Centralized and decentralized event-triggered control for group consensus with fixed topology in continuous time ☆

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ABSTRACT

The communication weights in traditional consensus problems are all positive, while the communication weights in group consensus are partly real numbers. In addition, with regard to energy consumption and communication constraints, event-triggered control has advantages over periodic control. Therefore, it is of great significance to investigate group consensus based on event-triggered control. We develop two event-triggered functions to decide when to activate the control input in centralized and decentralized cases, respectively. Additionally, the infamous Zeno behavior can be excluded in the centralized case. Moreover, in the decentralized case, we simplify the event-triggered function by calculating the maximum and minimum of the corresponding parameters, so as to save memory of the systems.

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1. Introduction

Multi-agent systems have been a hot topic with abundant literatures [1–6] in recent years. To adapt the advent of networks, the idea of distributed or decentralized algorithm can be traced back to [7,8]. Vicsek et al. [9] proposed a novel type of phase transition in a system of self-driven particles, which is the origin of nearest neighbor rules. Then according to Vicsek's model, Jadbabaie et al. [1] introduced nearest neighbor rules into the multi-agent systems. For more details, refer to survey papers [2,10,11] and the references cited therein.

Group consensus, which has attracted an increasing attention [12,13], is one aspect in extended consensus problems. The agents in the same sub-network can reach a consistent value, while no agreement can be achieved between any two different sub-networks. In addition, in discrete-time multi-agent systems, group consensus was

termed as cluster consensus [14,15] with similar definition. In this paper, we focus on group consensus with continuous time. In [16], Yu and Wang proposed a new distributed control protocol with group consensus, where the switching topologies were finite and the communication delays were bounded. Moreover, a double-tree-form transformation was introduced to reduce the order of the multi-agent systems. Then Tan et al. [17] relaxed the assumption proposed in [16], such that the sums of adjacent weights, from every node in one group to all nodes in another group, are identical. Based on these results, Shang [18] studied the group consensus with noises and time delays. In addition, the methods proposed in [19] can be extended to multi-group consensus. It is noted that the above papers all concentrate on the study of periodic control protocol, which is a serious problem when we consider energy consumption and communication constraints on wireless platform. Therefore, event-triggered control is an appropriate choice for solving this problem.

Event-triggered control, which has a long history dating back to [20,21], is aimed at improving the efficiency of control. In [22,23], some advantages of event-triggered control were emphasized and the motivation of the development of systematic designs was also provided. Heemels et al. [24] gave an overview of event-triggered and self-triggered control in recent years. Event-triggered control in multi-agent systems is both conceptually interesting because designing a distributed control protocol based on event-triggered technique requires only relative information from local neighbors [25,26], and practically interesting because it can solve real-time

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scheduling problem excluding the infamous Zeno behavior [27]. Furthermore, a periodic event-triggered control was proposed in [28], which further reduced the number of control executions and maintained the requirements of closed-loop performance. However, to the best of our knowledge, the communication weights in multi-agent systems, based on event-triggered control mentioned above, are all positive, while group consensus takes negative weights into consideration. Hence, it is of great importance to investigate group consensus based on event-triggered control.

Motivated by the above discussions, this paper investigates the conditions for achieving group consensus with centralized and decentralized event-triggered control, respectively. The multi-agent systems are modeled containing two sub-networks in continuous time with undirected topology. However, the communication weights between the two sub-networks are not simply zeros but with balanced in- and out-degree. By introducing a candidate Lyapunov function $V(t)$ for input-to-state stability (ISS), we can get the derivative $\dot{V}(t)$ and enforce it to be negative. Furthermore, with the help of appropriate inequality zooming technique, the event-triggered functions both in centralized and decentralized cases are deduced. Finally, numerical examples are provided to validate the effectiveness of the developed criteria.

The main contributions of this paper are listed as follows.

1. We introduce both centralized and decentralized event-triggered control to group consensus to deal with energy consumption and communication constraints considered in real physical implementations.
2. We discuss the event-triggered based group consensus in the presence of negative communication weights and the infamous Zeno behavior can be excluded from centralized cases.
3. In decentralized event-triggered cases, by calculating the maximum and minimum of the corresponding parameters, we simplify the event-triggered function with $\varepsilon_i^2(t) = \beta(\gamma_{\min}/\eta_{\max}) q_i^2(t)$, which will be clarified in Section 4 in order to save memory of the systems.

The remainder of this paper is organized as follows. Basic definitions of group consensus and algebraic graph theory are given in Section 2. Centralized and decentralized event-triggered control on group consensus are discussed in Sections 3 and 4, respectively. Implementations of three examples are conducted to demonstrate the validity of the developed criteria in Section 5. Closing remarks and the conclusion of the whole paper are given in Section 6.

The following notations are utilized throughout this paper: $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ and $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$ represents the Euclidian norm of vector x. $A \in \mathbb{R}^{m \times n}$ and $\|A\|$ represents its corresponding Frobenius norm. $\uparrow$ denotes the increase of the value of variants and $\downarrow$ denotes the decrease.

2. Backgrounds and preliminaries

2.1. Algebraic graph theory

A triplet $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ is called a weighted graph if $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ is the set of N nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges, and $\mathcal{A} = (\mathcal{A}_{ij}) \in \mathbb{R}^{N \times N}$ is the $N \times N$ matrix of the weights of $\mathcal{G}$. Here we denote $\mathcal{A}_{ij}$ as the element of the ith row and jth column of matrix $\mathcal{A}$. The ith node in graph $\mathcal{G}$ represents the ith agent, and a directed path from node i to node j is denoted as an ordered pair $(v_i, v_j) \in \mathcal{E}$, which means that agent i can directly transfer its information to agent j. $\mathcal{A}$ is called the adjacency matrix of graph $\mathcal{G}$ and we use the notation $\mathcal{G}(\mathcal{A}) : \mathcal{A}_{ij} \neq 0 \Leftrightarrow (v_j, v_i) \in \mathcal{E}$ to represent the graph $\mathcal{G}$

corresponding to $\mathcal{A}$. In this paper, we assume that $\mathcal{G}$ represents an undirected fixed topology. Note that self-loops will not be considered in this paper, i.e., $\mathcal{A}_{ii} = 0, i = 1, 2, \ldots, N$. $\mathcal{G}$ is called connected if there is a path between any two nodes of $\mathcal{G}$. Let

$$\mathcal{D} = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_N \end{bmatrix}$$

be the $N \times N$ diagonal matrix where $d_i = \sum_{v_j \in \mathcal{N}_i} \mathcal{A}_{ij}$ and $\mathcal{N}_i = \{v_j \in \mathcal{V} | (v_j, v_i) \in \mathcal{E}\}$ is the set of neighbor nodes of node $i, i = 1, 2, \ldots, N$. Then $\mathcal{D}$ is termed as the indegree matrix of $\mathcal{G}$. The Laplacian matrix is $\mathcal{L} = \mathcal{D} - \mathcal{A}$ corresponding to $\mathcal{G}$. In addition, for a connected graph, $\mathcal{L}$ has only one single zero eigenvalue [2]. We denote by $\lambda_N(\mathcal{G}) \geq \lambda_{N-1}(\mathcal{G}) \geq \cdots \geq \lambda_2(\mathcal{G}) \geq \lambda_1(\mathcal{G}) = 0$ the eigenvalues of $\mathcal{L}$ with $\lambda_2(\mathcal{G}) > 0$ if $\mathcal{G}$ is connected.

2.2. Consensus and group consensus

Given the network with N agents where $x_i \in \mathbb{R}^n$ represents the state of the ith agent. In physical implementations, the state of a node can represent the voltage or current of smart grid [29,30], temperature of rooms [31], and attitude of unmanned aerial vehicles [32,33], etc.

