from node j to node i. It is assumed that $e_{ij} \in \mathcal{E}_\mathcal{G} \Leftrightarrow \alpha_{ij} > 0$ and $e_{ij} \notin \mathcal{E}_\mathcal{G} \Leftrightarrow \alpha_{ij} = 0$. The neighborhood of v_i is defined by $\mathcal{N}_i = \{v_j | e_{ij} \in \mathcal{E}_\mathcal{G}\}$. If $v_j \in \mathcal{N}_i$, then v_j is called the parent node of v_i. The *in-degree* and *out-degree* of node i are defined by $\deg_{in}(v_i) = \sum_{j=1}^{N} \alpha_{ij}$ and $\deg_{out}(v_i) = \sum_{j=1}^{N} \alpha_{ji}$, respectively. The digraph $\mathcal{G}$ is called balanced if $\deg_{in}(v_i) = \deg_{out}(v_i)$ for $i = 1, \ldots, N$. The Laplacian matrix of $\mathcal{G}$ is defined by $\mathcal{L}_\mathcal{G} = \mathcal{D}_\mathcal{G} - \mathcal{A}_\mathcal{G}$ where $\mathcal{D}_\mathcal{G} = \mathrm{diag}(\deg_{in}(v_1), \ldots, \deg_{in}(v_N))$. In a digraph $\mathcal{G}$, a directed path from v_{i_1} to v_{i_n} is a sequence of end to end edges $\{e_{i_{j+1}i_j} \in \mathcal{E}_\mathcal{G},\ j = 1, \ldots, n-1\}$. A digraph $\mathcal{G}$ is said to contain a spanning tree if there exists a node from which there are directed paths to all other nodes.

Lemma 1 (Lemma 3.3 in [11]): The Laplacian matrix $\mathcal{L}_\mathcal{G}$ of a digraph $\mathcal{G}$ has at least one zero eigenvalue and all non-zero eigenvalues have positive real parts. And $\mathcal{L}_\mathcal{G}$ has only one zero eigenvalue with the associated eigenvector 1_N if and only if the digraph $\mathcal{G}$ has a spanning tree.

In practice, due to the link failure or packet loss, the communication topology is actually time-varying rather than time-invariant. To model this phenomenon, the communication topology can be described by a time-varying digraph $\mathcal{G}^{(\theta_t)} = \{\mathcal{V}_\mathcal{G}, \mathcal{E}_\mathcal{G}^{(\theta_t)}, \mathcal{A}_\mathcal{G}^{(\theta_t)} \triangleq (\alpha_{ij}^{(\theta_t)})\}$, where $\theta_t : [0, \infty) \to \mathcal{S} = \{1, 2, \ldots, s\}$ is a piecewise constant function and $\mathcal{S}$ denotes the index set of all possible graphs. The piecewise-constant function θ_t can be regarded as a switching signal. The communication topology is switched just at the instant that the value of θ_t is changed. In this technical note, this switching signal θ_t is modeled as a continuous-time Markovian chain adopted to the filtration $\{\mathcal{F}_t; t \ge 0\}$.

Once the communication topology is constructed, the agent can send its state to its neighbors for reaching consensus. However, the agent's full state $x_i(t)$ may not be unavailable due to the physical limitation or the implementation cost. To solve this problem, one possible way is to estimate the agent's state by its output. By this idea, the following observer is proposed:

$$\dot{\hat{x}}_i(t) = (A + K_eC)\hat{x}_i(t) + Bu_i(t) - K_ey_i(t) \tag{3}$$

where $K_e \in \mathbb{R}^{n\times r}$ is selected in such a way that $A + K_eC$ is a *Hurwitz* matrix. There must exist such a matrix K_e since (A, C) is detectable.

When transmitting the estimated state to neighbors via practical communication channels, the transmitted information is prone to be corrupted by communication noises. In this technical note, it is assumed that the estimated state received by agent i from agent j is $\nu_{ij}(t) = \hat{x}_j(t) + \Delta_{ij}\eta_{ij}(t)$ where $\eta_{ij}(t) = (\eta_{ij1}(t), \ldots, \eta_{ijn}(t))^T \in \mathbb{R}^n$ are the n-dimensional standard white noise, $\Delta_{ij} = \mathrm{diag}(\delta_{ij1}, \ldots, \delta_{ijn}) \in \mathbb{R}^{n\times n}$ and $\delta_{ijk} > 0$ $(i, j = 1, \ldots, N; k = 1, \ldots, n)$ are finite noise intensities. Additionally, it is assumed that $\{\eta_{ijk}(t); i, j = 1, \ldots, N; k = 1, \ldots, n\}$ are independent with each other and adapted to the filtration $\{\mathcal{F}_t, t \ge 0\}$.

Motivated by [1], [12], [13], the following protocol is proposed to deal with the consensus problem with communication noises:

$$u_i(t) = K_1\hat{x}_i(t) + a(t) \sum_{j\in\mathcal{N}_i} \alpha_{ij}^{(\theta_t)} K_2 \left(\nu_{ij}(t) - \hat{x}_i(t)\right) \tag{4}$$

where $a(t) \ge 0$ is the uniformly continuous gain function and $\alpha_{ij}^{(\theta_t)}$ is the ith row and the jth column element of $\mathcal{A}_\mathcal{G}^{(\theta_t)}$. The control gain matrices K_1 and K_2 have the following specific forms:

$$\begin{cases} K_1 = [-A_{v,e}, -A_{v,1}, -A_{v,2} - K_{1,1}, \ldots, -A_{v,v} - K_{1,v-1}] \\ K_2 = [\Theta, K_{2,1}, \ldots, K_{2,v-1}, I_{l_v}] \end{cases}$$

where $K_{1,i} = [K_{2,i}, \Theta] \in \mathbb{R}^{m\times l_{i+1}}$ and $K_{2,i} = [\mathrm{diag}(b_{i,1}I_{l_1}, b_{i,2}I_{l_2-l_1}, \ldots, b_{i,i}I_{l_i-l_{i-1}}), \Theta]^T \in \mathbb{R}^{m\times l_i}$, $i = 1, \ldots, v-1$. The parameters $\{b_{i,j} | i = 1, \ldots, v-1; j = 1, \ldots, i\}$ should be selected in such a way that $b_{i,j} = 1$ $(j \notin \mathcal{I};\ i = j, j+1, \ldots, v-1)$ and all roots of the following polynomials (we call them the "parameter polynomials") have negative real parts

$$s^{v-i} + b_{v-1,i}s^{v-i-1} + \cdots + b_{i+1,i}s + b_{i,i} = 0, \quad i \in \mathcal{I} \tag{5}$$

where $\mathcal{I} = \{1\} \bigcup \{i | 2 \le i \le v-1 \text{ and } l_{i-1} < l_i\}$. Therefore, one possible way to determine parameters $b_{i,j}$ in (4) is:

(i) Select $v - i$ negative real numbers $\{r_1, \ldots, r_{v-i}\}$ as the roots of (5).

