On Three Types of Covering-Based Rough Sets

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Abstract—Rough set theory is a useful tool for data mining. It is based on equivalence relations and has been extended to covering-based generalized rough set. This paper studies three kinds of covering generalized rough sets for dealing with the vagueness and granularity in information systems. First, we examine the properties of approximation operations generated by a covering in comparison with those of the Pawlak's rough sets. Then, we propose concepts and conditions for two coverings to generate an identical lower approximation operation and an identical upper approximation operation. After the discussion on the interdependency of covering lower and upper approximation operations, we address the axiomization issue of covering lower and upper approximation operations. In addition, we study the relationships between the covering lower approximation and the interior operator and also the relationships between the covering upper approximation and the closure operator. Finally, this paper explores the relationships among these three types of covering rough sets.

Index Terms—Rough sets, approximation, covering, data mining, reduct, fuzzy sets, granular computing, computing with words.

1 Introduction

Across a wide variety of fields, data are being collected and accumulated at a dramatic pace, especially in the age of the Internet. There is much useful information hidden in the accumulated voluminous data, but it is very hard for us to obtain it. Thus, there is an urgent need for a new generation of computational theories and tools to assist humans in extracting knowledge from the rapidly growing volumes of digital data; otherwise, these huge data are useless for us.

For data in an information system, the acquisition of knowledge and reasoning may involve vagueness, incompleteness, and granularity. In order to deal with the incomplete and vague information in classification, concept formulation, and data analysis, researchers have proposed many methods other than classical logic, for example, fuzzy set theory [53], rough set theory [21], [22], [23], [24], computing with words [26], [38], [54], [55], [56], [57], [58], granular computing [2], [7], [10], [16], [49], [50], formal concept analysis [40], quotient space theory [60], [61], and computational theory for linguistic dynamic systems [39]. The advantage of the rough set method is that it does not need any additional information about the data, like probability in statistics or membership in fuzzy set theory.

The main idea of the rough set theory comes from Pawlak's work [20]. Many researchers made contributions to this theory [15], [33], [46], [63]. Applications of the rough set method to process control, economics, medical diagnosis, biochemistry, environmental science, biology, chemistry, psychology, conflict analysis, and other fields can be found in the literature [25], [27], [28], [29], [36], [48], [52], [62], [64], [71].

In Pawlak's rough set theory [20], the lower and upper approximation operations are two key concepts. An equivalence relation, that is, a partition, is the simplest formulation of the lower and upper approximation operations, but many interesting and meaningful extensions have been made based on binary relations [6], [30], [31], [34], [35], [74] and coverings [30], [37], [59], [78]. In this paper, we study covering-based rough sets. Extensive research on this subject can be found in [3], [4], [5], [8], [9], [11], [12], [13], [18], [41], [42], [43], [67], [68], [69], [73]. Yao also studied a kind of covering-based rough sets, called neighborhood system, in [47], [51].

For the partition-based lower and upper approximation operations, different partitions of a set will generate different lower and upper approximation operations on this set. However, in the covering-based rough set model, different coverings may generate the same lower or the same upper approximation operation. Therefore, it is necessary to find the conditions under which two coverings generate the same lower or the same upper approximation operation.

If we omit a member of a partition, it will not be a partition anymore; thus, there is no redundancy problem for a partition. As for a covering, if we omit a member from it, it can still be a covering and, furthermore, the new covering and the original one can generate the same lower or the same upper approximation operation. This phenomenon shows that there may exist redundancy in a covering. As a result, how to get rid of the redundancy from a covering is an important research issue in the covering rough set theory.

Lower and upper approximation operations in Pawlak's rough set model are dual; therefore, the lower approximation operation uniquely determines the upper approximation operation and vice versa. However, in the covering-based rough set model, lower and upper approximation operations are no longer dual. The next question, of course, is whether the lower approximation operation can uniquely determine the upper approximation operation and vice versa.

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Another important problem is to find the essential properties for the lower and the upper approximation operations. In other words, there is a need to establish an axiomatic system for the lower and the upper approximation operations. Last, there are several possibilities to define upper approximation operations in a covering setting. In this paper, we present three types of upper approximation operations and examine the relationships among these three types of upper approximation operations.

