Structure design of two types of sliding-mode controllers for a class of under-actuated mechanical systems

W. Wang, X.D. Liu and J.Q. Yi

Abstract: On the basis of sliding-mode control, two sliding-mode controller models based on incremental hierarchical structure and aggregated hierarchical structure for a class of under-actuated systems are presented. The design steps of the two types of sliding-mode controllers and the principle of choosing parameters are given. At the same time, to guarantee the system's stability, two determinant theorems are presented. Then, by theoretical analysis, the two types of sliding-mode controllers are proved to be globally stable in the sense that all signals involved are bounded. The simulation results show the validity of the methods. Therefore an academic foundation for the development of high-dimension under-actuated mechanical systems is provided.

1 Introduction

Under-actuated mechanical systems are characterised by the fact that they have fewer actuators than degrees of freedom to be controlled. That is to say, if the system has n degrees of freedom and m actuators ($m < n$), then there are $n - m$ state-dependent equality constraints on the feasible acceleration of the system that are sometimes referred to as second-order non-holonomic constraints. Examples of such systems include robot manipulators with passive joints (such as the Pendubot and the Acrobot), spacecraft, underwater robots, overhead cranes and so on. It is obvious that under-actuated mechanical systems have many advantages that include decreasing the actuators' number, lightening the system, reducing costs and so on.

Many papers concerning the control of under-actuated mechanical system models have been published in the last few years. Bullo and Lynch [1] proposed a notion of kinematic controllability for second-order under-actuated mechanical systems and used the structure of the system dynamics to naturally decouple the problem into path planning followed by time scaling. Xin and Kaneda [2] presented a robust controller for the Acrobot and the simulation results proved the validity of the swing-up control. Fantoni *et al.* [3] solved the control of the Pendubot on the basis of an energy approach and the passivity properties of the system. The gain-scheduling controller for an overhead crane was studied by Corriga *et al.* [4]. Other under-actuated mechanical systems have been the subject of much recent research [5–14]. However, the control of nonlinear under-actuated mechanical systems has proved challenging because the techniques developed for fully actuated systems cannot be used directly. At the same time, there are many difficulties in the control of under-actuated mechanical systems because of the high nonlinearity, change of the parameters and multi-object to be controlled.

As a kind of highly robust variable structural control method, the sliding-mode controller (SMC) is able to respond quickly, invariant to systemic parameters and external disturbance. Therefore one can consider using SMC to implement the control of the under-actuated mechanical systems. The SMC [15–17], a kind of variable structural control system, is a nonlinear feedback control whose structure is intentionally changed to achieve the desired performance. Therefore the SMC method has gained in popularity in both theory and application. Usually, SMC laws include two parts: switching control law and equivalent control law. The switching control law is used to drive the system's states towards a specific sliding surface and the equivalent control law guarantees the system's states to stay on the sliding surface and converge to zero along the sliding surface. Levant [18] presented a universal single-input–single-output sliding-mode controller with finite-time convergence. But this method is not suitable for large-scale under-actuated mechanical systems. Poznyak *et al.* [19] adopted an integral sliding-mode idea to solve the control problem of multi-model linear uncertain systems. However, this method increased the computational complexity. With an increase of system scale, analysis of convergence and stability problems associated with the system states will become more and more difficult. Therefore the controller structure is very important for controlling complex large-scale nonlinear systems. Many researchers have worked on this problem including Wang [20], who presented a hierarchical fuzzy system. In the design part, he derived a gradient decent algorithm for tuning the parameters of the hierarchical fuzzy system to match the input–output pairs and the simulation results showed that the algorithm was effective. Yi *et al.* [21] presented a new fuzzy controller for anti-swing and position control of an overhead travelling crane

doi:10.1049/iet-cta:20050435

Paper first received 22nd August 2005

W. Wang and X.D. Liu are with the Department of Automatic Control, School of Information Science and Technology, Beijing Institute of Technology, 5 South, Zhongguancun Road, Haidian District, Beijing 100081, People's Republic of China

J.Q. Yi is with the Laboratory of Complex Systems and Intelligence Science, Institute of Automation, Chinese Academy of Sciences, P.O. Box 2728, Beijing 100080, People's Republic of China

E-mail: jianqiang.yi@mail.ia.ac.cn

based on 'single input rule modules' dynamically connected to a fuzzy inference model. Mon and Lin [22] presented a hierarchical sliding-mode controller. However, it only guaranteed that the second-layer sliding surface was stable and that the total control, including only one subsystem's equivalent control, could not guarantee that other subsystems' sliding surfaces were existent. As a result, the anti-disturbance ability of the SMC could be lost. Wang *et al.* [23] also proposed a hierarchical sliding-mode controller for a second-order under-actuated system, but the method was only suitable for simple under-actuated systems that only included two subsystems. For high-dimension under-actuated systems, it is difficult to guarantee the stability of the system according to the hypothesis proposed in that paper. That is, using proper controller structure will predigest the design process and the complex degree of the controller. A systematic way to obtain stabilising controllers for under-actuated mechanical systems with only one input needs to be studied.

This paper proposes two types of sliding-mode controllers based on the incremental hierarchical structure and the aggregated hierarchical structure for a class of under-actuated systems. For the incremental hierarchical structure sliding-mode controller (IHSSMC), the design steps are as follows: first, two states are chosen to construct the first-layer sliding surface. Second, the first-layer sliding surface and one of the left states are used to construct the second-layer sliding surface. This process continues until the last-layer sliding surface is obtained. For the aggregated hierarchical structure sliding-mode controller (AHSSMC), the idea behind this method are as follows: first, the under-actuated system is divided into several subsystems. For each part, we define a first-layer sliding surface. Then, the first-layer sliding surfaces are used to construct the second-layer sliding surface. By theoretical analysis, the conclusion is made that all sliding surfaces of the two SMC structures are asymptotically stable. Simulation results show the validity of the two methods.

2 Dynamic model of under-actuated systems

The general dynamic model of under-actuated mechanical systems with m actuated units from a total of n units can be expressed as follows

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + G(\boldsymbol{q}) = \tau \tag{1}$$

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{bmatrix} M_{11}(\boldsymbol{q}) & M_{12}(\boldsymbol{q}) \\ M_{21}(\boldsymbol{q}) & M_{22}(\boldsymbol{q}) \end{bmatrix} \tag{2}$$

$$\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{bmatrix} C_{11}(\boldsymbol{q}, \dot{\boldsymbol{q}}) & C_{12}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ C_{21}(\boldsymbol{q}, \dot{\boldsymbol{q}}) & C_{22}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \end{bmatrix} \tag{3}$$

$$G(\theta) = \begin{bmatrix} G_1(\boldsymbol{q}) \\ G_2(\boldsymbol{q}) \end{bmatrix}; \quad \boldsymbol{q} = \begin{bmatrix} \boldsymbol{q}_1 \\ \boldsymbol{q}_2 \end{bmatrix}; \quad \tau = \begin{bmatrix} \boldsymbol{\tau}_1 \\ 0 \end{bmatrix} \tag{4}$$

where $\boldsymbol{q} = [\boldsymbol{q}_1, \boldsymbol{q}_2]^{\mathrm{T}} \in R^n$ is the vector of state variables. Here, $\boldsymbol{q}_1 \in R^m$ represents the vector of the m actuated unit variables and $\boldsymbol{q}_2$ represents the vector of the $n - m$ under-actuated unit variables. $\boldsymbol{M}(\boldsymbol{q})$ is the $n \times n$ inertia matrix, $\boldsymbol{C}(\boldsymbol{q}, \boldsymbol{q})\dot{\boldsymbol{q}}$ the vector of the Coriolis and centripetal torques, $G(\boldsymbol{q})$ the gravitational term and $\boldsymbol{\tau}_1$ the vector of control torque.