(ii) Construct a polynomial $\prod_{k=1}^{v-i}(s - r_k) = 0$.

(iii) $b_{j,i}$ $(j = i, \ldots, v-1)$ can be set as the coefficient of s^{j-i} in $\prod_{k=1}^{v-i}(s - r_k)$.

Remark 1: Compared to the protocol in [12], the improvements of the proposed protocol are:

- A time-varying gain $a(t)$ is introduced in the proposed protocol to attenuate the noise's effect. Therefore, the closed-loop MAS becomes a time-variant system which cannot be dealt with by the analysis approach used in [12].
- K_1 and K_2 are only required to satisfy that all roots of (5) have the negative real parts, however, the minimum non-zero eigenvalue of $\mathcal{L}_\mathcal{G}$ should be known for the design of control gains in [12].

Finally, we give the mathematical definition of the mean square/almost sure consensus.

Definition 1: The protocol defined by (4) is said to be able to solve the mean square/almsot sure consensus problems of the linear MAS defined by (1) if there exists a random vector x^* satisfying $E\{\|x^*\|_2^2\} < \infty$ such that for $i = 1, \ldots, N$, $x_i(t) \xrightarrow{m.s.} x^*$ (mean square consensus) or $x_i(t) \xrightarrow{a.s.} x^*$ (almost sure consensus).

III. Main Results

Substituting (4) into (3) obtains that

$$\dot{\hat{X}}(t) = \left[I_N \otimes (A + BK_1) - a(t)\mathcal{L}_{\mathcal{G}}^{(\theta_t)} \otimes BK_2\right] \hat{X}(t) + a(t)\Sigma^{(\theta_t)}\eta(t) - (I_N \otimes K_eC)D(t)$$

where $\hat{X}(t) = (\hat{x}_1^T(t), \hat{x}_2^T(t), \ldots, \hat{x}_N^T(t))^T$, $\eta(t) = [\eta_{11}^T(t), \ldots, \eta_{1N}^T(t), \ldots, \eta_{NN}^T(t)]^T$, $\Sigma^{(\theta_t)} = \mathrm{diag}(\Sigma_1^{(\theta_t)}, \Sigma_2^{(\theta_t)}, \ldots, \Sigma_N^{(\theta_t)})$, $\Sigma_i^{(\theta_t)} = BK_2(\alpha_{i1}^{(\theta_t)}\Delta_{i1}, \alpha_{i2}^{(\theta_t)}\Delta_{i2}, \ldots, \alpha_{iN}^{(\theta_t)}\Delta_{iN})$, $\mathcal{L}_{\mathcal{G}}^{(\theta_t)}$ is the Laplacian matrix of $\mathcal{G}^{(\theta_t)}$, $D(t) = (d_1^T(t), \ldots, d_N^T(t))^T$ and $d_i(t) = x_i(t) - \hat{x}_i(t)$. By (1) and (3), it follows that $d_i(t) = e^{(A+K_eC)t}d_i(0) \to 0_n$.

By the definitions of K_1 and K_2, we can obtain that $K_2(A + BK_1) = \Theta$ and $K_2BK_2 = K_2$. Let $Z(t) \triangleq (z_1(t), \ldots, z_N(t))^T$ and $z_i(t) = K_2\hat{x}_i(t)$, then the following system is obtained:

$$\dot{Z}(t) = -a(t)\left[\left(\mathcal{L}_{\mathcal{G}}^{(\theta_t)} \otimes I_m\right)Z(t) - (I_N \otimes K_2)\Sigma^{(\theta_t)}\eta(t)\right] - \tilde{D}(t) \tag{6}$$

where $\tilde{D}(t) = (I_N \otimes (K_2K_eCe^{(A+K_eC)t}))D(0)$.

Before further discussion, the following four conditions are introduced, which play fundamental roles in the study of stochastic consensus.

(C1): All possible digraphs $\{\mathcal{G}^{(i)}; 1 \leq i \leq s\}$ are balanced and the union of all those digraphs has a spanning tree.

(C2): $\int_0^\infty a(t)dt = \infty$ and $\int_0^\infty a^2(t)dt < \infty$ (e.g. $a(t) = 1/(t+1)$).

(C3): $\{\theta_t, \mathcal{F}_t, t \geq 0\}$ is a homogeneous ergodic Markovian chain with right continuous trajectories, taking values on the set $\{1, 2, \ldots, s\}$. The transition matrix of this Markovian chain is denoted by $Q = [q_{ij}] \in \mathbb{R}^{s\times s}$ which is a "doubly stochastic generator matrix" (i.e., $\sum_{i,i\neq j} q_{ij} = \sum_{i,i\neq j} q_{ji}$).

(C4): All roots of the parameter polynomials defined by (5) have the negative real parts.

Theorem 1: If Conditions (C1), (C2), and (C3) hold, then there exists a random vector z^* with finite second-order moment such that $z_i(t) \xrightarrow{m.s.} z^*$ and $z_i(t) \xrightarrow{a.s.} z^*$, $i = 1, \ldots, N$.

Proof: See Appendix A. ■

Lemma 2: Consider the following stochastic differential equation:

$$\xi^{(p)}(t) + \sum_{i=1}^{p} b_i\xi^{(i-1)}(t) = \zeta(t)$$

where $\zeta(t)$ is a m.s. continuous random process, $\zeta(t) \xrightarrow{m.s.} \zeta^*$ ($\zeta(t) \xrightarrow{a.s.} \zeta^*$), and ζ^* is a random vector satisfying $E\{\|\zeta^*\|_2^2\} < \infty$.

(I) If all roots of $s^p + \sum_{i=1}^p b_is^{i-1} = 0$ have negative real parts, then $\xi(t) \xrightarrow{m.s.} \zeta^*/b_1$ ($\xi(t) \xrightarrow{a.s.} \zeta^*/b_1$) and $\xi^{(i)}(t) \xrightarrow{m.s.} 0$ ($\xi^{(i)}(t) \xrightarrow{a.s.} 0$), $i = 1, \ldots, p$.

(II) If $E\{\zeta^*\} \neq 0$ and for any initial state, there exists a random vector $v \in \mathbb{R}^{p+1}$ satisfying $E\{\|v\|_2^2\} < \infty$ such that $(\xi(t), \xi^{(1)}, \ldots, \xi^{(p)})^T \xrightarrow{m.s.} v$, then all roots of $s^p + \sum_{i=1}^p b_is^{i-1} = 0$ have negative real parts.