In this paper, we assume that each agent has the dynamics as follows:

$$\dot{x}_i(t) = u_i(t), \quad i = 1, 2, \ldots, N, \tag{1}$$

where $x_i \in \mathbb{R}$. The most popular distributed control protocol is the state feedback distributed control

$$u_i(t) = -\sum_{v_j \in \mathcal{N}_i} \mathcal{A}_{ij}(x_i(t) - x_j(t)) \tag{2}$$

proposed in [2,11,3], where $\mathcal{A}_{ij} \geq 0, \forall v_i, v_j \in \mathcal{V}$. Note that the first-order multi-agent systems (1) with distributed control protocol (2) can reach a consistent state asymptotically [2], i.e.,

$$\lim_{t \to +\infty} x_i(t) = \frac{1}{N} \sum_{j=1}^{N} x_j(0), \quad i = 1, 2, \ldots, N.$$

However, the multi-agent systems in physical implementations of cooperative control can reach more than one consistent state in complex networks, which can be called cluster consensus [14,15] or group consensus [16,19]. In this paper, we investigate the case that agents in a network can reach two consistent states asymptotically with event-triggered distributed control. For convenient use, we introduce the concepts of *group consensus* proposed in [16].

Suppose that the complex network $\mathcal{G}$ contains $N_1 + N_2$ $(N_1, N_2 > 0)$ agents consisting of two sub-networks $\mathcal{G}_1 = \{\mathcal{V}_1, \mathcal{E}_1, \mathcal{A}_1\}$ and $\mathcal{G}_2 = \{\mathcal{V}_2, \mathcal{E}_2, \mathcal{A}_2\}$, where $x^1 = (x_1, x_2, \ldots, x_{N_1})^T$ and $x^2 = (x_{N_1+1}, x_{N_1+2}, \ldots, x_{N_1+N_2})^T$ represent the states of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Thus, all the agents are divided into two groups with communication between the two groups. Furthermore, the whole graph is $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ and the corresponding state is $x = (x_1, x_2, \ldots, x_{N_1+N_2})^T$. Consequently, denote the index sets of sub-networks by $\mathcal{I}_1 = \{1, 2, \ldots, N_1\}$ and $\mathcal{I}_2 = \{N_1+1, N_1+2, \ldots, N_1+N_2\}$, and denote the node sets by $\mathcal{V}_1 = \{v_1, v_2, \ldots, v_{N_1}\}$ and $\mathcal{V}_2 = (v_{N_1+1}, v_{N_1+2}, \ldots, v_{N_1+N_2})$, where $\mathcal{I} = \mathcal{I}_1 \bigcup \mathcal{I}_2$ and $\mathcal{V} = \mathcal{V}_1 \bigcup \mathcal{V}_2$. More specifically, the neighbor sets of the corresponding sub-networks are $\mathcal{N}_{1i} = \{v_j \in \mathcal{V}_1 | (v_j, v_i) \in \mathcal{E}\}$ and $\mathcal{N}_{2i} = \{v_j \in \mathcal{V}_2 | (v_j, v_i) \in \mathcal{E}\}$, where $\mathcal{N}_i = \{v_j \in \mathcal{V} | (v_j, v_i) \in \mathcal{E}\} = \mathcal{N}_{1i} \bigcup \mathcal{N}_{2i}, \forall i \in \mathcal{I}$.

A new distributed control protocol is proposed as follows:

$$u_i(t) = \begin{cases} -\sum\limits_{v_j \in \mathcal{N}_{1i}} \mathcal{A}_{ij}(x_i(t) - x_j(t)) \\ \quad -\sum\limits_{v_j \in \mathcal{N}_{2i}} \mathcal{A}_{ij}x_j(t), \quad \forall i \in \mathcal{I}_1; \\ -\sum\limits_{v_j \in \mathcal{N}_{2i}} \mathcal{A}_{ij}(x_i(t) - x_j(t)) \\ \quad -\sum\limits_{v_j \in \mathcal{N}_{1i}} \mathcal{A}_{ij}x_j(t), \quad \forall i \in \mathcal{I}_2, \end{cases} \quad (3)$$

where $\mathcal{A}_{ij} \geq 0, \forall i, j \in \mathcal{I}_1$ and $\forall i, j \in \mathcal{I}_2$; $\mathcal{A}_{ij} \in \mathbb{R}, \forall (i,j) \in \Xi = \{(i,j) \mid i \in \mathcal{I}_1, j \in \mathcal{I}_2\} \bigcup \{(j,i) \mid i \in \mathcal{I}_1, j \in \mathcal{I}_2\}$.

Remark 1. Note that the communication weights between the two groups are real numbers. Therefore, distributed control protocol (3) is the extension of control protocol (2). In addition, the existence of negative weights complicates the dynamics of the multi-agent systems.

Definition 1. If the states of the agents in $\mathcal{G}$ satisfy the following two conditions:

$$\lim_{t \to +\infty} \|x_i(t) - x_j(t)\| = 0, \quad \forall i, j \in \mathcal{I}_1; \quad (4)$$

$$\lim_{t \to +\infty} \|x_i(t) - x_j(t)\| = 0, \quad \forall i, j \in \mathcal{I}_2, \quad (5)$$

then the multi-agent systems (1) are said to reach a *group consensus* asymptotically.

Note that given the set

$$\Gamma = \{x_1 = x_2 = \cdots = x_{N_1},$$
$$x_{N_1+1} = x_{N_1+2} = \cdots = x_{N_1+N_2}\},$$

then Γ is a globally attractive and invariant manifold if the group consensus can be reached. In the sequel, we will discuss the group consensus based on both centralized and decentralized event-triggered distributed control protocols.

3. Centralized event-triggered control for group consensus

For each agent i in graph $\mathcal{G}(\mathcal{A})$, we introduce a time-varying error function $\varepsilon_i(t)$, where $\varepsilon_i(t) = x_i(t_k) - x_i(t), t \geq 0$. t_k is in the sequence of event-triggered executions which are denoted by $t_0, t_1, \ldots$. Then the corresponding control sequence updates are $u_i(t_0), u_i(t_1), \ldots$, thus the value of the distributed control input $u_i(t)$ is in a zero-order hold form and piecewise constant between each pair of event times, such as $t \in [t_k, t_{k+1}), k = 0, 1, \ldots$. Furthermore, we introduce an error vector $\varepsilon(t) = (\varepsilon_1(t), \varepsilon_2(t), \ldots, \varepsilon_{N_1+N_2}(t))^T \in \mathbb{R}^{N_1+N_2}$, then

$$\varepsilon(t) = x(t_k) - x(t), \quad k = 0, 1, \ldots, \ t \in [t_k, t_{k+1}), \quad (6)$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_{N_1+N_2}(t))^T \in \mathbb{R}^{N_1+N_2}$.

Our purpose is to design an appropriate centralized event-triggered mechanism satisfying the group consensus in Definition 1. Therefore, the event-triggered distributed control protocol developed in the centralized case is shown in Fig. 1 and is defined analogous to
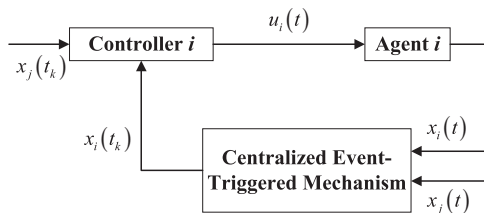

Fig. 1. Centralized event-triggered mechanism schematic.

(3) as follows:

$$u_i(t) = \begin{cases} -\sum\limits_{v_j \in \mathcal{N}_{1i}} \mathcal{A}_{ij}(x_i(t_k) - x_j(t_k)) \\ \quad -\sum\limits_{v_j \in \mathcal{N}_{2i}} \mathcal{A}_{ij}x_j(t_k), \quad \forall i \in \mathcal{I}_1; \\ -\sum\limits_{v_j \in \mathcal{N}_{2i}} \mathcal{A}_{ij}(x_i(t_k) - x_j(t_k)) \\ \quad -\sum\limits_{v_j \in \mathcal{N}_{1i}} \mathcal{A}_{ij}x_j(t_k), \quad \forall i \in \mathcal{I}_2, \end{cases} \quad (7)$$

where $t \in [t_k, t_{k+1}), k = 0, 1, \ldots$.