This kind of under-actuated mechanical system has the following property

(P1) The inertia matrix $\boldsymbol{M}(\boldsymbol{q})$ is symmetric and positive definite for all $\boldsymbol{q}$.

In this paper, we only consider single-input–multiple-output (SIMO) under-actuated mechanical systems such as the Pendubot, the Acrobot, multi-degree inverted pendulum, overhead crane, and so on. If we suppose that $m = 1$, the model of the under-actuated systems can then be converted as follows

$$\begin{aligned} \ddot{\boldsymbol{q}} &= \boldsymbol{M}(\boldsymbol{q})^{-1}[\tau - \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - G(\boldsymbol{q})] \\ &= -\boldsymbol{M}(\boldsymbol{q})^{-1}[\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + G(\boldsymbol{q})] + \boldsymbol{M}(\boldsymbol{q})^{-1}\tau \\ &= F(\boldsymbol{q}, \dot{\boldsymbol{q}}) + B\tau_1 \end{aligned} \tag{5}$$

Note that this paper works with a system processing only one input that appears many times in practice. The model of the SIMO under-actuated mechanical system can then be rewritten as

$$\begin{aligned} \ddot{\boldsymbol{q}}_1 &= f_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) + b_1\boldsymbol{\tau}_1 \\ \ddot{\boldsymbol{q}}_2 &= f_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) + b_2\boldsymbol{\tau}_1 \\ &\vdots \\ \ddot{\boldsymbol{q}}_n &= f_n(\boldsymbol{q}, \dot{\boldsymbol{q}}) + b_n\boldsymbol{\tau}_1 \end{aligned} \tag{6}$$

The control objective is to design a single input $\boldsymbol{\tau}_1$ to guarantee simultaneously the states q_i, $i = 1, \ldots, n$, to achieve the desired performance.

3 Design of the IHSSMC

For SIMO under-actuated mechanical systems, the mathematical model can be translated into the following form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f_1(\boldsymbol{X}) + b_1(\boldsymbol{X})u \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= f_2(\boldsymbol{X}) + b_2(\boldsymbol{X})u \\ &\vdots \\ \dot{x}_{2n-1} &= x_{2n} \\ \dot{x}_{2n} &= f_n(\boldsymbol{X}) + b_n(\boldsymbol{X})u \end{aligned} \tag{7}$$

where $\boldsymbol{X} = (x_1, x_2, \ldots, x_{2n})^{\mathrm{T}}$ is a state variable vector; $f_1(\boldsymbol{X}), \ldots, f_n(\boldsymbol{X})$ and $b_1(\boldsymbol{X}), \ldots, b_n(\boldsymbol{X})$ the nominal continuous nonlinear functions and u the control input. $f_1(\boldsymbol{X}), \ldots, f_n(\boldsymbol{X})$ and $b_1(\boldsymbol{X}), \ldots, b_n(\boldsymbol{X})$ are abbreviated as $f_1, \ldots, f_n$ and $b_1, \ldots, b_n$ in the following description. This class of under-actuated mechanical system belongs to a kind of SIMO nonlinear coupled system. Therefore we can divide this system into several subsystems and the system variable (x_{2i-1}, x_{2i}), $i = 1, \ldots, n$, can be treated as the states of the ith subsystem, respectively. The control objective is to design a single input u to simultaneously control the states $\boldsymbol{X} = (x_1, x_2, \ldots, x_{2n})^{\mathrm{T}}$ to achieve the desired performance. This form can be treated as a norm expression of a class of SIMO under-actuated systems (such as the Pendubot, the Acrobot, overhead crane, pendulum etc.).

To design stable IHSSMC, we make the following assumptions for plant (7)

(A1) $0 \le |f_i(\boldsymbol{X})| \le M_i, \quad \boldsymbol{X} \in A_d^c$
(A2) $0 < |b_i(\boldsymbol{X})| \le B_i, \quad \boldsymbol{X} \in A_d^c$

where M_i and B_i are finite positive constants and A_d^c is a set given as follows

$$A_d^c = \left\{ \boldsymbol{X} \| \boldsymbol{X} - \boldsymbol{X}_0 \|_{p,w} \le \Delta \right\} \tag{8}$$

where w is a set of weights and Δ is a positive constant that denotes all state variables' boundary. $\boldsymbol{X}_0 \in R^{2n}$ is a fixed point and $\|\boldsymbol{X}\|_{p,w}$ is a weighted p-norm, which is defined as

$$\|\boldsymbol{X}\|_{p,w} = \left[\sum_{i=1}^{2n} \left(\frac{x_i}{w_i} \right)^p \right]^{1/p} \tag{9}$$

If $p = \infty$

$$\|\boldsymbol{X}\|_{\infty,w} = \max\left(\frac{|x_1|}{w_1} \cdots \frac{|x_{2n}|}{w_{2n}} \right) \tag{10}$$

If $p = 2$ and $w = 1$, $\|\boldsymbol{X}\|_{p,w}$ will denote the Euclidean norm $\|\boldsymbol{X}\|$.

For the state variables (x_1, x_2), we can construct a suitable pair of sliding surfaces as the first layer

$$s_1 = c_1 x_1 + x_2 \tag{11}$$

where c_1 is a real positive constant. Then, the first-layer surface s_1 can be considered as a general state variable. The first-layer sliding mode variable and one of the left system state variables can be used to construct the second-layer surface s_2, which is expressed as

$$s_2 = c_2 x_3 + s_1 \tag{12}$$

where c_2 is a constant that can change its sign according to the states of the system. Similarly, the $(i-1)$th layer surface s_{i-1} can also be thought of as a general variable to construct the ith-layer surface s_i with one of the left system state variables, which can be written as

$$s_i = c_i x_{i+1} + s_{i-1} \tag{13}$$

where c_i is a constant that can change its sign according to the states of the system. In turn, we can obtain the $(2n-1)$th layer surface s_{2n-1} as

$$s_{2n-1} = c_{2n-1} x_{2n} + s_{2n-2} \tag{14}$$

From the definition of the sliding surfaces, it is clear that all the system's states will be eventually reflected in the last surface. The advantage of this idea is that it can change a traditional high-order sliding-mode surface into several first-order sliding mode surfaces. The coefficients of subsliding-mode surface are easy to design, whereas for high-order sliding mode-surfaces, the coefficients need to satisfy the Hurwitz polynomial.

A group of Lyapunov functions can be defined as

$$V_1 = \tfrac{1}{2} s_1^2, \ldots, V_i = \tfrac{1}{2} s_i^2, \ldots, V_{2n-1} = \tfrac{1}{2} s_{2n-1}^2$$

If we choose the coefficients to satisfy $c_i x_{i+1} \cdot s_{i-1} > 0$, $i = 2, \ldots, 2n-1$, we can obtain that $V_1 \le V_2 \le \cdots \le V_i \le \cdots \le V_{2n-1}$. Then, the coefficients of the sliding-mode surfaces can be chosen as

$$c_i = C_i \operatorname{sign}(x_{i+1} s_{i-1}) \tag{15}$$

where C_i is a positive constant. According to the conditions $s_i = c_i x_{i+1} + s_{i-1}$ and $c_i x_{i+1} \cdot s_{i-1} > 0$, we can obtain that s_i and s_{i-1} are of the same sign. Therefore (15) will become

$$c_i = C_i \operatorname{sign}(x_{i+1} s_1) \tag{16}$$

In the following, we will derive the SMC to guarantee the last layer to converge to zero. For the Lyapunov functions $V_{2n-1} = (1/2) s_{2n-1}^2$, the Lyapunov stability condition can be derived as follows

$$\begin{aligned}
\dot{V}_{2n-1} &= s_{2n-1} \dot{s}_{2n-1} \\
&= s_{2n-1}(c_{2n-1} \dot{x}_{2n} + \dot{s}_{2n-2}) \\
&= s_{2n-1}[c_{2n-1}(f_n + b_n u) + c_{2n-2} x_{2n} \\
&\quad + c_{2n-3}(f_{n-1} + b_{n-1} u) + \cdots + c_1 x_2 + f_1 + b_1 u] \\
&= s_{2n-1} \left\{ \sum_{i=2}^{n} (c_{2i-1} f_i + c_{2i-2} x_{2i}) + (f_1 + c_1 x_2) \right. \\
&\quad \left. + \left[\sum_{i=2}^{n} (c_{2i-1} b_i) + b_1 \right] u \right\}
\end{aligned} \tag{17}$$