Proof (**I**)*:* Let $\mathfrak{D}$ denote the differential operator, namely $\mathfrak{D}^n\xi(t) = \xi^{(n)}(t)$. Let $\{r_i | i = 1, \ldots, p\}$ denote the roots of $s^p + \sum_{i=1}^p b_is^{i-1} = 0$. Let $\xi_1(t) = \prod_{i=2}^p(\mathfrak{D} - r_i)\xi(t)$, then $\dot{\xi}_1(t) = r_1\xi_1(t) + \zeta(t)$. It follows that $\xi_1(t) = \xi_1(0)e^{r_1t} + \psi(t)$, where $\psi(t) = \int_0^t \zeta(t)e^{r_1(t-s)}ds$. According to the properties of m.s. integral, it can be obtained that

$$E\left\{\left|\psi(t) - \int_0^t \zeta^*e^{r_1(t-s)}ds\right|^2\right\} \leq \left\{\int_0^t E^{\frac{1}{2}}\left\{|(\zeta(t) - \zeta^*)|^2\right\}e^{\Re(r_1)(t-s)}ds\right\}^2.$$

By L'Hôpital's rule, it follows that:

$$\lim_{t\to\infty}\int_0^t E^{\frac{1}{2}}\left\{|(\zeta(s) - \zeta^*)|^2\right\}e^{\Re(r_1)(t-s)}ds = -\lim_{t\to\infty}\frac{E^{\frac{1}{2}}\left\{|(\zeta(t) - \zeta^*)|^2\right\}}{\Re(r_1)} = 0.$$

Meanwhile, it is noted that $\lim_{t\to\infty}\int_0^t\zeta^*e^{r_1(t-s)}ds = -\zeta^*/r_1$. Hence, $\xi_1(t) \xrightarrow{m.s.} -\zeta^*/r_1$. By repeating this procedure for p times, it can be proved that $\xi_i(t) \triangleq \prod_{j=i+1}^p(\mathfrak{D} - r_i)\xi(t) \xrightarrow{m.s.} \zeta^*/(\prod_{j=1}^i(-r_i))$, which leads to that $\xi(t) \xrightarrow{m.s.} \zeta^*/b_1$ and $\xi^{(i)}(t) \xrightarrow{m.s.} 0, i = 1, \ldots, p$.

If $\zeta(t)$ is a.s. convergent to ζ^*, then there exists a subset $\Omega_0 \subset \Omega$ (Ω is the sample space) such that $P\{\Omega_0\} = 0$ and $\zeta(t)$ is convergent to ζ^* on the set $\Omega \setminus \Omega_0$. Since all roots of $s^p + \sum_{i=1}^p b_is^{i-1} = 0$ have the negative real parts, by the knowledge of differential equation, $\lim_{t\to\infty}\xi(t) = \zeta^*/b_1$ and $\lim_{t\to\infty}\xi^{(i)}(t) = 0$ on $\Omega \setminus \Omega_0$, which means $\xi(t) \xrightarrow{a.s.} \zeta^*/b_1$ and $\xi^{(i)}(t) \xrightarrow{a.s.} 0, i = 1, \ldots, p$.

(**II**): It is clear that there exists a random variable ξ_1^* such that $E\{|\xi_1^*|_2^2\} < \infty$ and $\xi_1(t) \xrightarrow{m.s.} \xi_1^*$, where $\xi_1(t)$ is defined in Proof (I). This leads to that $\Re(r_1) \leq 0$. If $\Re(r_1) = 0$, then it is obtained that $E\{|\xi_1^*|^2\} \geq |E\{\xi_1^*\}|^2 = \lim_{t\to\infty}|\xi_1(0)e^{r_1t} + E\{\psi(t)\}|^2 = \infty$, which contradicts $E\{|\xi_1^*|_2^2\} < \infty$. Hence, $\Re(r_1) < 0$. By repeating this procedure for p times, it can be obtained that all roots of $s^p + \sum_{i=1}^p b_is^{i-1} = 0$ have negative real parts. ■

Theorem 2: The proposed protocol (4) can solve the m.s. and a.s. consensus problems of linear MASs defined by (1) if Conditions (C1), (C2), (C3), and (C4) hold.

Proof: Let $x_i(t) = ({}^ex_i^T(t), {}^1x_i^T(t), \ldots, {}^vx_i^T(t))^T$ where ${}^jx_i(t) = ({}^jx_{i1}(t), \ldots, {}^jx_{ij}(t))^T \in \mathbb{R}^{l_j}$, ${}^jx_{i1}(t) \in \mathbb{R}^{l_1}$ and ${}^jx_{ik}(t) \in \mathbb{R}^{l_k - l_{k-1}}$ ($i = 1, \ldots, j; j = 1, \ldots, v$). It is easy to see that ${}^e\dot{x}_i(t) = A_e{}^ex_i(t)$, which indicates that ${}^ex_i(t) = e^{A_et}({}^ex_i(0)) \to 0_{l_e}$ as t goes to ∞.

By Lemma 1, if Conditions (C1), (C2), and (C3) hold, then $K_2\hat{x}_i(t) \xrightarrow{m.s.} z^*$ and $K_2\hat{x}_i(t) \xrightarrow{a.s.} z^*$ $(i = 1, \ldots, N)$. This together with $\lim_{t\to\infty} x_i(t) - \hat{x}_i(t) = 0_n$ leads to that $K_2x_i(t) \xrightarrow{m.s.} z^*$ and $K_2x_i(t) \xrightarrow{a.s.} z^*$. Denote $z^* = (z_1^{*T}, \ldots, z_v^{*T})^T$, where $z_1^* \in \mathbb{R}^{l_1}$ and $z_i^* \in \mathbb{R}^{l_i - l_{i-1}}$ $(i = 2, \ldots, v)$. Then, $(\mathfrak{D}^{v-j} + \sum_{k=j}^{v-1} b_{k,j}\mathfrak{D}^{k-j})({}^jx_{ij}(t))$ is convergent to z_j^* in the m.s./a.s. sense, where $j \in \mathcal{I}$ and $\mathcal{I}$ is defined in (5). By Lemma 2, if Condition (C4) holds, then ${}^jx_{ij}(t)$ is convergent to z_j^*/b_{jj} and ${}^{j+k}x_{ij}(t) = \mathfrak{D}^k({}^jx_{ij}(t))$ $(k = 1, \ldots, v - j)$ are convergent to $0_{l_j - l_{j-1}}$ in the m.s./a.s. sense. Therefore, $x_i(t)$ is convergent to $x^* \triangleq (0_{l_e}, {}^1x^{*T}, \ldots, {}^vx^{*T})^T$ in the m.s./a.s. sense, where ${}^ix^* \triangleq (0_{l_{i-1}}^T, z_i^{*T}/b_{ii})^T$ $(i = 1, \ldots, v - 1)$ and ${}^vx^* = (0_{l_{v-1}}^T, z_v^{*T})^T$. ■

By Lemma 2 and Theorem 2, we know that the selection of $b_{i,j}$ can influence the convergence rate of consensus. Qualitatively speaking, the larger the distance between the roots of the parameter polynomials and the imaginary axis, the faster the convergence rate. Furthermore, $b_{i,j}$ can partially determine the mathematical expectation of final consensus value by the proof of Theorem 2.