The generalization of the classical rough set theory attracts many researchers. Yao extensively reviewed and compared the constructive and algebraic approaches in the study of rough sets and characterized several classes of rough set algebras by different sets of axioms [45]. Yao's axioms of approximation operations guarantee the existence of certain types of binary relations producing the same approximation operations. His focus was on binary-relation-based rough set models. In a sense, Yao's works [44], [45], [46], [47] had made an exhaustive study of the familiar binary relations on a universe of discourse and given the corresponding axiomatic systems for the generalized rough set models related to those binary relations. There is a one-to-one correspondence between equivalence relations and partitions on a universe of discourse, but, unfortunately, there is no such correspondence between all binary relations and all coverings on a universe of discourse [1]. Zakowski studied a set of axioms on approximation operations [59]. Lin and Liu gave a set of six axioms on approximation operations by the topological method [17]. Zhu and He presented a compact axiomatic system [65]. These axiomatic systems are all for Pawlak's rough set model.

Zakowski first extended Pawlak's rough set theory by using a covering of the domain, rather than a partition [59]. Based on the mutual correspondence of the concepts of extension and intension, Bonikowski et al. have formulated the necessary and sufficient conditions for the existence of operations on rough sets, which are analogous to classical operations on sets [4]. Pomykala studied the second type of the covering rough set model [30]. His main method was the interior operator from topology. The redundancy problem and the axiomization problem have not been considered in his work. A third type of covering generalized rough sets has been introduced in [37], but no properties of this new class of covering generalized rough sets have been discussed. Two other types of covering-based rough set models have been proposed in [73], [77], and [78], but we do not consider them in this paper.

The remainder of this paper is arranged as follows: In Section 2, we present the fundamental concepts and properties of the classical rough set theory originated from Pawlak, and these are all bases our discussion starts from. We also define some concepts that will be used in the remainder of this paper. Section 3 discusses the first type of covering generalized rough sets. In Section 3.1, by comparing with Pawlak's rough set theory, we detail the similar and the different points between those two rough set models. In Section 3.2, we propose one of the core concepts in this paper, reduct, to reduce a covering to its simplest form while not changing the covering lower approximation operation and the first type of covering upper approximation operation. Still, through this concept, we get a necessary and sufficient condition for two coverings to generate the same lower approximation operation or the same upper approximation operation and establish the interdependency between the lower and the upper approximation operations. Section 3.3 presents an algorithm to compute the reduct of a covering. Axiomization of the covering lower approximation operation is the topic in Section 3.4. We try to find the essential properties for the covering lower approximation operation, and come up with an axiomatic system for the covering lower approximation operation, which introduces a logical and algebraic structure of this operation. Contrary to the situation of the lower approximation, the current popular properties of the upper approximation operation are not sufficient to characterize such an operation, as proved by an example in Section 3.5. In Section 3.6, we study the relationships between the interior operator and the lower approximation operation and the relationships between the closure operator and the first type of upper approximation operation. In Sections 4 and 5, we investigate similar issues as in Section 3 for the second and third types of covering generalized rough sets. Section 6 presents some results about relationships among these three types of covering generalized rough sets discussed in this paper. We establish the conditions under which two types of upper approximation operations are identical. This paper concludes in Section 7 with remarks for future works.

2 BACKGROUNDS

In this section, we present concepts such as classical rough sets, coverings, and interior and closure operators.

2.1 Fundamentals of Pawlak's Rough Sets

Let U be a finite set and R be an equivalence relation on U. R will generate a partition $U/R = \{Y_1, Y_2, \dots, Y_m\}$ on U, where $Y_1, Y_2, \dots, Y_m$ are the equivalence classes generated by R. $\forall X \subseteq U$, the lower and upper approximations of X, are, respectively, defined as follows:

$$R_*(X) = \cup\{Y_i \in U/R | Y_i \subseteq X\}$$
$$R^*(X) = \cup\{Y_i \in U/R | Y_i \cap X \neq \phi\}.$$

Proposition 1. *Let ϕ be the empty set and $-X$ the complement of X in U. Pawlak's rough sets have the following properties:*