The total control law of the IHSSMC can be assumed as

$$u = u_{\text{eq}} + u_{\text{sw}} \tag{18}$$

where u_{sw} is the switching control of the IHSSMC. We can then obtain

$$\begin{aligned}
\dot{V}_{2n-1} &= s_{2n-1} \dot{s}_{2n-1} \\
&= s_{2n-1} \left\{ \sum_{i=2}^{n} (c_{2i-1} f_i + c_{2i-2} x_{2i}) + (f_1 + c_1 x_2) \right. \\
&\quad \left. + \left[\sum_{i=2}^{n} (c_{2i-1} b_i) + b_1 \right] (u_{\text{eq}} + u_{\text{sw}}) \right\} \\
&= s_{2n-1} \left\{ \sum_{i=2}^{n} (c_{2i-1} f_i + c_{2i-2} x_{2i}) + (f_1 + c_1 x_2) \right. \\
&\quad + \left[\sum_{i=2}^{n} (c_{2i-1} b_i) + b_1 \right] u_{\text{eq}} \\
&\quad \left. + \left[\sum_{i=2}^{n} (c_{2i-1} b_{2i}) + b_1 \right] u_{\text{sw}} \right\}
\end{aligned} \tag{19}$$

Let

$$u_{\text{sw}} = -\frac{[\eta \cdot \operatorname{sign}(s_{2n-1}) + k \cdot s_{2n-1}]}{\sum_{i=2}^{n} (c_{2i-1} b_i) + b_1} \tag{20}$$

$$u_{\text{eq}} = -\frac{\sum_{i=2}^{n} (c_{2i-1} f_i + c_{2i-2} x_{2i}) + (f_1 + c_1 x_2)}{\sum_{i=2}^{n} (c_{2i-1} b_i) + b_1} \tag{21}$$

Then, we have

$$\begin{aligned}
\dot{V}_{2n-1} &= -s_{2n-1} \cdot \eta \cdot \operatorname{sign}(s_{2n-1}) - k \cdot s_{2n-1}^2 \\
&= -\eta |s_{2n-1}| - k \cdot s_{2n-1}^2 \le 0
\end{aligned} \tag{22}$$

where k and η are positive constants.

Therefore the control laws (20) and (21) of the IHSSMC can guarantee that the last-layer sliding surface is stable and reachable in finite time.

Remark 1: When the last-layer sliding surface converges to zero, all other sliding surfaces will converge to zero because of the condition $0 \le V_1 \le V_2 \le \cdots \le V_i \le \cdots \le V_{2n-1}$. Therefore we can obtain that $x_3 = x_4 = \cdots = x_{2n} = s_1 = \cdots = s_{2n-1} = 0$. At the same time, the control law becomes $u_{\text{eq}} = -((f_1 + c_1 x_2)/b_1)$, which is equal to the first-layer sliding surface's equivalent control law and satisfies the reachable and stable condition of the SMC. Therefore the control law will drive this subsystem's states to converge to zero along the first-layer sliding surface.

4 Stability analysis of the IHSSMC

Theorem 1: Consider the SIMO under-actuated system (7) with the SMC law defined by (18), (20) and (21). Let the parameters of the incremental sliding surfaces be determined by (16) and let the assumptions (1) and (2) be true. Then, the overall IHSSMC is globally stable in the sense that all signals involved are bounded, with the errors converging to zero asymptotically.

Proof: Integrating both sides of (22) yields

$$\int_0^t \dot{V}_{2n-1}\,\mathrm{d}\tau = \int_0^t (-\eta|s_{2n-1}| - ks_{2n-1}^2)\,\mathrm{d}\tau \tag{23}$$

Hence

$$V_{2n-1}(t) = V_{2n-1}(0) - \int_0^t (\eta|s_{2n-1}| + ks_{2n-1}^2)\,\mathrm{d}\tau \geq 0 \tag{24}$$

Then, we can obtain that

$$\lim_{t\to\infty}\int_0^t (\eta|s_{2n-1}| + ks_{2n-1}^2)\,\mathrm{d}\tau \leq V_{2n-1}(0) < \infty \tag{25}$$

It is obvious that

$$0 \leq \int_0^\infty \eta|s_{2n-1}|\,\mathrm{d}\tau < \infty \tag{26}$$

$$0 \leq \int_0^\infty ks_{2n-1}^2\,\mathrm{d}\tau < \infty \tag{27}$$

If the parameters of IHSSMC satisfy (16), then we have

$$0 \leq V_i \leq V_{2n-1} \tag{28}$$

Then

$$\begin{aligned}\int_0^\infty V_i\,\mathrm{d}\tau &= \int_0^\infty \frac{1}{2}(c_i x_{i+1} + s_{i-1})^2\,\mathrm{d}\tau \leq \int_0^\infty \frac{1}{2}s_{2n-1}^2\,\mathrm{d}\tau \\ &= \int_0^\infty V_{2n-1}\,\mathrm{d}\tau\end{aligned} \tag{29}$$

Further

$$\int_0^\infty (c_i^2x_{i+1}^2 + 2c_ix_{i+1}s_{i-1} + s_{i-1}^2)\,\mathrm{d}\tau \leq \int_0^\infty s_{2n-1}^2\,\mathrm{d}\tau < \infty \tag{30}$$

Because $c_ix_{i+1}\cdot s_{i-1} > 0$, we can obtain

$$\int_0^\infty x_{i+1}^2\,\mathrm{d}\tau < \infty,\quad x_{i+1}\in L_2 \tag{31}$$

$$\int_0^\infty s_{i-1}^2\,\mathrm{d}\tau < \infty,\quad s_{i-1}\in L_2 \tag{32}$$

From (26), we have

$$\begin{aligned}\int_0^\infty |c_ix_{i+1} + s_{i-1}|\,\mathrm{d}\tau &= \int_0^\infty |c_ix_{i+1}|\,\mathrm{d}\tau + \int_0^\infty |s_{i-1}|\,\mathrm{d}\tau \\ &\leq \int_0^\infty |s_{2n-1}|\,\mathrm{d}\tau < \infty\end{aligned} \tag{33}$$

Therefore we can obtain

$$\int_0^\infty |x_{i+1}|\,\mathrm{d}\tau < \infty,\quad x_{i+1}\in L_1 \tag{34}$$

$$\int_0^\infty |s_{i-1}|\,\mathrm{d}\tau < \infty,\quad s_{i-1}\in L_1 \tag{35}$$

From (13), we have

$$|s_{i-1}| = \left|\sum_{j=3}^{i} c_{j-1}x_j + c_1x_1 + x_2\right| \leq \|X\|_{\infty,w},\quad s_{i-1}\in L_\infty \tag{36}$$

where

$$w = \left\{\frac{1}{c_1}, 1, \frac{1}{c_3}, \ldots, \frac{1}{c_i}, \ldots, \frac{1}{c_{2n}}\right\}$$

is a set of weights.

