A Modified Reachability Tree Approach to Analysis of Unbounded Petri Nets

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Abstract—Reachability trees, especially the corresponding Karp-Miller’s finite reachability trees generated for Petri nets are fundamental for systematically investigating many characteristics such as boundedness, liveness, and performance of systems modeled by Petri nets. However, too much information is lost in a FRT to render it useful for many applications. In this paper, modified reachability trees (MRT) of Petri nets are introduced that extend the capability of Karp–Miller’s FRTs in solving the liveness, deadlock, and reachability problems, and in defining or determining possible firing sequences. The finiteness of MRT is proved and several examples are presented to illustrate the advantages of MRT over FRT.

Index Terms—Analysis method, discrete event systems, Petri nets, reachability tree.

I. INTRODUCTION

OVER the last three decades, Petri nets have been proven to be very powerful in modeling, analysis, verification, simulation, performance evaluation, and control of discrete dynamic systems [1]–[3], [9]. Besides the graphical representation, a fundamental advantage of Petri nets is their capacity to systematically investigate many properties and characteristics of the modeled systems. Among various analysis methods and tools, the reachability tree of a Petri net, i.e., the tree representation of its reachability set, is a fundamental and powerful one for various properties including liveness, boundedness, conservativity, reachability, and coverability [2], [3].

Unfortunately, the application of reachability tree analysis method is greatly limited by the fact that the reachability tree of a Petri net may be an infinite tree for a given initial state or marking [2]. Efforts have been made to find some finite representations for reachability trees [4], [5]. Karp and Miller [4] developed the finite reachability tree (FRT) method by introducing a special symbol, \( \omega \), to represent an infinite component in markings resulting from some transition firing loops. This FRT is proved to be useful in determining such properties as safeness, boundedness, conservativeness, and coverability [2], [3], as well liveness and reversibility when the tree contains no \( \omega \), i.e., a system with only a finite number of markings [9].

However, due to the loss of information caused by \( \omega \) symbol, the FRT cannot be used to solve the liveness, deadlock, or reachability problems or to define or determine which firing sequences are possible for an unbounded Petri net that contains infinite number of different markings. The FRT of an unbounded Petri net contains \( \omega \) in at least one of its nodes [9]. A marking with an \( \omega \) component simply indicates that the number of tokens in the corresponding place is potentially infinite with no further information on the structural reason to cause it. Consider the examples given in [2]. Two Petri nets of Fig. 1, with identical FRTs as given in Fig. 1(c), are similar nets but have different reachability sets or reachability trees when they are fully explored. In Fig. 1(a), the number of tokens in place \( p_2 \) is always an even number (until \( t_1 \) fires), whereas in Fig. 1(b), it can be an arbitrary integer. The \( \omega \) symbol itself, however, does not allow this kind of information to be represented, preventing the use of the FRT from solving the reachability problem, i.e., whether a specific given marking can be reached. Fig. 2 gives two Petri nets whose common FRT is shown in Fig. 2(c). Note that the first net in Fig. 2(a) can lead to deadlock (after firing sequence \( t_1t_2t_3 \), for example), but the second one in Fig. 2(b) cannot. The FRT approach fails to distinguish these two cases, thereby making it invalid for deadlock detection.

Following Peterson’s suggestion [2] to use the expression \( a + bn \) rather than \( \omega \) to represent the value of the components of a marking, we presented a modified reachability tree (MRT) method for Petri nets [6]. Later, we developed a computer program for the automated generation of MRTs for Petri nets [7]. MRTs have been proven to be useful. The proposed method extends the capability of FRT in solving the liveness and reachability problems, and in defining or determining which firing sequences are possible.