Before proceeding, we introduce some key definitions and lemmas related to our main results.

Definition 2 (*cf. Yu and Wang [16]*). The communication topology $\mathcal{G} = \{\mathcal{G}_1, \mathcal{G}_2\}$ of the multi-agent systems is consisted of $N_1 + N_2$ nodes defined in Section 2.2. Given any $i \in \mathcal{I}_1$, the out-degree and in-degree of node v_i in $\mathcal{G}_1$ to $\mathcal{G}_2$ are defined as follows:

$$d_{\text{out}}(v_i, \mathcal{G}_2) = \sum_{j=N_1+1}^{N_1+N_2} \mathcal{A}_{ji}, \quad d_{\text{in}}(v_i, \mathcal{G}_2) = \sum_{j=N_1+1}^{N_1+N_2} \mathcal{A}_{ij}.$$

Given $i \in \mathcal{I}_1$, if $d_{\text{in}}(v_i, \mathcal{G}_2) = 0$ and $d_{\text{out}}(v_i, \mathcal{G}_2) = 0$, then we say $v_i \in \mathcal{V}_1$ is *in-degree balanced* and *out-degree balanced* to $\mathcal{G}_2$, respectively. Similarly, given $i \in \mathcal{I}_2$, if $d_{\text{in}}(v_i, \mathcal{G}_1) = 0$ and $d_{\text{out}}(v_i, \mathcal{G}_1) = 0$, then we say $v_i \in \mathcal{V}_2$ is in-degree balanced and out-degree balanced to $\mathcal{G}_1$, respectively. Furthermore, if all nodes in $\mathcal{G}_1(\mathcal{G}_2)$ are out(in)-degree balanced to $\mathcal{G}_2(\mathcal{G}_1)$, we say that $\mathcal{G}_1(\mathcal{G}_2)$ is out(in)-degree balanced to $\mathcal{G}_2(\mathcal{G}_1)$, and vice versa.

We use $\mathcal{L}$ to represent the Laplacian matrix of the communication topology $\mathcal{G}$, where $\mathcal{L} = (l_{ij}) \in \mathbb{R}^{(N_1+N_2) \times (N_1+N_2)}$ is defined as follows:

$$l_{ij} = \begin{cases} -\mathcal{A}_{ij}, & j \neq i; \\ \sum\limits_{k=1, k \neq i}^{N_1+N_2} \mathcal{A}_{ik}, & j = i. \end{cases}$$

Suppose $\mathcal{L}$ has a block form

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix}, \quad (8)$$

where $\mathcal{L}_{11} \in \mathbb{R}^{N_1 \times N_1}$ and $\mathcal{L}_{22} \in \mathbb{R}^{N_2 \times N_2}$, then the multi-agent systems (1) with u_i given in (3) is equivalent to the following form:

$$\begin{cases} \dot{x}^1(t) = -\mathcal{L}_{11}x^1 - \mathcal{L}_{12}x^2; \\ \dot{x}^2(t) = -\mathcal{L}_{21}x^1 - \mathcal{L}_{22}x^2, \end{cases} \quad (9)$$

where $\mathcal{L}_{12} = \mathcal{L}_{21}$.

Assumption 1. Considering the balance of the two sub-networks $\mathcal{G}_1$ and $\mathcal{G}_2$ mentioned in Definition 2, we propose three assumptions for the convenience of later proofs as follows:

(A1) $\quad \sum\limits_{j=N_1+1}^{N_1+N_2} \mathcal{A}_{ij} = 0, \forall i \in \mathcal{I}_1;$

(A2) $\quad \sum\limits_{j=1}^{N_1} \mathcal{A}_{ij} = 0, \forall i \in \mathcal{I}_2;$

(A3) $\quad (x^1)^T \mathcal{L}_{12}x^2$ is in the form of $(x_{i_1} - x_{j_1})(x_{i_2} - x_{j_2})$, where $(i_1, j_1) \in \mathcal{E}_1$ and $(i_2, j_2) \in \mathcal{E}_2$.

Lemma 1 (*cf. Yu and Wang [19]*). *With Assumptions (A1), (A2) and distributed control protocol (3), the multi-agent systems (1) can reach the group consensus asymptotically if and only if*

(i) $\mathcal{L}$ *has only two simple zero eigenvalues while the others have positive real parts;*

(ii) $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *are in-degree and out-degree balanced to each other.*

Theorem 1. *If the undirected communication graph $\mathcal{G}$ is connected where the corresponding Laplacian matrix $\mathcal{L}$ is satisfied with the condition (i) in Lemma 1, then with Assumption 1, given the multi-agent systems (1) with the distributed control protocol (7) and given centralized event-triggered mechanism*

$$\|\varepsilon(t)\| = \alpha \frac{\|\mathcal{L}x(t)\|}{\|\mathcal{L}\|} \tag{10}$$

where $\alpha \in (0,1)$, the multi-agent systems can asymptotically reach group consensus. Furthermore, for any initial conditions in $\mathbb{R}^{N_1+N_2}$ with $t \geq 0$, the inter-event times $\{t_{k+1} - t_k\}$ derived from the centralized event-triggered mechanism (10) are strictly positive, where the lower bounded time is denoted by $\tau = \alpha / \|\mathcal{L}\|(1+\alpha)$.

Proof. From Assumption (A1), we know that

$$\sum_{v_j \in N_{2i}} \mathcal{A}_{ij} = 0, \quad \forall i \in \mathcal{I}_1,$$

then the first part of control protocol (7) can be rewritten as

$$
\begin{aligned}
u_i(t) &= -\sum_{v_j \in \mathcal{N}_{1i}} \mathcal{A}_{ij}(x_i(t_k) - x_j(t_k)) - \sum_{v_j \in \mathcal{N}_{2i}} \mathcal{A}_{ij}x_j(t_k) \\
&= -\sum_{v_j \in \mathcal{N}_{1i}} \mathcal{A}_{ij}(x_i(t_k) - x_j(t_k)) \\
&\quad - \sum_{v_j \in \mathcal{N}_{2i}} \mathcal{A}_{ij}(x_i(t_k) - x_j(t_k)) \\
&= -\sum_{v_j \in \mathcal{N}_i} \mathcal{A}_{ij}(x_i(t_k) - x_j(t_k)),
\end{aligned}
$$

where $t \in [t_k, t_{k+1}), k = 0, 1, \ldots, \forall i \in \mathcal{I}_1$. Similarly, the second part of control protocol (7) can be rewritten in the same form. Therefore, the compact form of the event-triggered distributed control protocol is

$$u(t) = -\mathcal{L}x(t_k), \quad t \in [t_k, t_{k+1}), \quad k = 0, 1, \ldots. \tag{11}$$

Note that we discuss the centralized form of the event-triggered control, thus the dynamics of multi-agent systems (9) can be rewritten in the form

$$\dot{x}(t) = -\mathcal{L}x(t_k) = -\mathcal{L}x(t) - \mathcal{L}\varepsilon(t), \quad t \in [t_k, t_{k+1}).$$

We choose a candidate Lyapunov function for the closed-loop system as follows:

$$
\begin{aligned}
V &= \frac{1}{2}x^{\mathrm{T}}\mathcal{L}x \\
&= \frac{1}{2}\left[(x^1)^{\mathrm{T}}, (x^2)^{\mathrm{T}}\right]\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix}\begin{bmatrix} x^1 \\ x^2 \end{bmatrix} \\
&= \frac{1}{2}\left((x^1)^{\mathrm{T}}\mathcal{L}_{11}x^1 + (x^2)^{\mathrm{T}}\mathcal{L}_{22}x^2 + 2(x^1)^{\mathrm{T}}\mathcal{L}_{12}x^2\right).
\end{aligned} \tag{12}
$$