From (16), (20) and (21), we can obtain

$$u = u[\boldsymbol{X}, f(\boldsymbol{X})] \tag{37}$$

Therefore u is bounded. Then, we can define that

$$U_M = \sup_{X\in A_d^c}(u_{\mathrm{sw}} + u_{\mathrm{eq}}) \tag{38}$$

For $\dot{s}_{i-1}$, we can derive the following result

$$\begin{aligned}|\dot{s}_{i-1}| &= \left|\sum_{j=3}^{i} c_{j-1}\dot{x}_j + c_1\dot{x}_1 + \dot{x}_2\right| \\ &= \begin{cases}\left|\begin{array}{l}\sum_{j=2}^{i/2}(c_{2j-1}f_j + c_{2j-2}x_{2j}) \\ +(f_1 + c_1x_2) + \left[\sum_{j=2}^{i/2}(c_{2j-1}b_j) + b_1\right]u\end{array}\right|, & \text{if } i = \text{even} \\ \left|\begin{array}{l}\sum_{j=2}^{(i-1)/2}(c_{2j-1}f_j + c_{2j-2}x_{2j}) + (f_1 + c_1x_2) \\ +c_{i-1}x_{j+1} + \left[\sum_{j=2}^{(i-1)/2}(c_{2j-1}b_j) + b_1\right]u\end{array}\right|, & \text{if } i = \text{odd}\end{cases} \\ &\leq \begin{cases}\sum_{j=1}^{i} M_j + \|X\|_{\infty,w_i} + \sum_{j=1}^{i} B_j\cdot U_M, & \text{if } i = \text{even} \\ \sum_{j=1}^{i-1} M_j + \|X\|_{\infty,w_i} + \sum_{j=1}^{i} B_j\cdot U_M, & \text{if } i = \text{odd}\end{cases} \\ &\leq \sum_{j=1}^{i} M_j + \|X\|_{\infty,w_i} + \sum_{j=1}^{i} B_j\cdot U_M \\ &< \infty\end{aligned} \tag{39}$$

Therefore we have

$$\dot{s}_{i-1}\in L_\infty \tag{40}$$

From (32), (35), (36) and (40), and using the Barbalat lemma, we have $\lim_{t\to\infty} s_{i-1} = 0$, that is to say, s_{i-1}, $i = 2, \ldots, 2n-1$, are asymptotically stable.

Similarly, we can obtain that x_{i+1}, $i = 2, \ldots, 2n-1$, are also asymptotically stable.

For $s_1 = 0$, we can find that u becomes $u = u_1 = u_{\mathrm{eq}\ 1} = -((f_1 + c_1x_2)/b_1)$, which is equal to the equivalent law of the first layer. Therefore x_1 and x_2 will slide to zero along the surface of $s_1 = 0$.

Then, we have proved that all system states are stable and will converge to zero. □

5 Design of the AHSSMC

The dynamic model of the under-actuated mechanical system is shown as (7). The model can be divided into

several subsystems. Then, the AHSSMC can be designed as

$$s_1 = c_1 x_1 + x_2 \tag{41}$$

$$s_2 = c_2 x_3 + x_4 \tag{42}$$

$$\vdots$$

$$s_i = c_i x_{2i-1} + x_{2i} \tag{43}$$

$$\vdots$$

$$s_n = c_n x_{2n-1} + x_{2n} \tag{44}$$

where c_i, $i = 1, \ldots, n$, are the sliding-mode coefficients, which satisfy the Hurwitz polynomial. For the second-order system, the coefficients are real positive constants.

The second sliding surface can be obtained by combining the first sliding surfaces. This is expressed as

$$S = \alpha_1 s_1 + \alpha_2 s_2 + \cdots + \alpha_n s_n \tag{45}$$

where α_i, $i = 1, \ldots, n$ are constants.

From the definition of the sliding surfaces, it is clear that all the system states will be eventually reflected in the last surface. The advantage of this idea is that it only needs to construct a two-layer sliding surface for the whole system. The coefficients of the subsliding-mode surface are easy to design, whereas for a high-order sliding-mode surface, the coefficients need to satisfy the Hurwitz polynomial.

Using the equivalent control method, each subsystem's equivalent control law u_{eqi} can be obtained. The form is as follows

$$u_{\mathrm{eqi}} = -\frac{f_i(\boldsymbol{X}) + c_i x_{2i}}{b_i(\boldsymbol{X})} \tag{46}$$

To guarantee the system's states to slide along the sliding surfaces, the total control law needs to include the equivalent control law. Therefore we can adopt the total control law as follows

$$u = \sum_{i=1}^{n} u_{\mathrm{eqi}} + u_{\mathrm{sw}} \tag{47}$$

where u_{sw} is the switching control law.

According to the Lyapunov stabilisation theorem, we can construct the switching control law u_{sw}. The Lyapunov energy function is chosen as

$$V = \tfrac{1}{2} S^2 \tag{48}$$

Then, we can obtain

$$\begin{aligned} \dot{V} = S\dot{S} &= S(\alpha_1 \dot{s}_1 + \alpha_2 \dot{s}_2 + \cdots + \alpha_n \dot{s}_n) \\ &= S[\alpha_1(c_1 \dot{x}_1 + \dot{x}_2) + \alpha_2(c_2 \dot{x}_3 + \dot{x}_4) + \cdots \\ &\quad + \alpha_n(c_n \dot{x}_{2n-1} + \dot{x}_{2n})] \end{aligned}$$

$$= S\left[\begin{array}{l} \alpha_1\left(c_1 x_2 + f_1(\boldsymbol{X}) + b_1\left(\sum_{i=1}^{n} u_{\mathrm{eqi}} + u_{\mathrm{sw}}\right)\right) \\ + \alpha_2\left(c_2 x_4 + f_2(\boldsymbol{X}) + b_2\left(\sum_{i=1}^{n} u_{\mathrm{eqi}} + u_{\mathrm{sw}}\right)\right) \\ + \cdots + \alpha_n\left(c_n x_{2n} + f_n(\boldsymbol{X}) + b_n\left(\sum_{i=1}^{n} u_{\mathrm{eqi}} + u_{\mathrm{sw}}\right)\right) \end{array}\right]$$

$$= S\left\{\sum_{i=1}^{n}\left[\alpha_i b_i\left(\sum_{\substack{j=1 \\ j\neq i}}^{n} u_{\mathrm{eqj}}\right)\right] + \sum_{i=1}^{n} \alpha_i b_i u_{\mathrm{sw}}\right\} \tag{49}$$

Let

$$\sum_{i=1}^{n}\left[\alpha_i b_i\left(\sum_{\substack{j=1 \\ j\neq i}}^{n} u_{\mathrm{eqj}}\right)\right] + \sum_{i=1}^{n} \alpha_i b_i u_{\mathrm{sw}} = -\eta\ \mathrm{sign}(S) - kS \tag{50}$$

where η and k are positive constants. Therefore we have

$$u_{\mathrm{sw}} = -\left(\sum_{i=1}^{n} \alpha_i b_i\right)^{-1} \times \left\{\sum_{i=1}^{n}\left[\alpha_i b_i\left(\sum_{\substack{j=1 \\ j\neq i}}^{n} u_{\mathrm{eqj}}\right)\right] + \eta\,\mathrm{sign}(S) + kS\right\} \tag{51}$$

Therefore we choose the coefficient α_i to guarantee that $\sum_{i=1}^{n} \alpha_i b_i \neq 0$. Then, formula (49) becomes

$$\dot{V} = -\eta|S| - kS^2 \tag{52}$$

We can then ascertain that the second-layer sliding-mode surface is stable.