This paper formalizes and improves the results in [6], [7] with a proof for the finiteness of MRTs, its usefulness in solving reachability, deadlock, and liveness problems, as well as several examples to illustrate the advantages of MRTs over FRTs for Petri net analysis. Next section presents the preliminaries for this MRT method by introducing \( \omega \)-numbers, \( \omega \)-vectors, and their operations. Section III presents the MRT generation algorithm and several examples. Section IV provides the theoretical proof of the finiteness of the proposed algorithm and their property checking capability. Section V concludes this paper.
II. PRELIMINARIES

We assume that a reader has the basic knowledge of Petri nets and related definitions of their execution rules, boundedness, liveness, and reachability. Let $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}^+$ denote the set of integers, nonnegative integers, and positive integers, respectively. The original $\omega$ symbol is defined as in [2] with the properties

$$\omega + a = \omega, \quad a < \omega, \quad \omega \leq \omega, \quad \forall a \in \mathbb{Z}.$$  

Based on our previous work [6] and [7], we introduce the following notations and operations:

1) **$\omega$-number.** A subset of integers $S$ is called an $\omega$-number if and only if there exists $k \in \mathbb{N}^+$, $n$, $q \in \mathbb{Z}$ such that

$$S = \{ik + q \mid i \geq n\}.$$  

It can be shown that $\omega$-number $S$ can be expressed uniquely as

$$S = \omega(k, n, q) \equiv k\omega_n + q$$

$$\equiv \{ik + q \mid k \in \mathbb{N}^+, \quad n \in \mathbb{Z}, \quad 0 \leq q < k, \quad i \geq n\}$$

$\omega(k, n, q)$ or $k\omega_n + q$ is called a canonical $\omega$-number with $k$ as its base, $n$ the least bound, and $q$ the reminder.

Notice that the notion of $\omega$-numbers contains more information about the structure of the infiniteness than the notion $\omega$ does.

2) **Addition of $\omega$-numbers and integers.** For integer $a \in \mathbb{Z}$, it is defined that

$$\omega(k, n, q) + a = \omega(k, n + s, r)$$

where

$$q + a = sk + r, \quad s \in \mathbb{Z}, \quad 0 \leq r < k.$$  

Note that subtraction is defined since $a$ is allowed to be negative in this definition.

3) **Comparison of two $\omega$-numbers.** Let $Z_\omega(N_\omega)$ be the set of integers (nonnegative integers) and $\omega$-numbers. For any $a$, $b \in Z_\omega$, $a \leq b$ is defined as either

$$a, \quad b \in Z \quad \text{and} \quad a \leq b$$

or

$$a = \omega(k, m, q), \quad b = \omega(k, n, q) \quad \text{and} \quad m \leq n.$$  

Note that in the second case, $b$ is a subset of $a$ if both are viewed as sets.

4) **Comparison of two $\omega$-vectors.** A vector $x \in Z_\omega^n$ is called an $\omega$-vector if and only if at least one of its coordinates is an $\omega$-number. Clearly, an $\omega$-vector can be viewed as a set of vectors on $Z$ in an obvious way. An instance of an $\omega$-vector is any ordinary vector in the corresponding vector set. An $\omega$-vector $q$ contains $b$ if and only if $a$ and $b$ have the same non-$\omega$-number coordinates, and

$$a_i \leq b_i, \quad i = 1, 2, \ldots, n$$

where $a_i$ and $b_i$ are coordinates of $a$ and $b$, respectively.

5) **Calculation of the next-state function.** The next-state function, $\delta(\mu, t)$, is defined to be the marking resulting from firing transition $t$ at the current marking $\mu$. If $\mu$ is an $\omega$-marking, i.e., represented by an $\omega$-vector, then $\delta(\mu, t)$ is calculated in the same way as for ordinary markings by using the addition of $\omega$-numbers and integers defined in Item 2.

6) **Association of $\omega$-number components.** Two $\omega$-number components in an $\omega$-marking are called associated if and only if they are reached at the same firing of a transition. The associated $\omega$-number components are synchronized.
in the sense that their integers must be paired orderly (i.e., no cross-products among the associated \( \omega \)-number components).

7) **Conditionally enabled.** If some \( \omega \)-number components of the resulting marking by firing a transition have negative least bounds, the transition is called conditionally enabled by the \( \omega \)-marking.

8) **Elimination of negative markings.** Since no marking with negative components should be allowed in the reachability tree for a Petri net, the resulting marking by firing a conditionally enabled transition is modified as: for an \( \omega \)-number component with a negative least bound \( -s \), setting its least bound to zero and increasing the least bounds of all the associated components by \( s \), and repeating the procedure until all the \( \omega \)-number components have nonnegative least bounds. Note that one may keep markings with negative components to simplify the MRT at the expense of lacking their proper physical explanation.