Owing to the fact that $\mathcal{L}$ has two zero eigenvalues and the rest are with positive real numbers, without loss of generality we denote the spectrum of $\mathcal{L}$ by $\lambda(\mathcal{L}) = \{\lambda_1, \lambda_2, \ldots, \lambda_{N_1+N_2}\}$, where $\lambda_1 = 0, \lambda_2 = 0$. In addition, $\mathcal{G}$ is undirected. Thus, $\lambda_3, \lambda_4, \ldots, \lambda_{N_1+N_2}$ are all positive real numbers. Therefore, $\mathcal{L}$ can be diagonalized with a matrix $U \in \mathbb{R}^{(N_1+N_2)\times(N_1+N_2)}$, i.e.,

$$\mathcal{L} = U^{\mathrm{T}}\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{N_1+N_2} \end{bmatrix}U = U^{\mathrm{T}}DU.$$

Thus, V can be rewritten as

$$V = \frac{1}{2}(Ux)^{\mathrm{T}}D(Ux) = \frac{1}{2}\tilde{x}^{\mathrm{T}}D\tilde{x} = \frac{1}{2}\sum_{i=1}^{N_1+N_2}\lambda_i\tilde{x}_i^2 \geq 0.$$

With $\mathcal{L} = \mathcal{L}^{\mathrm{T}}$, the derivative of the Lyapunov function is

$$
\begin{aligned}
\dot{V} &= \frac{1}{2}(\dot{x}^{\mathrm{T}}\mathcal{L}x + x^{\mathrm{T}}\mathcal{L}\dot{x}) \\
&= \frac{1}{2}(x^{\mathrm{T}}\mathcal{L}^{\mathrm{T}}\dot{x} + x^{\mathrm{T}}\mathcal{L}\dot{x}) \\
&= x^{\mathrm{T}}\mathcal{L}\dot{x} \\
&= -x^{\mathrm{T}}\mathcal{L}\mathcal{L}(x+\varepsilon) \\
&= -\|\mathcal{L}x\|^2 - x^{\mathrm{T}}\mathcal{L}\mathcal{L}\varepsilon \\
&\leq -\|\mathcal{L}x\|^2 + \|\mathcal{L}x\|\|\mathcal{L}\|\|\varepsilon\|.
\end{aligned}
$$

In order to make $\dot{V} < 0$, we choose ε to satisfy

$$\|\varepsilon\| \leq \alpha\frac{\|\mathcal{L}x\|}{\|\mathcal{L}\|} \tag{13}$$

where $\alpha \in (0,1)$, then $\dot{V} \leq (\alpha-1)\|\mathcal{L}x\|^2 < 0$. Thus, the event-triggered times $t_k, k = 0, 1, \ldots$, are derived from $\|\varepsilon(t_k)\| = \alpha(\|\mathcal{L}x(t_k)\|/\|\mathcal{L}\|)$. If the event-triggered function (10) is satisfied, then $\lim_{t \to +\infty}V(t) = 0$. Considering the form of (12),

$$(x^1)^{\mathrm{T}}\mathcal{L}_{11}x^1 = \frac{1}{2}\sum_{(i,j)\in\mathcal{E}_1}\mathcal{A}_{ij}(x_i - x_j)^2 \tag{14}$$

and

$$(x^2)^{\mathrm{T}}\mathcal{L}_{22}x^2 = \frac{1}{2}\sum_{(i,j)\in\mathcal{E}_2}\mathcal{A}_{ij}(x_i - x_j)^2 \tag{15}$$

are both in quadratic form. In addition, with (14), (15), (A3) and condition (i) in Lemma 1, $2(x^1)^{\mathrm{T}}\mathcal{L}_{12}x^2$ is in the form of $(x_{i_1} - x_{j_1})(x_{i_2} - x_{j_2})$, which can be transformed into the form of $[(x_{i_1} - x_{j_1}) + (x_{i_2} - x_{j_2})]^2$, where $(i_1, j_1) \in \mathcal{E}_1$ and $(i_2, j_2) \in \mathcal{E}_2$. We will show it later with Example 1 in Section 5. Therefore, with $\dot{V}(t) < 0$, $V(t)$ is in the quadratic form subject to $\lim_{t \to +\infty}x(t) \in \Gamma$ defined in Section 2. Additionally, Γ is a globally attractive and invariant manifold. Thus, group consensus can be asymptotically reached.

In the sequel, we will demonstrate that the inter-event times $\{t_{k+1} - t_k\}$ are strictly positive by a lower bounded time $\tau = \alpha/\|\mathcal{L}\|(1+\alpha)$. First of all, the study of the time derivative of $\|\varepsilon(t)\|/\|\mathcal{L}x(t)\|$ is essential:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\|\varepsilon\|}{\|\mathcal{L}x\|}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{(\varepsilon^{\mathrm{T}}\varepsilon)^{1/2}}{[(\mathcal{L}x)^{\mathrm{T}}(\mathcal{L}x)]^{1/2}}\right) \\
&= \frac{(\varepsilon^{\mathrm{T}}\varepsilon)^{-1/2}\varepsilon^{\mathrm{T}}\dot{\varepsilon}[(\mathcal{L}x)^{\mathrm{T}}(\mathcal{L}x)]^{1/2}}{\|\mathcal{L}x\|^2} \\
&\quad - \frac{[(\mathcal{L}x)^{\mathrm{T}}(\mathcal{L}x)]^{-1/2}(\mathcal{L}x)^{\mathrm{T}}(\mathcal{L}\dot{x})(\varepsilon^{\mathrm{T}}\varepsilon)^{1/2}}{\|\mathcal{L}x\|^2} \\
&= -\frac{\varepsilon^{\mathrm{T}}\dot{x}}{\|\varepsilon\|\|\mathcal{L}x\|} - \frac{(\mathcal{L}x)^{\mathrm{T}}(\mathcal{L}\dot{x})\|\varepsilon\|}{\|\mathcal{L}x\|^3} \\
&\leq \frac{\|\varepsilon\|\|\dot{x}\|}{\|\varepsilon\|\|\mathcal{L}x\|} + \frac{\|\mathcal{L}x\|\|\mathcal{L}\|\|\dot{x}\|\|\varepsilon\|}{\|\mathcal{L}x\|^3} \\
&= \left(1 + \frac{\|\mathcal{L}\|\|\varepsilon\|}{\|\mathcal{L}x\|}\right)\frac{\|-\mathcal{L}(x+\varepsilon)\|}{\|\mathcal{L}x\|} \\
&\leq \left(1 + \frac{\|\mathcal{L}\|\|\varepsilon\|}{\|\mathcal{L}x\|}\right)\left(\frac{\|\mathcal{L}x\| + \|\mathcal{L}\varepsilon\|}{\|\mathcal{L}x\|}\right) \\
&\leq \left(1 + \frac{\|\mathcal{L}\|\|\varepsilon\|}{\|\mathcal{L}x\|}\right)^2.
\end{aligned} \tag{16}
$$

Letting $p = \|\varepsilon(t)\|/\|\mathcal{L}x(t)\|$, with (16) we get $\dot{p} \leq (1+\|\mathcal{L}\|p)^2$, then $p(t) \leq \psi(t, \psi_0)$ such that

$$\dot{\psi} = (1+\|\mathcal{L}\|\psi)^2 \tag{17}$$

and $\psi(0, \psi_0) = \psi_0$. Furthermore, according to (10) we have $\psi(\tau, 0) = \alpha/\|\mathcal{L}\|$. The solution of (17) is $\psi(\tau, 0) = \tau/(1 - \tau\|\mathcal{L}\|)$. Therefore, $\alpha/\|\mathcal{L}\| = \tau/(1 - \tau\|\mathcal{L}\|)$, i.e., $\tau = \alpha/\|\mathcal{L}\|(1+\alpha)$ is the lower bounded time. □

Remark 2. Note that once the error function (10) is triggered, we have $\varepsilon(t_k) = x(t_k) - x(t_k) = 0$ and (13) is naturally satisfied. Hence,

Table 1
Algorithm for centralized event-triggered control.