IV. Discussions on Fixed Communication Topologies

In Section III, all possible digraphs are required to be balanced. In fact, this requirement is unnecessary for MASs with fixed communication topologies. In this section, we discuss the stochastic consensus problem under the fixed topology. Define four sets: $\mathcal{T}=\{\mathcal{G}|\mathcal{G}$ is a digraph which has N nodes$\}$, $\mathcal{T}_1=\{\mathcal{G}\in\mathcal{T}|\mathcal{G}$ does not contain any spanning tree$\}$, $\mathcal{T}_2=\{\mathcal{G}\in\mathcal{T}|\mathcal{G}$ contains a spanning tree and everynode of $\mathcal{G}$ has at least one parent node$\}$ and $\mathcal{T}_3=\{\mathcal{G}\in\mathcal{T}|\mathcal{G}$ contains a spanning tree and there exists one node of $\mathcal{G}$ which has no parent node$\}$. In the literature, the agent denoted by the node without any parent node is called the leader. It is easy to see that $\mathcal{T}=\mathcal{T}_1\bigcup\mathcal{T}_2\bigcup\mathcal{T}_3$ and $\mathcal{T}_i\bigcap\mathcal{T}_j=\varnothing$, $1\le i\neq j\le 3$. When the MAS works under the topology $\mathcal{G}\in\mathcal{T}_2$, the MAS is called the leaderless MAS. When the MAS works under the topology $\mathcal{G}\in\mathcal{T}_3$, the MAS is called the leader-follower MAS.

Since the communication topology is fixed, the protocol is changed into

$$u_i(t)=K_1\hat{x}_i(t)+a(t)\sum_{j\in\mathcal{N}_i}\alpha_{ij}K_2\left(\nu_{ij}(t)-\hat{x}_i(t)\right). \tag{7}$$

Consequently, the auxiliary system (6) becomes the following form:

$$\dot{Z}(t)=-a(t)\left[(\mathcal{L}_{\mathcal{G}}\otimes I_m)Z(t)-(I_N\otimes K_2)\Sigma\eta(t)\right]-\tilde{D}(t). \tag{8}$$

By Lemma 2 and the proof of Theorem 2, the m.s. consensus problem is solved if and only if Condition (C4) holds and there exists a random vector $z^*\in\mathbb{R}^m$ such that $E\{\|z^*\|_2^2\}<\infty$ and $Z(t)\xrightarrow{m.s.}1_N\otimes z^*$.

Case I—$\mathcal{G}\in\mathcal{T}_1$: The m.s. consensus problem cannot be solved.

Proof: By Lemma 1, there exist at least two left eigenvectors $v_1,v_2\in\mathbb{R}^N$ of $\mathcal{L}_{\mathcal{G}}$ associated with zero eigenvalue such that $v_1\neq v_2$ and $v_1^T1_N=v_2^T1_N=1$. Let $\mu(t)=((v_1-v_2)^T\otimes I_m)E\{Z(t)\}$, then $\mu(t)=\mu(0)-((v_1-v_2)^T\otimes I_m)\int_0^t\tilde{D}(s)ds$. Assume $Z(t)\xrightarrow{m.s.}1_N\otimes z^*$, where $z^*\in\mathbb{R}^m$ is a random vector. It is easy to see that

$$\begin{aligned}\lim_{t\to\infty}\mu(t)&=\mu(0)-\left((v_1-v_2)^T\otimes I_m\right)\int_0^\infty\tilde{D}(s)ds\\&=(v_1-v_2)^T1_NE\{z^*\}=0_m.\end{aligned}$$

This together with the arbitrariness of the initial states $Z(0)$ and $D(0)$ leads to $v_1=v_2$ which contradicts $v_1\neq v_2$. Hence, the m.s. consensus problem can not be solved. ■

Case II—$\mathcal{G}\in\mathcal{T}_2$: The m.s. consensus problem can be solved if and only if Conditions (C2) and (C4) hold.

Proof (Sufficiency): By Itô's integral formula, the solution to (8) is $Z(t)=\Xi_1(t)+\Xi_2(t)$, where $\Xi_1(t)=\Phi(t,0)Z(0)-\int_0^t\Phi(t,s)\tilde{D}(s)ds$; $\Xi_2(t)=\int_0^t\Phi(t,s)a(s)(I_N\otimes K_2)\Sigma dW(s)$; $\Phi(t,t_0)$ is the state transition matrix of (8); and $\{W(t),\mathcal{F}_t\}$ is the nN^2-dimensional standard Brownian motion. Let $\Psi(t,t_0)$ denote the state transition matrix of differential equation $\dot{\xi}(t)=-a(t)\mathcal{L}_{\mathcal{G}}\xi(t)$. Then, it is easy to see that $\Phi(t,t_0)=\Psi(t,t_0)\otimes I_m$. According to [14], if $\mathcal{G}\in\mathcal{T}_2$ and Condition (C2) holds, then $\lim_{t\to\infty}\Psi(t,t_0)=1_N\kappa^T\triangleq\Psi_\infty$ where κ is the left eigenvector of $\mathcal{L}_{\mathcal{G}}$ associated with eigenvalue zero and $\kappa^T1_N=0$. Moreover, $\Psi(t,t_0)$ is uniformly continuous and uniformly bounded with respect to time.

Therefore

$$\begin{aligned}\lim_{t\to\infty}\Xi_1(t)&=(\Psi_\infty\otimes I_m)Z(0)-\int_0^\infty(\Psi_\infty\otimes I_m)\tilde{D}(s)ds\\&=\left[\Psi_\infty\otimes\left(K_2K_3C\int_0^\infty e^{(A+K_3C)t}dt\right)\right]D(0).\end{aligned}$$

By the same approach used in [6], it can be proved that there exists a random vector $\nu^*\in\mathbb{R}^m$ such that $E\{\|\nu^*\|_2^2\}<\infty$ and $\lim_{t\to\infty}E\times\{\|\Xi_2(t)-1_N\otimes\nu^*\|_2^2\}=0$. Let $z^*=(\kappa^T\otimes I_m)Z(0)+[\kappa\otimes(K_2K_3C\int_0^\infty e^{(A+K_3C)t}dt)]D(0))+\nu^*$, then $E\{\|z^*\|_2^2\}<\infty$ and $Z(t)\xrightarrow{m.s.}1_N\otimes z^*$.