9) **Classification of nodes.** To describe an MRT, some new names for the nodes are defined: interior, terminal, duplicate, \( \omega \)-duplicate, and frontier nodes. An interior node is a node in MRT with subnodes. A terminal node is a node corresponding to a dead marking without any enabled transitions. A duplicate node is a node with a marking that had previously appeared in the tree along the same path. An \( \omega \)-duplicate node is a node with an \( \omega \)-marking that is contained by another node that appears previously in the tree along the same path. All the other nodes in the tree are frontier nodes.

Notations and operations defined in items 1) to 4) involve \( \omega \)-numbers only, while those defined in items 5) to 9) deal with firings of Petri nets and resulting markings.

III. **GENERATION OF MODIFIED REACHABILITY TREES**

The procedure to construct the MRT for a Petri net can now be precisely stated in terms of the notations and operations introduced in the previous section. Let \( T \) be the set of transitions and \( P \) be the set of places of a Petri net with initial marking \( \mu_0 \). The procedure begins by defining the initial marking to be the root of the MRT and, initially, a frontier node. As long as a frontier node exists, it is converted into an interior, terminal, duplicate, or \( \omega \)-duplicated node by the following algorithm.

**MRT Generation Algorithm:**

1) Let the initial marking \( \mu_0 \) be a frontier node.

2) If no frontier node exists, terminate. Otherwise, let \( x \) be a frontier node to be processed.

   2.1) If there exists another node \( y \) in the tree which is not a frontier node, and has the same marking associated with it, \( \mu[x] = \mu[y] \), then node \( x \) is an \( \omega \)-duplicate node; goto 2.

   2.2) If there exists another node \( y \) on the path from the root to node \( x \) with a marking that contains \( \mu[x] \), \( x \) is an \( \omega \)-duplicate node; goto 2.

   2.3) If no transition is enabled by marking \( \mu[x] \), then \( x \) is a terminal node; goto 2.

   2.4) For all transitions \( t_j \in T \) which are enabled by \( \mu[x] \), let \( \delta(\mu[x], t_j) \) be the modified resulting marking by firing \( t_j \), which is calculated according to rules 5 to 8 in Section II. Create a new node \( z \) in the MRT. The marking \( \mu[z] \) associated with the new node is, for each place \( p \in P \):

   2.4.1) If there exists a node \( y \) on the path from the root node to \( x \) with \( \mu[y] \leq \delta(\mu[x], t_j) \), \( \mu[y] < \delta(\mu[x], t_j) \), and \( \delta(\mu[x], t_j) \in N^+ \), then

   \[
   \mu[z],(k, n, q),
   \]

   where \( k = 8(\mu[x], t_j) - \mu[y] \), and

   \[
   \delta(\mu[x], t_j) = nk + q, \quad 0 \leq q < k;
   \]

   2.4.2) Otherwise, \( \mu[z] = \delta(\mu[x], t_j) \).

   2.4.3) A dash arc, labeled \( t_j \), is directed from \( x \) to node \( z \) in the MRT if \( t_j \) is conditionally enabled by \( \mu[x] \). Otherwise a solid arc, labeled \( t_j \), is directed from \( x \) to \( z \). Node \( z \) now becomes a frontier one.

2.5) Node \( x \) is redefined as an interior node; goto 2.

The generation procedure is based on [6] with the major modification that the modified next-state function \( \delta(\mu[x], t) \) is calculated according to rules 5) to 8) defined in Section II. Hence, no markings with negative components can be produced by the procedure. This ensures that the MRT contains only but all reachable markings from the initial marking of a Petri net. The new MRT generation algorithm enables the usefulness of MRTs in solving the liveness and reachability problems for Petri nets, which is proved in the next section.

In addition, a major error has been found in [6]. In Step 2.2 of the algorithm, an \( \omega \)-duplicate node is originally defined as: “if there exists another node \( y \) in the tree which contains \( \mu[x] \).” This should be changed to “if there exists another node \( y \) on the path from the root node to \( x \) with a marking contains \( \mu[x] \).” If we search through the whole tree for the possible \( \omega \)-duplicate node, there will be a problem that some terminal node might contain the current marking \( x \).