Step 1:	Given the initial conditions with $x(t_0) = x_0$, $t_0 = 0$ and the communication topology $\mathcal{G}$, where x_0 is the initial state of the multi-agent systems
Step 2:	**While** $\|\varepsilon(t)\| \geq \epsilon$ where $\epsilon \in \mathbb{R}$ is the given small bounded error, **goto Step 3**. Else **goto Step 4**
Step 3:	If $\|\varepsilon(t)\| < \alpha \|\mathcal{L}x(t)\| / \|\mathcal{L}\|$, then $u(t) = -\mathcal{L}x(t_k)$ and **goto Step 2**. Else $t_k = t_{k+1}$, where $\varepsilon(t_{k+1}) = 0$ and **goto Step 2**
Step 4:	Terminate the algorithm

the error function always runs below an upper boundary and the mechanism that resets error to zero is the core of event-triggered design. The second part of Theorem 1 guarantees non-existence of the infamous Zeno behavior which usually occurred in hybrid system [34]. The algorithm for centralized event-triggered control is shown in Table 1.

Remark 3. According to the form of τ, we know that it is proportional to $\alpha \in (0,1)$ and inversely proportional to $\|\mathcal{L}\|$. Thus, if $\alpha\uparrow$, then $\tau\uparrow$ and the error tolerance of $\|\varepsilon\|$ will increase and vice versa. Moreover, $\|\mathcal{L}\|$ represents the strength of communications among the multi-agent systems. Therefore, if $\|\mathcal{L}\|\uparrow$, then $\tau\downarrow$, which shows that the frequency of updates increases with stronger connectivity in the multi-agent systems. All the analyses mentioned above are in accordance with our intuitions and will be illustrated in Section 5.2.

4. Decentralized event-triggered control for group consensus

The centralized event-triggered control requires a global error function $\varepsilon(t)$ to decide when to trigger the condition (10). However, this is impractical in physical implementations for the complexity of scale in multi-agent systems. Hence, we introduce a decentralized event-triggered mechanism to solve the group consensus. Particularly, each agent will update its own control input $u_i(t)$ at event times decided by information from itself and from its neighbors. We denote these event times by $t_0^i, t_1^i, \ldots, t_k^i, \ldots, \forall i \in \mathcal{I}$. Defining the error measurement function for agent i as

$$\varepsilon_i(t) = x_i(t_k^i) - x_i(t), \quad t \in [t_k^i, t_{k+1}^i], \quad k = 0, 1, \ldots. \quad (18)$$

The distributed control protocol (3) can be written in decentralized event-triggered form as

$$u_i(t) = \begin{cases} -\sum\limits_{v_j \in \mathcal{N}_{1i}} \mathcal{A}_{ij}(x_i(t_k^i) - x_j(t_{\tilde{k}(t)}^j)) \\ \quad -\sum\limits_{v_j \in \mathcal{N}_{2i}} \mathcal{A}_{ij} x_j(t_{\tilde{k}(t)}^j), \quad \forall i \in \mathcal{I}_1; \\ -\sum\limits_{v_j \in \mathcal{N}_{2i}} \mathcal{A}_{ij}(x_i(t_k^i) - x_j(t_{\tilde{k}(t)}^j)) \\ \quad -\sum\limits_{v_j \in \mathcal{N}_{1i}} \mathcal{A}_{ij} x_j(t_{\tilde{k}(t)}^j), \quad \forall i \in \mathcal{I}_2, \end{cases} \quad (19)$$

where $\tilde{k}(t) \triangleq \arg\min_{h \in \mathbb{N}: t \geq t_h^j} \{t - t_h^j\}$ and $t_{\tilde{k}(t)}^j$ is the latest event time of agent j within $t \in [t_k^i, t_{k+1}^i)$.

Remark 4. From (19) we note that the control updates of agent i depend not only on its own triggering times $t_0^i, t_1^i, \ldots$, but also on the triggering times of its neighbors $t_0^j, t_1^j, \ldots$, where $v_j \in \mathcal{N}_i$.

Suppose $\mathcal{L}x \triangleq q = (q_1, q_2, \ldots, q_{N_1+N_2})^T$. Then, $q_i(t) = \sum_{v_j \in \mathcal{N}_i} \mathcal{A}_{ij}(x_i(t) - x_j(t)), \forall i \in \mathcal{I}$. We introduce a notation $|N_i^A| = \sum_{v_j \in \mathcal{N}_i} \mathcal{A}_{ij}$ to simplify the expression of the following proof.

Theorem 2. If the undirected communication graph $\mathcal{G}$ is connected where the corresponding Laplacian matrix $\mathcal{L}$ is satisfied with the condition (i) in Lemma 1, then with the Assumption 1, given the multi-agent systems (1) with the distributed control protocol (19) and

decentralized event-triggered mechanism

$$\varepsilon_i^2(t) = \beta \frac{\gamma_{\min}}{\eta_{\max}} q_i^2(t), \quad (20)$$

the multi-agent systems (1) can asymptotically reach group consensus. In addition, $\beta \in (0, 1)$, $\gamma_{\min} = \min_{i \in \mathcal{I}} \{1 - c|N_i^A|\}$, $\eta_{\max} = \max_{i \in \mathcal{I}} \{|N_i^A|/c\}$ and $c \in \bigcap_{i \in \mathcal{I}} (0, 1/|N_i^A|)$. Moreover, for any initial conditions in $\mathbb{R}^{N_1+N_2}$ with $t \geq 0$, $\exists l \in \mathcal{I}$, such that the next inter-event time τ_D is strictly positive.

Proof. With Assumptions (A1) and (A2), we know that

$$\sum_{v_j \in N_{2i}} \mathcal{A}_{ij} = 0, \quad \forall i \in \mathcal{I}_1 \quad \text{and} \quad \sum_{v_j \in N_{1i}} \mathcal{A}_{ij} = 0, \quad \forall i \in \mathcal{I}_2.$$

Then by the similar mathematical operations in the proof of Theorem 1, we can transform the decentralized event-triggered control protocol (19) into the following form:

$$u_i(t) = -\sum_{v_j \in \mathcal{N}_i} \mathcal{A}_{ij}(x_i(t_k^i) - x_j(t_{\tilde{k}(t)}^j)), \quad \forall i \in \mathcal{I}.$$

Thus,

$$\begin{aligned} \dot{x}_i(t) &= u_i(t) \\ &= -\sum_{v_j \in \mathcal{N}_i} \mathcal{A}_{ij}(x_i(t_k^i) - x_j(t_{\tilde{k}(t)}^j)) \\ &= -\sum_{v_j \in \mathcal{N}_i} \mathcal{A}_{ij}(x_i(t) - x_j(t)) \\ &\quad -\sum_{v_j \in \mathcal{N}_i} \mathcal{A}_{ij}(\varepsilon_i(t) - \varepsilon_j(t)), \quad \forall i \in \mathcal{I}. \end{aligned} \quad (21)$$

Furthermore, we rewrite (21) in a compact vector form as

$$\dot{x} = -\mathcal{L}x(t) - \mathcal{L}\varepsilon(t). \quad (22)$$

We again choose a candidate Lyapunov function for the closed-loop system as follows:

$$V = \frac{1}{2}x^T \mathcal{L}x.$$

Then,

$$\dot{V} = x^T \mathcal{L}\dot{x} = -x^T \mathcal{L}\mathcal{L}(x + \varepsilon) = -q^T q - q^T \mathcal{L}\varepsilon.$$