(*Necessity*) Let $v\in\mathbb{R}^N$ denote a left eigenvector of $\mathcal{L}_{\mathcal{G}}$ associated with a nonzero eigenvalue λ. It is easy to see that $v^T1_N=0$. By (8), it is obtained that

$$\begin{aligned}(v^T\otimes I_m)\lim_{t\to\infty}E\{Z(t)\}&=e^{-\lambda\int_0^\infty a(s)ds}(v^T\otimes I_m)Z(0)\\&\quad-\lim_{t\to\infty}\int_0^t e^{-\lambda\int_s^t a(\tau)d\tau}(v^T\otimes I_m)\tilde{D}(s)ds\\&=v^T1_NE\{z^*\}=0_m.\end{aligned}$$

This together with the arbitrariness of the initial state $Z(0)$ and $D(0)$ leads to $\int_0^\infty a(s)ds=\infty$.

Let $\kappa\in\mathbb{R}^N$ denote the left eigenvector of $\mathcal{L}_{\mathcal{G}}$ associated eigenvalue zero. Denote $\mu(t)=(\kappa\otimes 1_m)^TZ(t)$, then $D\{\mu(t)\}=(\kappa^T\otimes 1_m^TK_2)\Sigma\Sigma^T(\kappa\otimes K_2^T1_m)\int_0^ta^2(s)ds$. If the m.s. consensus problem is solved, then $\lim_{t\to\infty}D\{\mu(t)\}=(\kappa^T\otimes 1_m^TK_2)\Sigma\Sigma^T(\kappa\otimes K_2^T1_m)\int_0^\infty a^2(s)ds<\infty$, which leads to that $\int_0^\infty a^2(s)ds<\infty$.

The necessity of Condition (C4) can be easily proved by Lemma 2. ■

Case III—$\mathcal{G}\in\mathcal{T}_3$: The m.s. consensus problem can be solved if and only if Condition (C4) and the following Condition (C2') hold.

(C2'). $\int_0^\infty a(t)dt=\infty$ and $\lim_{t\to\infty}a(t)=0$.

Proof (Sufficiency): Without loss of generality, assume that node 1 is the agent without any parent node. Then the Laplacian matrix has the following form:

$$\mathcal{L}_{\mathcal{G}}=\begin{bmatrix}0 & 0_{N-1}^T\\ L_1 & L_2\end{bmatrix} \tag{9}$$

which follows that $\dot{z}_1(t)=-K_2K_3Ce^{(A+K_3C)t}d_1(0)$. Thus, $z_1(t)$ is convergent to $z_1^*\triangleq-\int_0^\infty K_2K_3Ce^{(A+K_3C)t}d_1(0)dt$.

Let $\delta(t)=(\hat{z}_2^T(t),\hat{z}_3^T(t),\ldots,\hat{z}_N^T(t))^T$ where $\hat{z}_i(t)=z_i(t)-z_1(t)$, then it can be obtained that $\delta(t)=\Xi_3(t)+\Xi_4(t)$, where $\Xi_3(t)=(\Phi(t,0)\otimes I_m)\delta(0)-\int_0^t(\Phi(t,s)\otimes I_m)\hat{D}(s)ds$, $\Xi_4(t)=\int_0^t(\Phi(t,s)\otimes I_m)a(s)(I_{N-1}\otimes K_2)\hat{\Sigma}d\hat{W}(s)$, $\Phi(t,t_0)$ is the state transition matrix of $\dot{\xi}(t)=-a(t)L_2\xi(t)$, $\hat{\Sigma}=\mathrm{diag}(\Sigma_2,\ldots,\Sigma_N)$, $\hat{D}(t)=(I_{N-1}\otimes K_2K_eC)\mathrm{diag}(d_2(t),\ldots,d_N(t))$ and $\{\hat{W}(t),\mathcal{F}_t\}$ is the $nN(N-1)$-dimensional standard Brownian motion. By [14], we know if Condition (C2') holds, then $\lim_{t\to\infty}\Phi(t,t_0)=\Theta$ and $\Phi(t,t_0)$ is uniformly bounded. Hence, it can be proved that $\lim_{t\to\infty}\Xi_3(t)=0_{m(N-1)}$. By using the same techniques in the proof of Theorem 1 in [14], it can be obtained that $\Xi_4(t)\xrightarrow{m.s.}0_{m(N-1)}$. Therefore, $z_i(t)\xrightarrow{m.s.}z_1^*$, $i=1,\ldots,N$.

(*Necessity*) If the m.s. consensus problem is solved, then $\delta(t)\xrightarrow{m.s.}0_{m(N-1)}$ which follows that: $\lim_{t\to\infty}\Phi(t,0)=\Theta$. According to [14], if $\int_0^\infty a(t)dt<\infty$, then $\lim_{t\to\infty}\Phi(t,0)\neq\Theta$. Therefore, $\int_0^\infty a(t)dt=\infty$ is necessary.

Let p denote a eigenvector of L_2 associated with eigenvalue λ. If the m.s. consensus problem is solved, then it can be obtained that $\lim_{t\to\infty}E\{|(p\otimes 1_m)^T\delta(t)|^2\}=\lim_{t\to\infty}\|(p^T\otimes 1_m)K_2\hat{\Sigma}\|_2^2\int_0^te^{-2\lambda\int_\tau^ta(s)ds}a^2(\tau)d\tau=0$, which implies that $g(t)=\int_0^te^{-2\lambda\int_\tau^ta(s)ds}a^2(\tau)d\tau\to 0$ as t goes to ∞. Since $a(t)\ge 0$ is a uniformly continuous function, by the proof of Theorem 1 in [14], if $a(t)\nrightarrow 0$, then $g(t)\nrightarrow 0$. Therefore, $\lim_{t\to\infty}a(t)=0$.

The necessity of Condition (C4) can be easily proved by Lemma 2. ■

Since $a(t)\ge 0$ is a uniformly continuous function, it is easy to see that $\int_0^\infty a^2(t)dt<\infty$ implies that $\lim_{t\to\infty}a(t)=0$. Therefore, Condition (C2') is weaker than Condition (C2).

Remark 2: By Condition (C2) and the properties of $\Phi(t, t_0)$, it is clear that $\{\Xi_2(t)|\mathcal{F}_t\}$, $\{\Xi_4(t)|\mathcal{F}_t\}$ are martingales and $\sup_{t\geq 0} E\{\|\Xi_2(t)\|_2^2\} < \infty$, $\sup_{t\geq 0} E\{\|\Xi_4(t)\|_2^2\} < \infty$. By the martingale convergence theorem, it can be proved that $\Xi_2(t)$ and $\Xi_4(t)$ are a.s. convergent to some variable vectors. This together with the proofs of Case II and Case III follows that: (i) for the leaderless MAS, the a.s. consensus problem can be solved if Conditions (C2) and (C4) hold; (ii) for the leader-follower MAS, the a.s. consensus problem can be solved if Conditions (C2) and (C4) hold.