6 Stability analysis of the AHSSMC

From the earlier design process, we can find that the second-layer sliding-mode surface is stable. Theorem 2 will prove that the first-layer sliding-mode surfaces are not only stable, but also asymptotically stable.

Theorem 2: Consider the SIMO under-actuated system (7) with the SMC law defined by (41–44). Let assumptions (1) and (2) be true. Then, the overall aggregated SMC system is globally stable in the sense that all signals involved are bounded with the errors converging to zero asymptotically.

Proof: Integrating both sides of (52) yields

$$\int_0^t \dot{V}\,\mathrm{d}\tau = \int_0^t (-\eta|S| - kS^2)\,\mathrm{d}\tau \tag{53}$$

Then, we have

$$V(t) - V(0) = \int_0^t (-\eta|S| - kS^2)\,\mathrm{d}\tau \tag{54}$$

We can find that

$$V(t) = \frac{1}{2}S^2 = V(0) - \int_0^\infty (\eta|S| + kS^2)\,\mathrm{d}\tau \le V(0) < \infty \tag{55}$$

Therefore we can obtain that $S \in L_\infty$, that is

$$\sup_{t\ge 0} |S| = \|S\|_\infty < \infty \tag{56}$$

At the same time, from (49) we can find that

$$\dot{V} = S\dot{S} \le -\eta|S| - kS^2 < \infty \tag{57}$$

It is obvious that $\dot{S} \in L_\infty$, that is

$$\sup_{t\ge 0} |\dot{S}| = \|\dot{S}\|_\infty < \infty \tag{58}$$

From (43), we have

$$|s_i| = |c_i x_{2i-1} + x_{2i}| \le \|\boldsymbol{X}\|_{\infty,w} \tag{59}$$

where $w = \{1/c_{2i-1},\ 1\}$ is a set of weights. Similarly, we have

$$\left|\sum_{\substack{j=1\\ j\neq i}}^{n} s_j\right| = \left|\sum_{\substack{j=1\\ j\neq i}}^{n} (c_j x_{2j-1} + x_{2j})\right| < \|\boldsymbol{X}\|_{\infty,w} \tag{60}$$

At the same time, from (43) we can find that

$$\begin{aligned} |\dot{s}_i| &= |c_i \dot{x}_{2i-1} + \dot{x}_{2i}| \\ &= |c_i x_{2i} + f_i + b_i u| \\ &\le M_i + \|\boldsymbol{X}\|_{\infty,w_i} + B_j \cdot U_M < \infty \end{aligned} \tag{61}$$

where $U_M = \sup_{X\in A_d^c}(u_{\mathrm{sw}} + u_{\mathrm{eq}})$. Hence, we can obtain that $s_i \in L_\infty$ and $\dot{s}_i \in L_\infty$, that is

$$\sup_{t\ge 0} |s_i| = \|s_i\|_\infty < \infty, \quad \sup_{t\ge 0} |\dot{s}_i| = \|\dot{s}_i\|_\infty < \infty \tag{62}$$

For the second-layer sliding-mode surface, we can rewrite formula (45) as

$$S = \alpha_i s_i + \sum_{\substack{j=1\\ j\neq i}}^{n} \alpha_j s_j \tag{63}$$

From the deriving process of the AHSSMC, we can find that α_i does not influence the stability of the system. Hence, we can construct two sliding surfaces as follows

$$\begin{aligned} S_1 &= \left(\alpha_{i1} s_i + \sum_{\substack{j=1\\ j\neq i}}^{n} \alpha_j s_j\right) \\ S_2 &= \left(\alpha_{i2} s_i + \sum_{\substack{j=1\\ j\neq i}}^{n} \alpha_j s_j\right) \end{aligned} \tag{64}$$

where α_{i1} and α_{i2} are arbitrary positive constants and $\alpha_{i1} \neq \alpha_{i2}$. Hence, $S_1 \neq S_2$. We might as well suppose that $\infty > \int_0^\infty S_1^2\,\mathrm{d}\tau > \int_0^\infty S_2^2\,\mathrm{d}\tau \ge 0$. From (55), we have

$$0 \le \int_0^\infty S_1^2\,\mathrm{d}\tau = \int_0^\infty \left(\alpha_{i1} s_i + \sum_{\substack{j=1\\ j\neq i}}^{n} \alpha_j s_j\right)^2 \mathrm{d}\tau < \infty \tag{65}$$

$$0 \le \int_0^\infty S_2^2\,\mathrm{d}\tau = \int_0^\infty \left(\alpha_{i2} s_i + \sum_{\substack{j=1\\ j\neq i}}^{n} \alpha_j s_j\right) \mathrm{d}\tau < \infty \tag{66}$$

Hence, we have

$$\begin{aligned} 0 &< \int_0^\infty (S_1^2 - S_2^2)\,\mathrm{d}\tau \\ &= \int_0^\infty \left((\alpha_{i1}^2 - \alpha_{i2}^2) s_i^2 + 2(\alpha_{i1} - \alpha_{i2}) \cdot s_i \cdot \sum_{\substack{j=1\\ j\neq i}}^{n} \alpha_j s_j\right) \mathrm{d}\tau < \infty \end{aligned} \tag{67}$$

Further, we can obtain

$$\begin{aligned} \int_0^\infty (S_1^2 - S_2^2)\,\mathrm{d}\tau &= \int_0^\infty \Bigg((\alpha_{i1}^2 - \alpha_{i2}^2) s_i^2 \\ &\quad + 2(\alpha_{i1} - \alpha_{i2}) \cdot s_i \cdot \sum_{\substack{j=1\\ j\neq i}}^{n} \alpha_j s_j\Bigg) \mathrm{d}\tau \\ &= \int_0^\infty \big((\alpha_{i1}^2 - \alpha_{i2}^2) s_i^2 \\ &\quad + 2(\alpha_{i1} - \alpha_{i2}) \cdot s_i (S_1 - \alpha_{i1} s_i)\big)\,\mathrm{d}\tau \\ &= \int_0^\infty -(\alpha_{i1} - \alpha_{i2})^2 s_i^2\,\mathrm{d}\tau \\ &\quad + \int_0^\infty 2(\alpha_{i1} - \alpha_{i2}) s_i S_1\,\mathrm{d}\tau > 0 \end{aligned} \tag{68}$$

From (55), we know that

$$0 \le \frac{1}{2} S^2 = V(0) - \int_0^\infty (\eta|S| + kS^2)\,\mathrm{d}\tau \tag{69}$$

Further, we can obtain

$$\int_0^\infty (\eta|S| + kS^2)\,\mathrm{d}\tau = \int_0^\infty \eta|S|\,\mathrm{d}\tau + \int_0^\infty kS^2\,\mathrm{d}\tau \le V(0) < \infty \tag{70}$$

Then, we have $\int_0^\infty \eta|S|\,\mathrm{d}\tau \ge 0$ and $\int_0^\infty kS^2\,\mathrm{d}\tau \ge 0$. If the summing of two positive numbers is finite, then the two positive numbers are also finite. Therefore we can obtain $0 \le \eta\int_0^\infty |S|\,\mathrm{d}\tau = \|S\|_1 < \infty$, $S \in L_1$ (absolute integral). Hence from (68), we have

$$\begin{aligned} \int_0^\infty (\alpha_{i1} - \alpha_{i2})^2 s_i^2\,\mathrm{d}\tau &< \int_0^\infty 2(\alpha_{i1} - \alpha_{i2}) s_i S_1\,\mathrm{d}\tau \\ &\le 2\int_0^\infty |(\alpha_{i1} - \alpha_{i2}) s_1 S_1|\,\mathrm{d}\tau \\ &\le 2|\alpha_{i1} - \alpha_{i2}| \int_0^\infty \|s_i\|_\infty |S_1|\,\mathrm{d}\tau \\ &= 2|\alpha_{i1} - \alpha_{i2}| \cdot \|s_i\|_\infty \|S_1\|_1 < \infty \end{aligned} \tag{71}$$

Therefore

$$\int_0^\infty s_i^2\,\mathrm{d}\tau < \infty \tag{72}$$

From (72), we have $s_i \in L_2$ (square integral). Because $s_i \in L_\infty$ and $\dot{s}_i \in L_\infty$, according to the Barbalat lemma, $\lim_{t\to\infty} s_i = 0$. □

In summary, the first-layer subsystems' sliding surfaces s_i, $i = 1, \ldots, n$, are not only stable, but also asymptotically stable.