Before proceeding we introduce a basic inequality

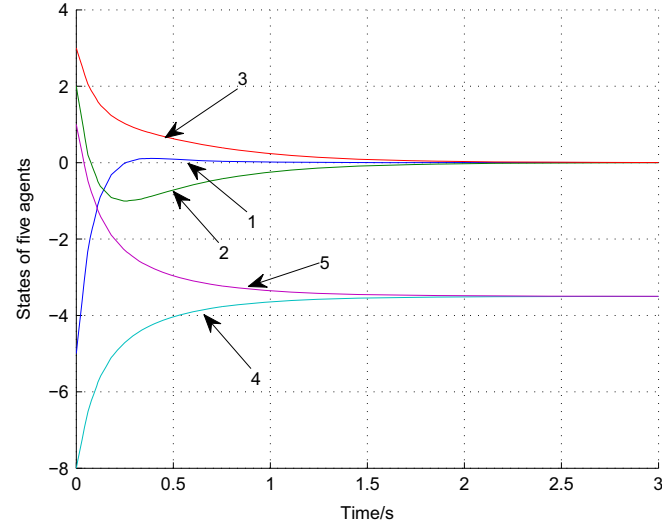$$|rs| \leq \frac{c}{2}r^2 + \frac{1}{2c}s^2, \quad r \in \mathbb{R}, \ s \in \mathbb{R}, \ \forall c > 0 \quad (23)$$

to better clarify our following proof. In the sequel, we will deduce the first part of our conclusion in Theorem 2:

$$\begin{aligned} \dot{V} &= -\sum_{i \in \mathcal{I}} q_i^2 - \sum_{i \in \mathcal{I}}\sum_{v_j \in \mathcal{N}_i} \mathcal{A}_{ij} q_i(\varepsilon_i - \varepsilon_j) \\ &= -\sum_{i \in \mathcal{I}} q_i^2 - \sum_{i \in \mathcal{I}}\sum_{v_j \in \mathcal{N}_i} \mathcal{A}_{ij} q_i \varepsilon_i + \sum_{i \in \mathcal{I}}\sum_{v_j \in \mathcal{N}_i} \mathcal{A}_{ij} q_i \varepsilon_j \\ &\leq -\sum_{i \in \mathcal{I}} q_i^2 + \sum_{i \in \mathcal{I}}\sum_{v_j \in \mathcal{N}_i} \mathcal{A}_{ij}\left(\frac{c}{2}q_i^2 + \frac{1}{2c}\varepsilon_i^2\right) \\ &\quad + \sum_{i \in \mathcal{I}}\sum_{v_j \in \mathcal{N}_i} \mathcal{A}_{ij}\left(\frac{c}{2}q_i^2 + \frac{1}{2c}\varepsilon_j^2\right) \quad \text{(cf. (23))} \\ &= -\sum_{i \in \mathcal{I}} q_i^2 + c\sum_{i \in \mathcal{I}} |N_i^A| q_i^2 + \frac{1}{2c}\sum_{i \in \mathcal{I}} |N_i^A| \varepsilon_i^2 \end{aligned}$$

Table 2
Algorithm for decentralized event-triggered control.

Step 1:	Given the initial conditions with $x(t_0) = x_0$, $t_0 = 0$ and the communication topology $\mathcal{G}$, where x_0 is the initial state of the multi-agent systems
Step 2:	Choose the appropriate parameters $c, \beta, \gamma_{\min}$ and $\eta_{\max}$ according to the given initial conditions
Step 3:	**While** $|\varepsilon_i(t)| \geq \epsilon$ for any $i \in \mathcal{I}$ where $\epsilon \in \mathbb{R}$ is the given bounded error, **goto Step 4**. Else **goto Step 6**
Step 4:	If $\varepsilon_i^2(t) < \beta \frac{\gamma_{\min}}{\eta_{\max}} q_i^2(t)$, $\forall i \in \mathcal{I}$, then $u_i(t) = -\sum_{v_j \in \mathcal{N}_i} \mathcal{A}_{ij}(x_i(t_k^i) - x_j(t_{k(t)}^i))$ and **goto Step 3**. Else **goto Step 5**
Step 5:	Suppose at $t = t_{k+1}^i$ for any $i \in \mathcal{I}$, $\varepsilon_i^2(t_{k+1}^i) = \beta \frac{\gamma_{\min}}{\eta_{\max}} q_i^2(t_{k+1}^i)$. Then $t_k^i = t_{k+1}^i$ where $\varepsilon_i(t_{k+1}^i) = 0$, and all the agents $j \in \mathcal{N}_i \cup i$ update their control protocols
	$u_j(t)$. **Goto Step 3**
Step 6:	Terminate the algorithm
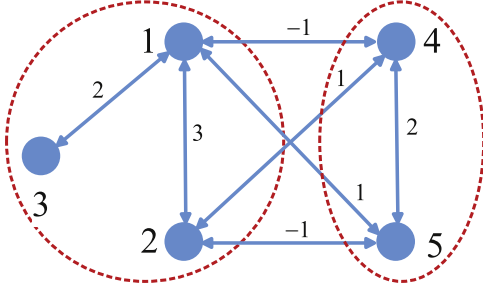
Fig. 2. Topology of five agents for Example 1.

Fig. 4. Group consensus of five agents by centralized event-triggered control.

Fig. 3. Group consensus of five agents.

Fig. 5. Error trajectories in centralized case.

$$+ \frac{1}{2c} \sum_{i \in \mathcal{I}} \sum_{v_j \in \mathcal{N}_i} \mathcal{A}_{ij} \varepsilon_j^2.$$

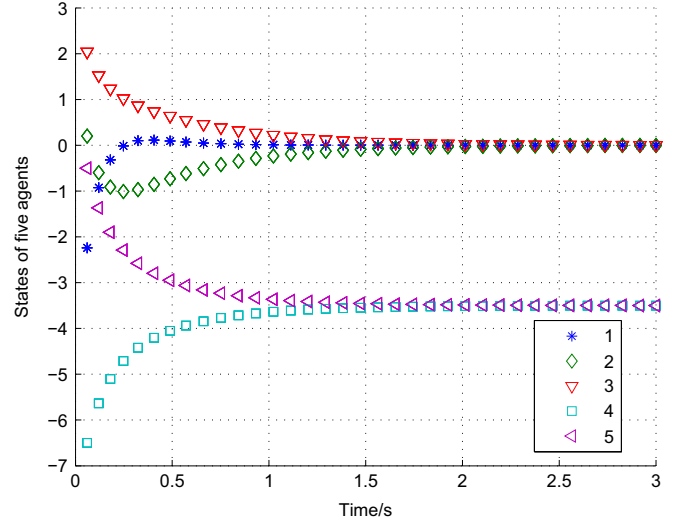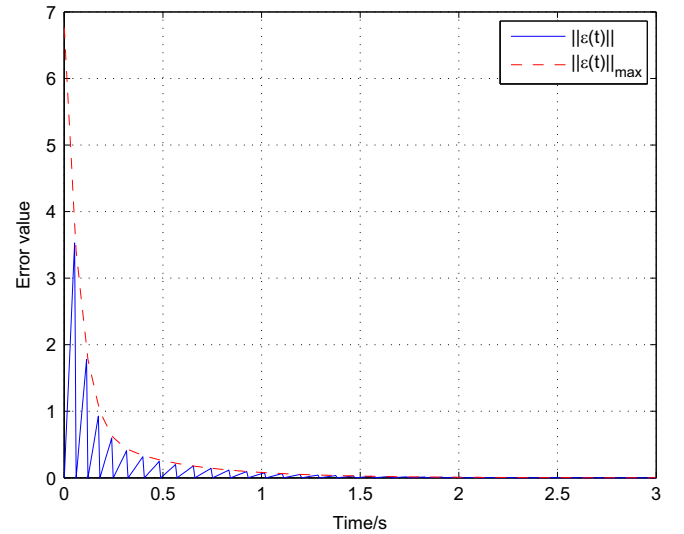Due to the symmetry of $\mathcal{L}$, the last term above can be rewritten as

$$\frac{1}{2c} \sum_{i \in \mathcal{I}} \sum_{v_j \in \mathcal{N}_i} \mathcal{A}_{ij} \varepsilon_j^2 = \frac{1}{2c} \sum_{i \in \mathcal{I}} \sum_{v_j \in \mathcal{N}_i} \mathcal{A}_{ij} \varepsilon_i^2 = \frac{1}{2c} \sum_{i \in \mathcal{I}} |N_i^{\mathcal{A}}| \varepsilon_i^2.$$