V. Conclusion

In this technical note, a dynamic output-feedback based protocol is proposed to solve the stochastic consensus problem of generic linear MIMO MASs with communication noises and Markovian switching topologies. It is shown that the mean square and almost sure consensus problems can be solved by the proposed protocol if the time-varying gain $a(t)$ satisfies $\int_0^\infty a(t)dt = \infty$ and $\int_0^\infty a^2(t)dt < \infty$; all roots of "parameter polynomials" have the negative real parts; all possible topology graphs are balanced and the union of them has a spanning tree; and the continuous-time Markovian chain is homogeneous and ergodic, whose transition matrix is a doubly stochastic generator matrix. Furthermore, for the fixed topology case, the sufficient and necessary conditions are obtained for ensuring the mean square consensus of the leaderless MAS and the leader-follower MAS, respectively.

Appendix

The Mean Square Case: The proof is divided into three parts.

Part I: This part is motivated by [15], and some techniques used in [15] are borrowed here to prove this theorem. Let $\delta(t) = Z(t) - JZ(t) = (I_{mN} - J)Z(t)$, $V(t) = E\{\delta(t)\delta^T(t)\}$ and $V_i = E\{\delta(t)\delta^T(t)1_{[\theta_t=i]}\}$, where $J = (1/N)1_N 1_N^T \otimes I_m$. Then it is obtained by the Itô's formula and [16, Lemma 4.2] that

$$\dot{V}_i(t) = -a(t)\left(\left(\mathcal{L}_\mathcal{G}^{(i)} \otimes I_m\right)V_i(t) + V_i(t)\left(\mathcal{L}_\mathcal{G}^{(i)T} \otimes I_m\right)\right) + \sum_{j=1}^s q_{ji}V_j(t) + \gamma_i(t) + \gamma_i^T(t) + a^2(t)R_i(t)$$

where $\gamma_i(t) = -E\{\delta(t)1_{[\theta_t=i]}\}\tilde{D}^T(t)(I_{mN} - J)$, $R_i(t) = (I_{mN} - J)(I_N \otimes K_2)\Sigma^{(i)}\Sigma^{(i)T}(I_{mN} \otimes K_2^T)(I_{mN} - J)p_i(t)$ and $p_i(t) = P(\theta_t = i)$.

Let $\bar{V}(t) = [V_1(t), V_2(t), \ldots, V_s(t)]^T$, then

$$\dot{\varphi}\left(\bar{V}(t)\right) = \left(-a(t)\Gamma + Q^T \otimes I_{m^2N^2}\right)\varphi\left(\bar{V}(t)\right) + \varphi\left(\gamma(t)\right) + a^2(t)\varphi\left(R(t)\right)$$

where $\Gamma = \mathrm{diag}((\mathcal{L}_\mathcal{G}^{(1)} \otimes I_m) \oplus (\mathcal{L}_\mathcal{G}^{(1)} \otimes I_m), \ldots, (\mathcal{L}_\mathcal{G}^{(s)} \otimes I_m) \oplus (\mathcal{L}_\mathcal{G}^{(s)} \otimes I_m))$, $\gamma(t) = [\gamma_1(t) + \gamma_1^T(t), \ldots, \gamma_s(t) + \gamma_s^T(t)]$, $R(t) = [R_1(t), \ldots, R_s(t)]$. Therefore, it can be obtained that

$$\begin{aligned} d&\left\|\varphi\left(\bar{V}(t)\right)\right\|_2^2 \\ &= 2\varphi^T\left(\bar{V}(t)\right)\left(-a(t)\hat{\Gamma} + \hat{Q} \otimes I_{m^2N^2}\right)\varphi\left(\bar{V}(t)\right)dt \\ &\quad + 2\varphi^T\left(\bar{V}(t)\right)\varphi\left(\gamma(t)\right)dt + 2a^2(t)\varphi^T\left(\bar{V}(t)\right)\varphi\left(R(t)\right)dt \end{aligned} \tag{10}$$

where $\hat{\Gamma} = (\Gamma + \Gamma^T)/2$ and $\hat{Q} = (Q + Q^T)/2$.

By Conditions (C1), (C3), Lemma 1 and Lemma 3.5 in [17], it is easy to see that the null space of $\hat{\Gamma} - \hat{Q} \otimes I_{N^2}$ is $\mathfrak{N}_1 = \{1_{sN} \otimes v_1 \otimes 1_N \otimes v_2 | v_1, v_2 \in \mathbb{R}^m\}$. Therefore, it follows from [18, p. 178] that

$$\min_{x\neq 0, x\in\mathfrak{N}_1^\perp}\left\{\frac{x^T(\hat{\Gamma} - \hat{Q} \otimes I_{m^2N^2})x}{\|x\|_2^2}\right\} = \lambda_2$$

where λ_2 denotes the smallest nonzero eigenvalue of $\hat{\Gamma} - \hat{Q} \otimes I_{m^2N^2}$. It is noted that $(1_{sN} \otimes v_1 \otimes 1_N \otimes v_2)^T\varphi(\bar{V}(t)) = 0$, hence it is obtained that

$$\varphi^T\left(\bar{V}(t)\right)(-\hat{\Gamma} + \hat{Q} \otimes I_{m^2N^2})\varphi\left(\bar{V}(t)\right) \leq -\lambda_2\left\|\varphi\left(\bar{V}(t)\right)\right\|_2^2. \tag{11}$$

By Condition (C2), there must exist a constant $t_0 > 0$ such that $\forall t \geq t_0$, $a(t) \leq \min\{1, \lambda_2\}$. Therefore, $\forall t \geq t_0$, the following inequality can be derived from (10) and (11):

$$\frac{d\left\|\varphi^T\left(\bar{V}(t)\right)\right\|_2^2}{dt} \leq -a(t)\lambda_2\left\|\varphi\left(\bar{V}(t)\right)\right\|_2^2 + H(t) \tag{12}$$

where $H(t) = 2\varphi^T(\bar{V}(t))\varphi(\gamma(t)) + a^2(t)\|\varphi(R(t))\|_2^2$. This together with the comparison theorem leads to

$$\left\|\varphi^T\left(\bar{V}(t)\right)\right\|_2^2 \leq I_1(t, t_0)\left\|\varphi\left(\bar{V}(t_0)\right)\right\|_2^2 + I_2(t, t_0) + I_3(t, t_0)$$

where $I_1(t, t_0) = e^{-\lambda_2\int_{t_0}^t a(\tau)d\tau}$, $I_2(t, t_0) = \int_{t_0}^t I_1(t, \tau)a^2(\tau)\|\varphi(R(\tau))\|_2^2 d\tau$ and $I_3(t, t_0) = 2\int_{t_0}^t I_1(t, \tau)\varphi^T(\bar{V}(t))\varphi(\gamma(\tau))d\tau$.