The procedure is implemented in C. It has 797 lines including the comments. The input to the program is the structure of the Petri net containing the number of places, number of transitions, input matrix, output matrix, and initial marking. The output of the program describes the structure again, and then lists all the markings in the tree each attached with its parent marking and type of nodes. After that, all the arcs are listed each with the node it starts, the node it ends, and its type.

Figs. 3 and 4 give the MRTs of the Petri nets in Figs. 1 and 2, respectively. The examples of changing from non-\( \omega \)-numbers to \( \omega \)-numbers, and calculating an \( \omega \)-numbers from another are presented. They indicate that Petri nets in Figs. 1 and 2 have different MRTs, even though their corresponding FRTs are same. It should be noted that Fig. 3 shows clearly that the number of tokens in places \( p_2 \) in Fig. 1(a) is always an even number, unless \( t_1 \) fires [see node (1, 2 \( \omega \), +1, 0, 0)], whereas the number of tokens in places \( p_2 \) in Fig. 1(b) can be an arbitrary nonnegative integer. Similarly, Fig. 4 shows that the Petri net in Fig. 2(a) can lead to deadlock since its MRT contains an interior node with all its subnodes linked by dash arcs, while the net in Fig. 2(b) cannot. Clearly, instance (1, 0, 0) of the marking represented by
Fig. 3. MRTs for Petri nets in Fig. 1 with different reachability.

Fig. 4. MRTs of Petri nets in Fig. 2 with different liveness property.

this interior node leads to deadlock. These examples illustrate that the MRT representation for the reachability set of Petri nets is more powerful than the FRT, and expresses all the necessary information lost in the FRT in those cases.

Fig. 5 presents another Petri net example. Its MRT is given in Fig. 6 and shows that the Petri net is possible to be deadlocked since it contains two interior nodes with all their subnodes linked by dash arcs. It is easy to see that three interior nodes include the instances of (0, 2, 0), (0, 3, 0), (0, 4, 0) where \( i > 3 \), respectively, which all lead to deadlock. For comparison, Fig. 5(b) gives the corresponding FRT for the same Petri net. Clearly, it is not possible to find the deadlock problem for this net if only its FRT is used.

IV. Finiteness, Liveness and Reachability of MRTs

The following theorems guarantee the finiteness of MRTs and their usefulness in analyzing reachability, deadlock, and liveness of unbounded Petri nets.

Theorem 1 (Finiteness): The modified reachability tree of a Petri net is finite.

The proof of the theorem is an extension of that used by Hack [8] for the finiteness of FRTs. We need the following lemmas.

Lemma 1: In any infinite directed tree in which each node has only a finite number of direct successors, there is an infinite path leading from the root.

The proof for the lemma is obvious.

Lemma 2: Every infinite sequence of numbers over \( N^+ \) and \( k \)-based \( \omega \)-numbers contains an infinite nondecreasing subsequence.

Proof: The numbers in this case can be classified into one of the \( k + 1 \) types, i.e., nonnegative integers, \( k \)-based \( \omega \)-numbers with remainder 0, \( k \)-based \( \omega \)-numbers with remainder 1, \( k \)-based \( \omega \)-numbers with remainder \( k \). Since the number of types is finite, every infinite sequence of numbers over them has to contain at least one infinite subsequence over a single type of number. It can be shown easily that this infinite subsequence over the single type contains an infinite nondecreasing subsequence.

Q.E.D.

Lemma 3: Every infinite sequence of \( \eta \)-vectors over \( N^+ \) and \( k \)-based \( \omega \)-numbers (different coordinates may have different bases) contains an infinite nondecreasing subsequence.

Proof: For \( \eta = 1 \), the proof is given in Lemma 2. Assuming the lemma is true for \( n = m \), \( m > 1 \), we prove the lemma is true for \( n = m + 1 \). Consider the first coordinate. By Lemma 2, an infinite subsequence of vectors that is nondecreasing in their first coordinate can be found.