Therefore,

$$\dot{V} \leq -\sum_{i \in \mathcal{I}} (1 - c|N_i^{\mathcal{A}}|) q_i^2 + \sum_{i \in \mathcal{I}} \frac{|N_i^{\mathcal{A}}|}{c} \varepsilon_i^2.$$

Consequently, in order to enforce $\dot{V} < 0$, we suppose that $\beta \in (0, 1)$, $\gamma_{\min} = \min_{i \in \mathcal{I}} \{1 - c|N_i^{\mathcal{A}}|\}$, $\eta_{\max} = \max_{i \in \mathcal{I}} \{|N_i^{\mathcal{A}}|/c\}$ and $c \in \bigcap_{i \in \mathcal{I}} (0, 1/|N_i^{\mathcal{A}}|)$. Then if

$$\varepsilon_i^2(t) \leq \beta \frac{\gamma_{\min}}{\eta_{\max}} q_i^2(t), \quad \forall i \in \mathcal{I}, \tag{24}$$

we can obtain that

$$\dot{V} \leq -\sum_{i \in \mathcal{I}} (1 - c|N_i^{\mathcal{A}}|) q_i^2 + \sum_{i \in \mathcal{I}} \frac{|N_i^{\mathcal{A}}|}{c} \varepsilon_i^2$$

$$\leq -\sum_{i \in \mathcal{I}} \gamma_{\min} q_i^2 + \sum_{i \in \mathcal{I}} \frac{|N_i^{\mathcal{A}}|}{c} \beta \frac{\gamma_{\min}}{\eta_{\max}} q_i^2$$

$$\leq -\sum_{i \in \mathcal{I}} \gamma_{\min} q_i^2 + \sum_{i \in \mathcal{I}} \eta_{\max} \beta \frac{\gamma_{\min}}{\eta_{\max}} q_i^2$$

$$= (\beta - 1) \sum_{i \in \mathcal{I}} \gamma_{\min} q_i^2$$
$$< 0.$$

In Theorem 1 we have demonstrated that $V(t)$ is in the quadratic form so that $\lim_{t \to +\infty} x(t) \in \Gamma$. Furthermore, Γ is a globally attractive and invariant manifold. Therefore, group consensus can be asymptotically reached with distributed control protocol (19) and decentralized event-triggered mechanism (20).

In what follows, we will demonstrate that for any initial conditions in $\mathbb{R}^{N_1 + N_2}$ with $t \geq 0$, $\exists l \in \mathcal{I}$, such that the next inter-event time τ_D is strictly positive. Suppose that there exists a t' such that all errors $\varepsilon_i(t') = 0$, $\forall i \in \mathcal{I}$, or there is at least one agent that can evolve with the increase of measurement error function (18). Letting $l = \arg \max_{i \in \mathcal{I}} |q_i|$, we have

$$\frac{|\varepsilon_l|}{(N_1 + N_2)|q_l|} \leq \frac{|\varepsilon_l|}{\|q\|} \leq \frac{\|\varepsilon\|}{\|q\|} = \frac{\|\varepsilon\|}{\|\mathcal{L}x\|}.$$

With the proof of Theorem 1 and the event-triggered mechanism (19), we can deduce that the next inter-event interval of agent l is bounded by τ_D, where

$$\frac{(N_1 + N_2)\tau_D}{1 - \tau_D \|\mathcal{L}\|} = \sqrt{\beta \frac{\gamma_{\min}}{\eta_{\max}}}.$$

Therefore, this inter-event time can be explicitly expressed as

$$\tau_D = \sqrt{\beta \frac{\gamma_{\min}}{\eta_{\max}}} \Big/ \left(\|\mathcal{L}\| \sqrt{\beta \frac{\gamma_{\min}}{\eta_{\max}}} + N_1 + N_2 \right)$$

and this completes the proof. □

Remark 5. Note that $q_i(t) = \sum_{v_j \in \mathcal{N}_i} \mathcal{A}_{ij}(x_i(t) - x_j(t))$, $\forall i \in \mathcal{I}$, only includes the relative state information of agent i's neighbors and its own state information. Therefore, it is a decentralized control protocol. Seeing the proof, we simplify the event-triggered function with only three parameters β, $\gamma_{\min}$ and $\eta_{\max}$, so as to save memory of the systems.

Remark 6. Note that τ_D is more complicated than τ in centralized case because the triggering times in decentralized case depend on every agent's dynamics. Thus, the decentralized event-triggered control updates more frequently than the centralized one. Moreover, when the scale of the multi-agent systems grows larger, i.e., $N_1 + N_2 \uparrow$, then $\tau_D \downarrow$. The algorithm for decentralized event-triggered control is shown in Table 2.

5. Examples and performance analysis

5.1. Examples

Example 1 (*Centralized event-triggered control with group consensus*). Given the multi-agent systems with five agents and the communication topology in Fig. 2. Agents 1, 2 and 3 are in one
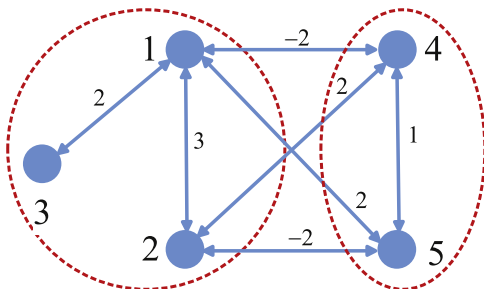

Fig. 6. Topology of five agents for Example 2.

group, while agents 4 and 5 are in another group. Then

$$\mathcal{L}_1 = \begin{bmatrix} 5 & -3 & -2 & 1 & -1 \\ -3 & 3 & 0 & -1 & 1 \\ -2 & 0 & 2 & 0 & 0 \\ 1 & -1 & 0 & 2 & -2 \\ -1 & 1 & 0 & -2 & 2 \end{bmatrix}.$$

The eigenvalues of $\mathcal{L}_1$ are $\lambda_1 = \lambda_2 = 0, \lambda_3 = 2, \lambda_4 = 3.55$ and $\lambda_5 = 8.45$, which are all positive real numbers. $x_0 = (-5, 2, 3, -8, 1)^{\mathrm{T}}$ and $\alpha = 0.85$. In this example, $x^1 = (x_1, x_2, x_3)^{\mathrm{T}}$,
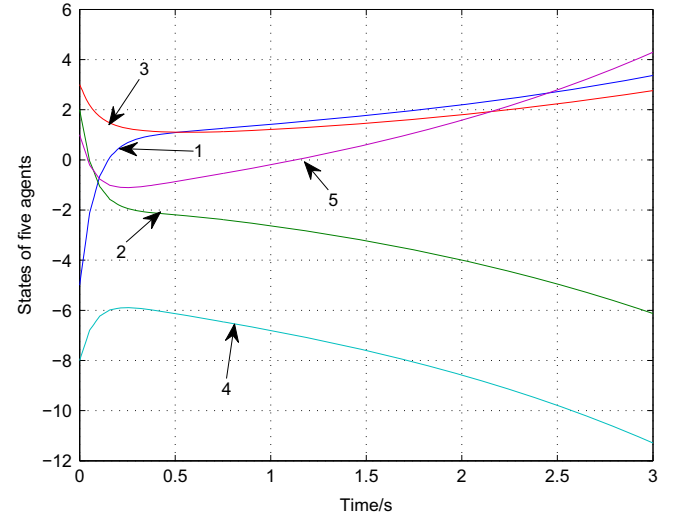

Fig. 7. Five agents without group consensus.
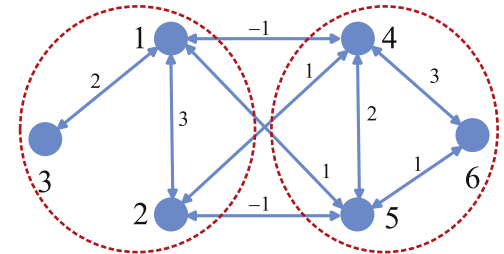