By Condition (C2), it is easy to see that $\lim_{t\to\infty} I_1(t, t_0) = 0$. And, it has been proved that $\lim_{t\to\infty} I_2(t, t_0) = 0$ in [15]. Next, it is proved that $I_3(t, t_0)$ is convergent to zero as well. Before further analysis, an assumption is made first:

(A1): There exist three finite positive constants M_1, M_2 and c such that $\|\varphi(\gamma(\tau))\|_2 \leq M_1 e^{-ct}$ and $\|\varphi(\bar{V}(t))\|_2 \leq M_2$.

If Assumption (A1) holds, we have that

$$\begin{aligned} |I_3(t, t_0)| &\leq 2\int_{t_0}^t I_1(t, \tau)\left\|\varphi^T\left(\bar{V}(t)\right)\right\|_2\left\|\varphi\left(\gamma(\tau)\right)\right\|_2 d\tau \\ &\leq 2M_1M_2\int_{t_0}^t e^{-\lambda_2\int_\tau^t a(s)ds}e^{-c\tau}d\tau \to 0\ (t \to \infty). \end{aligned}$$

Therefore, if Assumption (A1) holds, $\lim_{t\to\infty}\|\varphi(\bar{V}(t))\|_2^2 = 0$ which indicates that $\lim_{t\to\infty} E\{\|\delta(t)\|_2^2\} = 0$.

Part II: In this part, it is proved that Assumption (A1) holds. The first step is to prove that $\bar{\gamma}_i(t) = E\{\delta(t)1_{[\theta_t=i]}\}$ is bounded for all $t \in \mathbb{R}^+$. By Proposition 3.28 in [19], it is obtained that

$$d\bar{\gamma}_i(t) = -a(t)\left(\mathcal{L}_\mathcal{G}^{(i)} \otimes I_m\right)\bar{\gamma}_i(t)dt + \sum_{j=1}^s q_{ji}\bar{\gamma}_j(t)dt - (I_{mN} - J)\tilde{D}(t)p_i(t)dt. \tag{13}$$

Let $\bar{\gamma}(t) = [\bar{\gamma}_1^T(t), \bar{\gamma}_2^T(t), \ldots, \bar{\gamma}_s^T(t)]^T$ and $V_\gamma(t) = \|\bar{\gamma}(t)\|_2^2$, then it is obtained by (13) that

$$\begin{aligned} \dot{V}_\gamma(t) = 2\bar{\gamma}^T(t)&\left(-a(t)\hat{\mathcal{L}}_\mathcal{G} + \hat{Q} \otimes I_{mN}\right)\bar{\gamma}(t) \\ &- 2\bar{\gamma}^T(t)\left(P(t) \otimes [(I_{mN} - J)\tilde{D}(t)\right) \end{aligned}$$

where $\hat{\mathcal{L}}_\mathcal{G} = \mathrm{diag}(\mathcal{L}_\mathcal{G}^{(1)} + \mathcal{L}_\mathcal{G}^{(1)T}, \ldots, \mathcal{L}_\mathcal{G}^{(s)} + \mathcal{L}_\mathcal{G}^{(s)T}) \otimes I_m/2$ and $P(t) = (p_1(t), p_2(t), \ldots, p_s(t))^T$.

By Conditions (C1), (C3), Lemma 1 and Lemma 3.5 in [17], it is easy to see that the null space of $\hat{\mathcal{L}}_\mathcal{G} - \hat{Q} \otimes I_{mN}$ is $\mathfrak{N}_2 = \{1_{sN} \otimes v | v \in \mathbb{R}^m\}$. Therefore, it follows from [18, p. 178] (pp. 178) that

$$\min_{x\neq 0, x\in\mathfrak{N}_2^\perp}\left\{\frac{x^T(\hat{\mathcal{L}}_\mathcal{G} - \hat{Q} \otimes I_{mN})x}{\|x\|_2^2}\right\} = \hat{\lambda}_2$$

where $\hat{\lambda}_2$ is the smallest nonzero eigenvalue of $\hat{\mathcal{L}}_\mathcal{G} - \hat{Q} \otimes I_{mN}$.

By the definitions of $\tilde{D}(t)$ and $P(t)$, there exist two positive finite constants M_3 and b such that $\|P(t) \otimes ((I_{mN} - J)\tilde{D}(t))\|_2 \leq M_3 e^{-bt}$. It is noted that $(1_N \otimes v)^T\bar{\gamma}(t) = 0$, hence $\forall t \geq t_0$, $\dot{V}_\gamma(t) \leq (V_\gamma(t) + 1)M_3 e^{-bt}$. By comparison theorem, it is obtained that

$$V_\gamma(t) \leq e^{\int_0^t M_3 e^{-b\tau}d\tau}\left(V_\gamma(0) + \int_0^t M_3 e^{-b\tau}d\tau\right) < \infty$$

which follows that $\bar{\gamma}_i(t)$ is bounded. This together with the definitions of $\tilde{D}(t)$ and $\gamma(t)$ leads to that there exist two finite positive constants c and M_1 such that $\|\varphi(\gamma(t))\|_2 \leq M_1 e^{-ct}$.

For $\forall t > t_0$, it is obtained form (12) that $d\|\varphi(\bar{V}(t))\|_2^2/dt \leq M_1 e^{-ct}\|\varphi(\bar{V}(t))\|_2^2 + M_1 e^{-bt} + a^2(t)\|\varphi(R(t))\|_2^2$. By the comparison theorem, it is obtained that

$$\left\|\varphi\left(\bar{V}(t)\right)\right\|_2^2 \le e^{\int_0^t M_1 e^{-c\tau} d\tau}\left(\left\|\varphi\left(\bar{V}(0)\right)\right\|_2^2 + \int_0^t \left(M_1 e^{-b\tau} + a^2(\tau)\left\|\varphi\left(R(\tau)\right)\right\|_2^2\right) d\tau\right) < \infty.$$

Therefore, there exists a positive constant $M_2 < \infty$ such that $\|\varphi(\bar{V}(t))\|_2 < M_2$ for all $t \in \mathbb{R}^+$.

Part III: It is noted that $J(\mathcal{L}_{\mathcal{G}}^{(\theta_t)} \otimes I_m) = 0_{mN}$. Then multiplying both sides of (6) with J leads to $d(JZ(t)) = a(t)J(I_{mN} \otimes K_2)\Sigma^{(\theta_t)} dW(t) - J\tilde{D}(t)dt$, which implies that $JZ(t) = JZ(0) + \int_0^t a(\tau)J(I_N \otimes K_2)\Sigma^{(\theta_\tau)} dW(\tau) - \int_0^t J\tilde{D}(\tau)d\tau$, where $\{W(t)|\mathcal{F}_t\}$ is the nN^2-dimensional standard Brownian motion.