Remark 2: Although both the IHSSMC and the AHSSMC are hierarchical, there is some difference between them. First, the layer number is different. The IHSSMC has a multi-layer structure, whereas the AHSSMC has a two-layer structure. Secondly, the parameters of the AHSSMC are less than those of the IHSSMCs. Finally, the sliding-mode surface parameters of the AHSSMC are constant, whereas the sliding-mode surface parameters of the IHSSMC will change according to the system's states. In summary, the

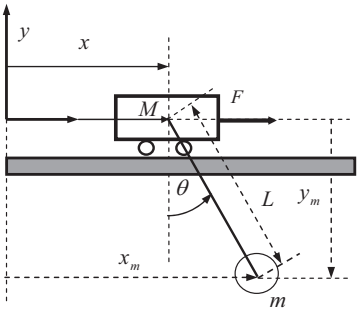

Fig. 1 *Overhead crane system*

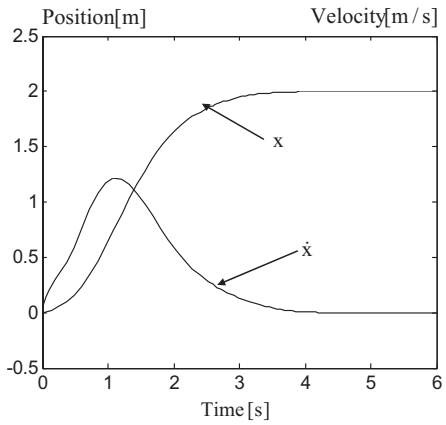

Fig. 2 *Output curve of displacement subsystem*

structure of the AHSSMC is simpler than that of the IHSSMC. But the design of the IHSSMC is more intuitionistic. The effects of the two sliding-mode controllers will be shown in the following section.

7 Simulation results

To assess the proposed IHSSMC and AHSSMC developed in this paper, a simulation example is given. An overhead crane system (shown as Fig. 1) is a typical under-actuated system. The control objective of the overhead crane is to move the trolley to its destination and complement anti-swing of the load at the same time.

For simplicity, in this paper, the following assumptions are made: (a) the trolley and the load can be regarded as point masses; (b) friction force that may exist in the trolley can be neglected; (c) elongation of the rope because of tension force is neglected and (d) the trolley moves along the rail and the load moves in the x–y plane.

From Fig. 1 we can find that $x_m = x + L\sin\theta$ and $y_m = -L\cos\theta$. Using Lagrange's method, we can obtain the model of the overhead crane system as

$$x: (m+M)\ddot{x} + mL(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) = F \tag{73}$$

$$\theta: \ddot{x}\cos\theta + L\ddot{\theta} + g\sin\theta = 0 \tag{74}$$

where M and m are the masses of the trolley and the load, respectively. θ is the sway angle of load and L is the length of suspension rope.

In summary, we can obtain f_1, b_1, f_2 and b_2 from (7)

$$f_1 = \frac{mL\dot{\theta}^2\sin\theta + mg\sin\theta\cos\theta}{M + m\sin^2\theta} \tag{75}$$

$$b_1 = \frac{1}{M + m\sin^2\theta} \tag{76}$$

$$f_2 = -\frac{(m+M)g\sin\theta + mL\dot{\theta}^2\sin\theta\cos\theta}{(M + m\sin^2\theta)L} \tag{77}$$

$$b_2 = -\frac{\cos\theta}{(M + m\sin^2\theta)L} \tag{78}$$

where $x_1 = e = x^d - x$, $x_2 = \dot{x}^d - \dot{x}$, $x_3 = \theta$ and $x_4 = \dot{\theta}$ are the displacement error of the trolley in the horizontal direction, the velocity error of the trolley in the horizontal direction, the sway angle of the load and the sway angle velocity of the load, respectively.