Fig. 8. Topology of six agents for Example 3.
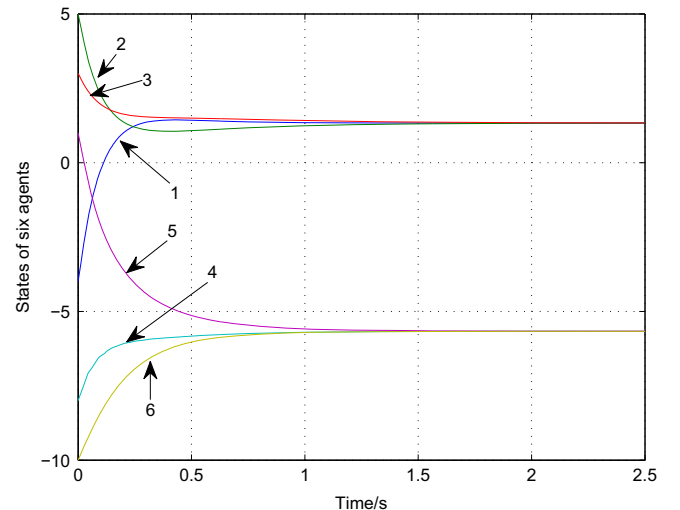

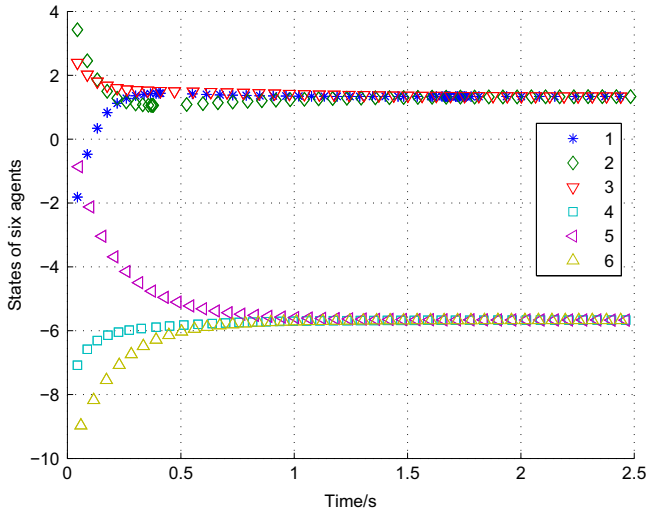**Fig. 9.** Group consensus of six agents.

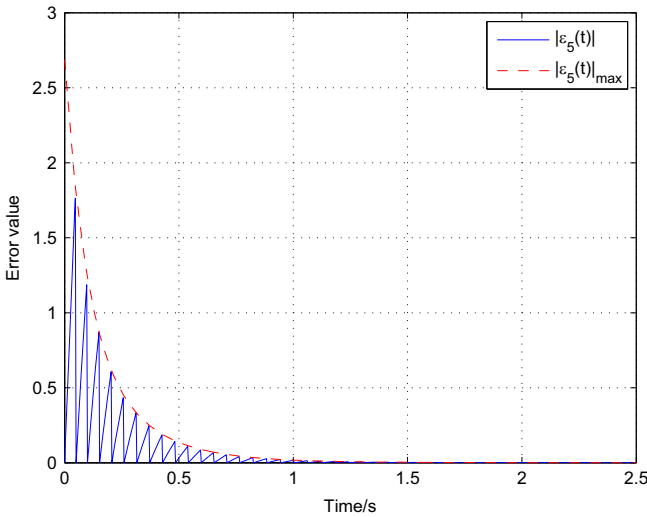**Fig. 10.** Group consensus of six agents by decentralized event-triggered control.

Fig. 11. Error trajectories of agent 5.

Table 3
Comparison of centralized case.

α	Event-triggered times	τ_{mean} (s)
0.25	100	0.0300
0.45	59	0.0508
0.65	46	0.0652
0.85	35	0.0857

$x^2 = (x_4, x_5)^T$ and

$$\mathcal{L}_{12} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then, $(x^1)^T \mathcal{L}_{12} x^2 = (x_1 - x_2)(x_4 - x_5)$ is in accordance with the proof in Theorem 1. The group consensus is shown in Fig. 3. In addition, from Fig. 4 we can see that centralized event-triggered control contains fewer control updates than the periodic control in Fig. 3.

Therefore, the event-triggered control developed in group consensus can reduce energy consumption and communication constraints. In Fig. 5, $\|\varepsilon(t)\|_{max} = \alpha(\|\mathcal{L}x(t)\| / \|\mathcal{L}\|)$, at every triggering time $\varepsilon(t)$ will become zero. Thus, the error accumulates gradually but always below the boundary of $\|\varepsilon(t)\|_{max}$.

Example 2 (*Centralized event-triggered control without group consensus*). Given the multi-agent systems with five agents and the communication topology in Fig. 6. Agents 1, 2 and 3 are in one group, while agents 4 and 5 are in another group. Then

$$\mathcal{L}_2 = \begin{bmatrix} 5 & -3 & -2 & 2 & -2 \\ -3 & 3 & 0 & -2 & 2 \\ -2 & 0 & 2 & 0 & 0 \\ 2 & -2 & 0 & 1 & -1 \\ -2 & 2 & 0 & -1 & 1 \end{bmatrix}.$$

The eigenvalues of $\mathcal{L}_2$ are $\lambda_1 = \lambda_2 = 0, \lambda_3 = -0.437, \lambda_4 = 2.87$ and $\lambda_5 = 9.57$, $x_0 = (-5, 2, 3, -8, 1)^T$ and $\alpha = 0.85$. Note that there is one negative eigenvalue $\lambda_3 = -0.437$ which is not satisfied with our requirement for Laplacian matrix. Thus, the group consensus cannot be reached as shown in Fig. 7.

Example 3 (*Decentralized event-triggered control with group consensus*). Given the multi-agent systems with six agents and the communication topology in Fig. 8. Agents 1, 2 and 3 are in one group, while agents 4, 5 and 6 are in another group. Then

$$\mathcal{L}_3 = \begin{bmatrix} 5 & -3 & -2 & 1 & -1 & 0 \\ -3 & 3 & 0 & -1 & 1 & 0 \\ -2 & 0 & 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 5 & -2 & -3 \\ -1 & 1 & 0 & -2 & 3 & -1 \\ 0 & 0 & 0 & -3 & -1 & 4 \end{bmatrix}.$$

The eigenvalues of $\mathcal{L}_3$ are $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 2.15$, $\lambda_4 = 3.92$, $\lambda_5 = 6.70$ and $\lambda_6 = 9.23$, which are all positive real numbers. $x_0 = (-4, 5, 3, -8, 1, -10)^T$, $c = 0.1$, $\beta = 0.5$, $\gamma_{min} = 0.5$ and $\eta_{max} = 50$. In Fig. 10, it can be seen that the decentralized event-triggered control requires fewer control updates than the periodic control in Fig. 9. Furthermore, each agent updates on its own triggering time in Fig. 10. In Fig. 11, $|\varepsilon_5(t)|_{max} = \sqrt{\beta\gamma_{min}/\eta_{max}}|q_5(t)|$, it illustrates that $|\varepsilon_5(t)|$ will not exceed the boundary of $|\varepsilon_5(t)|_{max}$ in dot line.

5.2. Performance analysis

In Table 3, τ_{mean} represents the mean time between each pair of triggering times with different parameter α and the total running times are the same. We can infer that τ_{mean} will increase with the increase of α, which is in accordance with Remark 3. Therefore, in physical implementations, tuning the parameter α can change the performance of the multi-agent systems.

6. Conclusions

This paper establishes both centralized and decentralized event-triggered control protocols for group consensus to deal with energy consumption and communication constraints considered in physical implementations. In the presence of negative communication weights, we develop two event-triggered functions to decide when to activate the control input in centralized and decentralized cases, respectively. In addition, the infamous Zeno behavior can be excluded in centralized case. Moreover, in decentralized event-triggered case, we simplify the event-triggered function by calculating the maximum and minimum of the corresponding parameters so as to save memory of the systems. In future work, we will focus

on the situations with communication noises and time delays, which are more suitable to the physical world.

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