Let $z^* = (1_N^T \otimes I_m/N)Z(0) + (1/N)\int_0^\infty a(\tau)(1_N^T \otimes K_2)\Sigma^{(m_\tau)} dW(\tau) - (1_N^T \otimes I_m/N)\int_0^\infty \tilde{D}(\tau)d\tau$, then $JZ(t) - 1_N \otimes z^* = I_4(t) + I_5(t)$, where $I_4(t) = -\int_t^\infty a(\tau)J(I_N \otimes K_2)\Sigma^{(\theta_\tau)} dW(\tau)$ and $I_5(t) = \int_t^\infty J\tilde{D}(\tau)d\tau$. It has been proved in [15] that $\lim_{t\to\infty} E\{\|I_4(t)\|_2^2\} = 0$. And it is easy to see that $\lim_{t\to\infty} I_5(t) = 0_N$. Therefore, it is proved that $JZ(t) \xrightarrow{m.s.} 1_N \otimes z^*$.

By combining the above three parts, it is proved that $z_i(t) \xrightarrow{m.s.} z^*$, $i = 1, \ldots, N$. Moreover, the statistic properties of z^* can be calculated as

$$E\{z^*\} = \frac{1_N^T \otimes I_m}{N} Z(0) + \int_0^\infty \frac{1_N^T \otimes I_m}{N} \tilde{D}(\tau)d\tau$$

and

$$D\{z^*\} \le \max_{1\le i\le s}\left(\frac{\left\|(1_N \otimes K_2)\Sigma^{(i)}\right\|_2^2}{N^2}\right)\int_0^\infty a^2(\tau)d\tau < \infty$$

which implies $E\{\|z^*\|_2^2\} < \infty$.

The Almost Sure Case: Let $\delta(t) = Z(t) - 1_N \otimes z^*$ and $V(t) = \delta^T(t)\delta(t)$, then by (6) and Itô's formula, it follows that:

$$\begin{aligned} dV(t) =& -a(t)\delta^T(t)\bar{\mathcal{L}}_{\mathcal{G}}^{(\theta_t)}\delta(t)dt - 2\delta^T(t)\tilde{D}(t)dt \\ &+ a^2(t)\mathrm{tr}\left((I_N \otimes K_2)\Sigma^{(\theta_t)}\Sigma^{(\theta_t)^T}\left(I_N \otimes K_2^T\right)\right)dt \\ &+ 2a(t)\delta^T(t)(I_N \otimes K_2)\Sigma^{(\theta_t)} dW(t) \end{aligned}$$

where $\bar{\mathcal{L}}_{\mathcal{G}}^{(\theta_t)} = (\mathcal{L}_{\mathcal{G}}^{(\theta_t)} + \mathcal{L}_{\mathcal{G}}^{(\theta_t)^T}) \otimes I_m$. Integrating both sides of the above equation from t_0 to t gives that $V(t) = V(t_0) - I_6(t, t_0) - 2I_7(t, t_0) + I_8(t, t_0) + 2I_9(t, t_0)$, where $I_6(t, t_0) = \int_{t_0}^t a(\tau)\delta^T(\tau)\bar{\mathcal{L}}_{\mathcal{G}}^{(\theta_\tau)}\delta(\tau)d\tau$, $I_7(t, t_0) = \int_{t_0}^t \delta^T(\tau)\tilde{D}(\tau)d\tau$, $I_8(t, t_0) = \int_{t_0}^t a^2(\tau)\mathrm{tr}((I_N \otimes K_2)\Sigma^{(\theta_\tau)}\Sigma^{(\theta_\tau)^T}(I_N \otimes K_2^T))d\tau$ and $I_9(t, t_0) = \int_{t_0}^t a(\tau)\delta^T(\tau)(I_N \otimes K_2)\Sigma^{(\theta_\tau)} dW(\tau)$.

Let $I_{7+}(t, t_0) = \int_{t_0}^t (\delta^T(\tau)\tilde{D}(\tau))^+ d\tau$ and $I_{7-}(t, t_0) = \int_{t_0}^t (\delta^T(\tau)\tilde{D}(\tau))^- d\tau$, then $I_7(t, t_0) = I_{7+}(t, t_0) - I_{7-}(t, t_0)$. It can be proved that $\{I_{7+}(t, t_0)|\mathcal{F}_t\}$ and $\{I_{7-}(t, t_0)|\mathcal{F}_t\}$ are submartigale. Moreover, it is easy to see that $E\{I_{7+}^+(t, t_0)\} < \infty$ and $E\{I_{7-}^+(t, t_0)\} < \infty$. Therefore, by the submartingale convergence theorem, $I_7(t, t_0)$ is a.s. convergent to certain random variable.

For $t > s \ge t_0$, we have

$$E\{I_8(t, t_0) - I_8(s, t_0)|\mathcal{F}_s\} = E\left\{\int_s^t a^2(\tau)\mathrm{tr} \times\left((I_N \otimes K_2)\Sigma^{(\theta_\tau)}\Sigma^{(\theta_\tau)^T}\left(I_N \otimes K_2^T\right)\right)d\tau|\mathcal{F}_s\right\} > 0.$$

Hence, $\{I_8(t, t_0)|\mathcal{F}_t\}$ is a submartingale. Moreover, by Condition (C2), it is obtained that $\sup_{t\ge t_0} E\{I_8^+(t, t_0)\} < \infty$. According to the submartingale convergence theorem, $I_8(t, t_0)$ is a.s. convergent to certain random variable.

Since $\delta(t) \xrightarrow{m.s.} 0_{mN}$, it is obtained that $E\{\|\delta(t)\|_2^2\} < \infty$. This together with Condition (C2) leads to that $\{I_9(t, t_0)|\mathcal{F}_t\}$ is a martingale. Let $\hat{V}(t) = V(0) - I_6(t, 0) + 2I_9(t, 0)$, then it is obtained that $E\{\hat{V}(t) - \hat{V}(s)|\mathcal{F}_s\} = -E\{I_6(t, s)|\mathcal{F}_s\} \le 0$. Therefore, $\hat{V}(t)$ is a supermartingale. It is noted that

$$\begin{aligned} \sup_{t\ge t_0} & E\left\{\hat{V}^-(t, t_0)\right\} \\ &= \sup_{t\ge t_0} E\left\{\left(V(t, t_0) + 2I_7(t, t_0) - I_8(t, t_0)\right)^-\right\} \\ &\le \sup_{t\ge t_0} E\left\{V^-(t, t_0) + 2I_7^-(t, t_0) + I_8^+(t, t_0)\right\} < \infty. \end{aligned}$$

By the supermartingale convergence theorem, $\hat{V}(t, t_0)$ is a.s. convergent to a random variable.

From the above discussion, it is shown that $V(t)$ is a.s. convergent to a random variable. Meanwhile, it is noted that $\lim_{t\to\infty} E\{V(t)\} = \lim_{t\to\infty} E\{\delta^T(t)\delta(t)\} = 0$. Hence, $z_i(t) \xrightarrow{a.s.} z^*$, $i = 1, \ldots, N$.

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