Applying the induction hypothesis on the sequence of \( m \)-vectors obtained by ignoring the first component of the
Proof of the Theorem 1: The theorem can be proved by contradiction. Assume there exists an infinite MRT. Then by Lemma 1 there is an infinite path $x_0, x_1, x_2, \ldots$ from the root $x_0$. Let $\mu[x_0], \mu[x_1], \mu[x_2], \ldots$ be the infinite sequence of $\omega$-vectors on $N^W$. From the construction algorithm of MRT, it is clear that all $\omega$-numbers in each coordinate of the infinite $\omega$-vector sequence has the same base. By Lemma 3 this infinite sequence of $\omega$-vectors has an infinite nondecreasing subsequence $\mu[x_0] \leq \mu[x_1] \leq \mu[x_2] \leq \ldots$ But by construction, we cannot have $\mu[x_i] = \mu[x_j]$, since then it would be a duplicate node and would have no successors. Thus, we must have an infinite strictly increasing sequence $\mu[x_0] < \mu[x_1] < \mu[x_2] < \ldots$. If $\mu[x_i]$ contains non-$\omega$-number components, then by construction, $\mu[x_i] < \mu[x_i+1]$. Therefore, $\mu[x_1]$ has at least one component that is an $\omega$-number, $\mu[x_2]$ has at least two $\omega$-number components, ..., and $\mu[x_n]$ has at least $n$ $\omega$-number components, i.e., a pure $\omega$-marking. But, since $\mu[x_0] \neq \mu[x_1]$, $\mu[x_2] \neq \mu[x_3]$, ... an $\omega$-duplicating node by definition, hence it would be no successors by construction. This is a contradiction, therefore the assumption that an infinite MRT would have existed is incorrect.

Q.E.D.

Theorem 2 (Reachability): The modified reachability tree of a Petri net consists of only but all reachable markings from its initial marking.

The proof of the theorem is obvious from the generation algorithm for the MRT. Note that in this case we consider an $\omega$-marking as a subset of markings represented by an $\omega$-vector, where integers of associated $\omega$-number components are synchronized while integers of nonassociated ones are listed in the form of cross-products. Clearly, all markings from the subset are reachable and all reachable markings from the initial markings are included in the MRT. Therefore, MRT can be used for the reachability analysis of systems modeled by Petri nets.

Before we introduce Theorem 3, we call a node in a MTR with all its subnodes linked by dash arcs a full conditional node, or a partial conditional node if only some but not all of its subnodes are linked by dash arcs. Both full and partial conditional nodes have to be associated with an $\omega$-marking. For a full conditional node, due to the synchronization of associated $\omega$-number components and the cross-product of the nonassociated $\omega$-number components, it is clear from the MRT generation algorithm that some instances of its $\omega$-marking are actually terminal markings [see Fig. 4(a) for example]. Based on this and Theorem 2, we have,

Theorem 3 (Deadlock): A Petri net has deadlocks if and only if its modified reachability tree contains terminal nodes and/or full conditional nodes.

Theorem 4 (Liveness): A transition in a Petri net is live at level 1 (i.e., it is potentially fireable from the initial marking) if and only if it appears as a label to some arc (dash or solid).

Definitions of liveness at different levels can be found in [2, page 86]. Liveness analysis at other levels can be conducted with MRT's too, but more complicated check schemes would be needed for this purpose.

Based on those theorems, it is obvious that MRTs can also be used to determine which sequences of transitions are possible or can be planned.

V. CONCLUSIONS

By introducing $\omega$-number, a notion which contains much more information about the structure of infiniteness than $\omega$ symbol, the MRT offers a more general and powerful representation for the reachability set of unbounded Petri nets and can be used to solve the liveness, deadlock, and reachability problems. Clearly, the concept of MRTs is a straightforward generalization from that of FRTs and the generation algorithms for both basically have the same computational complexity.
While FRTs are very useful for boundedness, safeness, conservation, and coverability, but are invalid for liveness, deadlock, and reachability analysis, which are more important and critical for many applications. Those problems now can be solved easily with MRTs developed in this paper.

REFERENCES


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