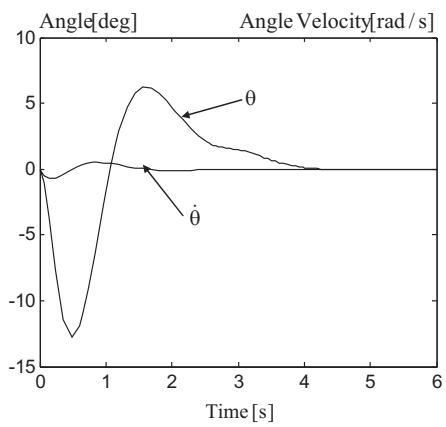

Fig. 3 *Output curve of angle subsystem*

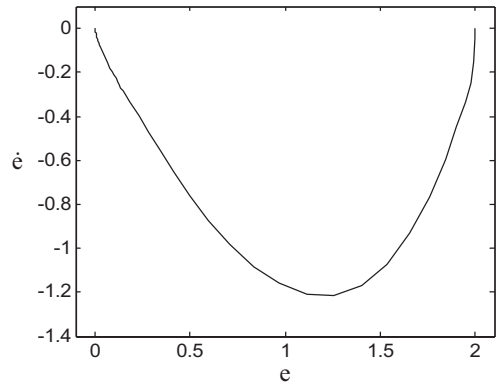

Fig. 4 *Phase curve of displacement error*

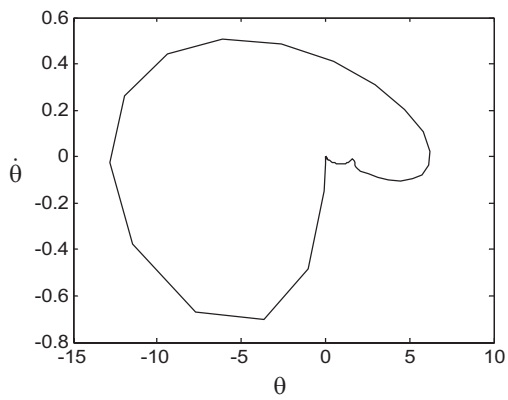

Fig. 5 *Phase curve of angle subsystem*

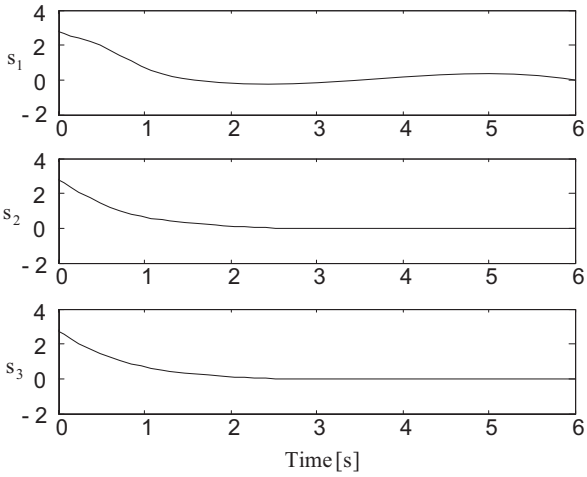

Fig. 6 *Convergent curve of all the sliding surfaces*

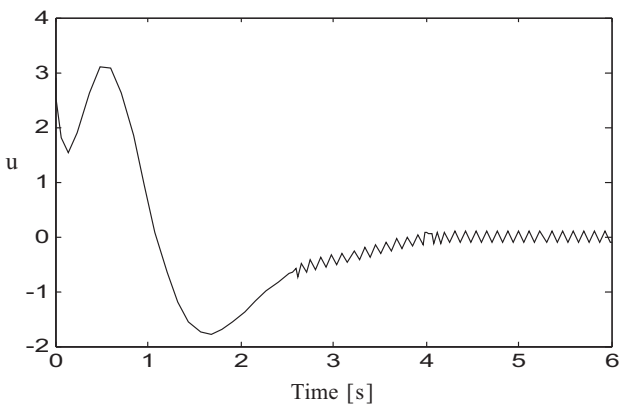

Fig. 7 *Output torque of the IHSSMC*

7.1 Simulation results of the IHSSMC

The parameters of the overhead crane are chosen as [23]: $M = 1$ kg, $m = 0.8$ kg and $L = 0.305$ m, and the parameters of the IHSSMC are chosen as $c_1 = C_1 = 1.4$, $C_2 = 0.2$, $C_3 = 0.1$, $k = 0.1$ and $\eta = 1$.

The initial conditions of the overhead crane system are $(x_0, \dot{x}_0) = (0, 0)$ and $(\theta_0, \dot{\theta}_0) = (0, 0)$ and the expectations are $x^d = 2m$, $\dot{x}^d = 0$, $\theta^d = 0$ and $\dot{\theta}^d = 0$, where x^d, $\dot{x}^d$, θ^d and $\dot{\theta}^d$ are the expected displacement and velocity of the trolley in the horizontal direction and the expected swing angle and swing angular velocity of the load, respectively.

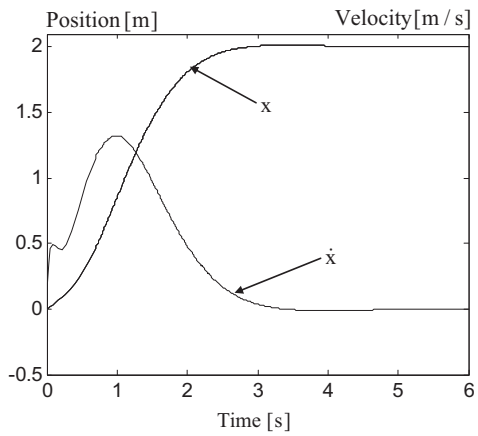

Fig. 8 *Output curve of displacement subsystem*

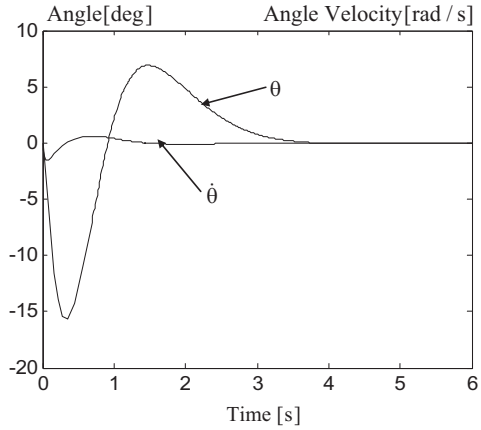

Fig. 9 *Output curve of angle subsystem*

Fig. 2 shows the displacement and the velocity of the overhead crane system and Fig. 3 shows the swing angle of the load and its angle velocity with the IHSSMC. The simulation results show that the IHSSMC can control the trolley to its destination and implement anti-sway control at the same time. Figs. 4 and 5 show the phase plane curve of the first-layer sliding surface. We can find that the first-layer sliding surface is existent and the first subsystem's states can converge to zero along the sliding surface. Fig. 6 shows the convergent curve of all the sliding surfaces. Fig. 7 shows the output torque of the controller. The simulation results show the validity of the IHSSMC.

7.2 Simulation results of the AHSSMC

The parameters of the AHSSMC are chosen as $c_1 = 0.8$, $c_2 = 35$, $\alpha_1 = 10$, $\alpha_2 = 1$, $\eta = 3.5$ and $k = 6$. Fig. 8 shows the displacement and the velocity of the overhead crane system and Fig. 9 shows the swing angle of the load and its angle velocity with the AHSSMC. The simulation results show that the AHSSMC can control the trolley to its destination and implement anti-sway control at the same time. Figs. 10 and 11 show the phase plane curve of the first-layer sliding surface. We can find that the first-layer sliding surface is existent and the first subsystem's states can converge to zero along the sliding surface. Fig. 12 shows the convergent curve of all the sliding surfaces. Fig. 13 shows the output torque of the controller. The simulation results show the validity of the AHSSMC.

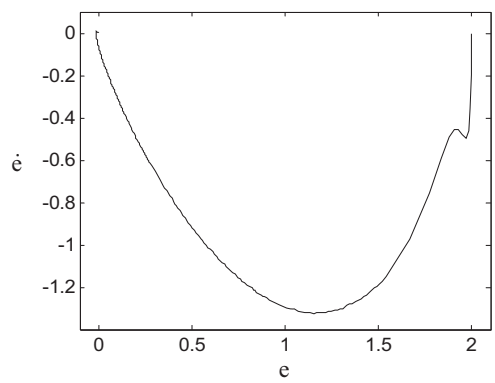

Fig. 10 *Phase curve of displacement error*

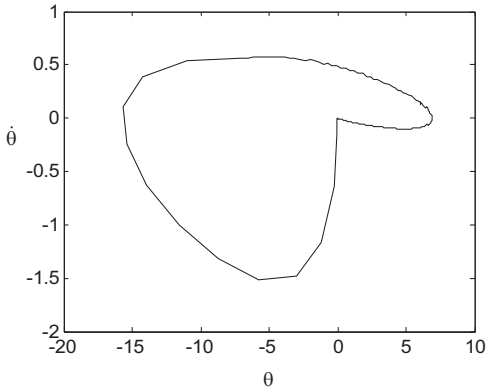

Fig. 11 *Phase curve of angle subsystem*

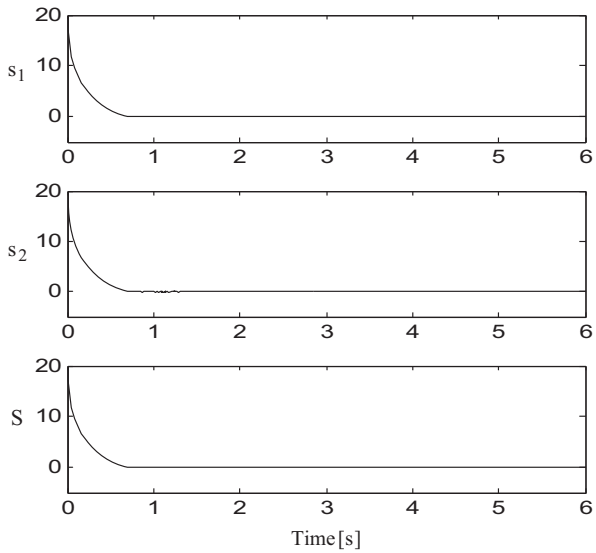

Fig. 12 *Convergent curve of all the sliding surfaces*

Remark 3: From the simulation results, we find that the control effects of the two sliding-mode controllers are different. For the AHSSMC, although its structure is two-layered, the control output torque is larger than that of the IHSSMCs. It is noticeable that the AHSSMC has a rapid response speed and a big initial swing angle. It requires that the controller has a larger output and the controlled object has a firm structure. It follows, therefore, that the AHSSMC suits a fast situation whereas the IHSSMC adapts to the slow situation that requires safety.

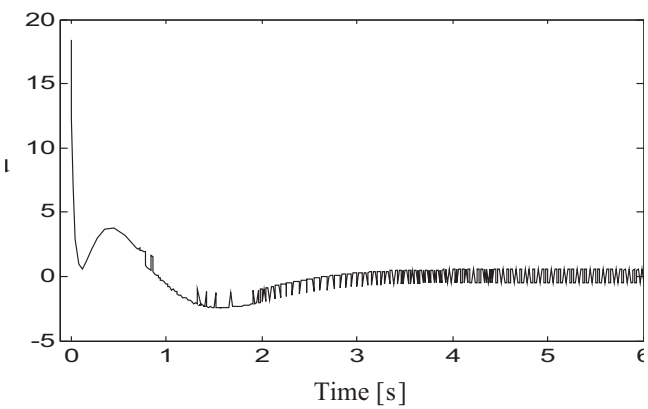

Fig. 13 *Output torque of the AHSSMC*

8 Conclusion

Two types of sliding-mode controller models based on incremental hierarchical structure and aggregated hierarchical structure for a class of SIMO under-actuated mechanical systems are presented in this paper. This paper has proved that the last-layer sliding surface is stable and all other sliding surfaces and system states can converge to zero asymptotically. At the same time, both the IHSSMC and the AHSSMC can reduce the dimension of the sliding surface and predigest the stability analysis. The simulation results also show the validity of the methods. In general, for the classical sliding-mode control methodology, a unique surface yielding a very hard algorithm needs to be defined and may be impossible to apply for some practical problems, whereas this work divides the problem into several layers (very simple ones) making the calculation very easy. The ideas of this paper are to simplify and to obtain a systematic tool for stabilising mechanical systems, in general, where no constraint on the kinematics is imposed, such as the non-holonomic ones, for instance. Therefore, this paper yields a systematic way to obtain stabilising controllers for under-actuated mechanical systems with only one input where it is possible to see how the presented methodology converges in the limit to the classical SMC process.

9 Acknowledgment

This work was supported by the National Nature Sciences Fund of China (grant no. 60575047 and no. 10402003) and the Chinese Postdoctoral Fund.

10 References

1 Bullo, F., and Lynch, K.M.: 'Kinematic controllability for decoupled trajectory planning in underactuated mechanical systems', *IEEE Trans. Robot. Autom.*, 2001, **17**, (4), pp. 402–411

2 Xin, X., and Kaneda, M.: 'A robust control approach to the swing up control problem for the Acrobot'. Proc. 2001 IEEE/RSJ Int. Conf. on Intelligent Robots and Systems, 2001, pp. 1650–1655

3 Fantoni, I., Lozano, R., and Spong, M.W.: 'Energy control of the Pendubot', *IEEE Trans. Autom. Control*, 2000, **45**, (4), pp. 725–729

4 Corriga, G., Giua, A., and Usai, G.: 'An implicit gain-scheduling controller for cranes', *IEEE Trans. Control Syst. Technol.*, 1998, **6**, (1), pp. 15–20

5 Yi, J., Yubazaki, N., and Hirota, K.: 'A new fuzzy controller for stabilization of parallel-type double inverted pendulum system', *Fuzzy Sets Syst.*, 2002, **126**, (1), pp. 105–119

6 Singhose, W., Porter, L., Kenison, M., and Kriikku, E.: 'Effect of hoisting on the input shaping control of gantry cranes', *Control Eng. Pract.*, 2000, **8**, (10), pp. 1159–1165

7 Mazenc, F., Pettersen, K., and Nijmeijer, H.: 'Global uniform asymptotic stabilization of an underactuated surface vessel', *IEEE Trans. Autom. Control*, 2002, **47**, (10), pp. 1759–1762

8 Martinez, S., Cortes, J., and Bullo, F.: 'Analysis and design of oscillatory control systems', *IEEE Trans. Autom. Control*, 2003, **48**, (7), pp. 1164–1177

9 Sanposh, P., Tarn, T.J., and Cheng, D.: 'Theory and experimental results on output regulation for nonlinear systems'. Proc. Am. Control Conf., Anchorage, AK, 8–10 May 2002, pp. 96–101

10 Choudhury, P., Stephens, B., and Lynch, K.M.: 'Inverse kinematics-based motion planning for underactuated systems'. Proc. 2004 IEEE Int. Conf. on Robotics and Automation, New Orleans, LA, April 2004, pp. 2242–2248

11 Fukao, T., Fujitani, K., and Kanade, T.: 'Image-based tracking control of a blimp'. Proc. 42nd IEEE Conf. on Decision and Control, Maui, Hawaii, USA, December 2003, pp. 5414–5419

12 Xin-Sheng, G., Li-Qun, C., and Yan-Zhu, L.: 'Attitude control of underactuated spacecraft through flywheels motion using genetic algorithm with wavelet approximation'. Proc. 5th World Congress on Intelligent Control and Automation, Hangzhou, P.R. China, 15–19 June 2004, pp. 5466–5470

13 Martinez, S., Cortes, J., and Bullo, F.: 'A catalog of inverse-kinematics planners for underactuated systems on matrix Lie groups'. Proc. 2003 IEEE/RSJ Int. Conf. on Intelligent Robots and Systems, Las Vegas, NV, October 2003, pp. 625–630
14 De Luca, A., and Iannitti, S.: 'A simple STLC test for mechanical systems underactuated by one control'. Proc. IEEE Int. Conf. on Robotics and Automation, Washington, DC, 2002, pp. 1735–1740
15 Yoo, B., and Ham, W.: 'Adaptive fuzzy sliding mode control of nonlinear system', *IEEE Trans. Fuzzy Syst.*, 1998, **6**, (2), pp. 315–321
16 Ha, Q.P., Nguyen, Q.H., and Durrant-Whyte, H.F.: 'Fuzzy sliding-mode controller with applications', *IEEE Trans. Ind. Electron.*, 2001, **48**, (1), pp. 38–45
17 Palm, R., and Driankov, D.: 'Design of a fuzzy gain scheduler using sliding mode control principles', *Fuzzy Sets Syst.*, 2001, **121**, (1), pp. 13–23
18 Levant, A.: 'Universal single-input–single-output (SISO) sliding-mode controllers with finite-time convergence', *IEEE Trans. Autom. Control*, 2001, **46**, (9), pp. 1447–1451
19 Poznyak, A.S., Fridman, L., and Bejarano, F.J.: 'Mini–max integral sliding-mode control for multi-model linear uncertain systems', *IEEE Trans. Autom. Control*, 2004, **49**, (1), pp. 97–102
20 Wang, L.-X.: 'Analysis and design of hierarchical fuzzy systems', *IEEE Trans. Fuzzy Syst.*, 1999, **17**, (5), pp. 617–624
21 Yi, J., Yubazaki, N., and Hirota, K.: 'Anti-swing and positioning control of overhead traveling crane', *Inf. Sci.*, 2003, **155**, pp. 19–42
22 Mon, Y.-J., and Lin, C.-M.: 'Hierarchical fuzzy sliding-mode control'. IEEE World Congress on Computational Intelligence, 2002, pp. 656–661
23 Wang, W., Yi, J.Q., Zhao, D.B., and Liu, D.T.: 'Design of a new stable sliding-mode controller of a class of second-order underactuated systems', *IEE Proc., Control Theory Appl.*, 2004, **151**, (6), pp. 683–690