$$\begin{array}{lll}
(1L) & R_*(U) = U & (Conormality) \\
(1H) & R^*(U) = U & (Conormality) \\
(2L) & R_*(\phi) = \phi & (Normality) \\
(2H) & R^*(\phi) = \phi & (Normality) \\
(3L) & R_*(X) \subseteq X & (Contraction) \\
(3H) & X \subseteq R^*(X) & (Extension) \\
(4L) & R_*(X \cap Y) = R_*(X) \cap R_*(Y) & (Multiplication) \\
(4H) & R^*(X \cup Y) = R^*(X) \cup R^*(Y) & (Addition) \\
(5L) & R_*(R_*(X)) = R_*(X) & (Idempotency) \\
(5H) & R^*(R^*(X)) = R^*(X) & (Idempotency) \\
(6) & R_*(-X) = -R^*(X) & (Duality) \\
(7L) & X \subseteq Y \Rightarrow R_*(X) \subseteq R_*(Y) & (Monotone) \\
(7H) & X \subseteq Y \Rightarrow R^*(X) \subseteq R^*(Y) & (Monotone) \\
(8L) & R_*(-R_*(X)) = -R_*(X) & (Complement) \\
(8H) & R^*(-R^*(X)) = -R^*(X) & (Complement) \\
(9L) & \forall K \in U/R, R_*(K) = K & (Granularity) \\
(9H) & \forall K \in U/R, R^*(K) = K & (Granularity).
\end{array}$$

Properties (3L), (4L), and (8L) are characteristic properties for the lower approximation operation [17], [65], [66], [72], that is, all other properties of the lower approximation operation can be deduced from these three properties. Correspondingly, (3H), (4H), and (8H) are characteristic properties for the upper approximation operation.

2.2 Basic Concepts

In this paper, we will investigate three types of covering-based rough set models. They have the same lower approximation but different upper approximations. To begin with, we present the basic concepts we need in this paper.

Definition 1 (Covering). *Let U be a domain of discourse and $\mathbf{C}$ a family of nonempty subsets of U. If $\cup\mathbf{C} = U$, $\mathbf{C}$ is called a covering of U.*

It is clear that a partition of U is certainly a covering of U, so the concept of a covering is an extension of a partition.

In the following discussion, unless stated to the contrary, the coverings are considered to be finite, that is, coverings consist of a finite number of sets in them.

First, we list some definitions about coverings to be used in this paper.

Definition 2 (Covering approximation space). *Let U be a nonempty set and $\mathbf{C}$ a covering of U. We call the ordered pair $< U, \mathbf{C} >$ a covering approximation space.*

Definition 3 (Minimal description). *Let $< U, \mathbf{C} >$ be a covering approximation space, $x \in U$.*

$$Md(x) = \{K \in \mathbf{C} | x \in K \wedge (\forall S \in \mathbf{C} \wedge x \in S \wedge S \subseteq K \Rightarrow K = S)\}$$

is called the minimal description of x.

In order to describe an object, we need only the essential characteristics related to this object, not all the characteristics for this object. That is the purpose of the minimal description concept.

Corresponding to the minimal description, we define the concept of the maximal description.

Definition 4 (Maximal description). *Let $< U, \mathbf{C} >$ be a covering approximation space, $K \in \mathbf{C}$. If no other element of $\mathbf{C}$ contains K, K is called a maximal description in $\mathbf{C}$. All maximal descriptions in $\mathbf{C}$ are denoted as $MaximalD(\mathbf{C})$.*

Definition 5 (Indiscernible neighborhood). *Let $< U, \mathbf{C} >$ be a covering approximation space. $\forall x \in U$, $\cup\{K | x \in K \in \mathbf{C}\}$ is called the indiscernible neighborhood of x and denoted as $Friends(x)$.*

Definition 6 (Close friends). *Let $< U, \mathbf{C} >$ be a covering approximation space, $x \in U$, $\cup\{Md(x)\}$ is called the close friends of x and denoted as $CFriends(x)$.*

Definition 7 (Unary). *Let $\mathbf{C}$ be a covering of a set U. $\mathbf{C}$ is called unary if $\forall x \in U$, $|Md(x)| = 1$.*

Definition 8 (Pointwise covered). *Let $\mathbf{C}$ be a covering of U. If $\forall K \in \mathbf{C}$ and $x \in K$, $K \subseteq CFriends(x)$, we call $\mathbf{C}$ a pointwise-covered covering.*

The following two concepts, interior and closure operators, are from topology. They are essential concepts in topology.

Definition 9 (Interior and closure operators). *For an operation $c : P(U) \to P(U)$, where $P(U)$ is the power set of U, if it satisfies the following axioms, then we call it a closure operator on U. $\forall X, Y \subseteq U$:*

Axiom I. $c(X \cup Y) = c(X) \cup c(Y)$.
Axiom II. $X \subseteq c(X)$.
Axiom III. $c(\phi) = \phi$.
Axiom IV. $c(c(X)) = c(X)$.

For an operation $i : P(U) \to P(U)$, if it satisfies the following axioms, then we call it an interior operator on U. $\forall X, Y \subseteq U$:

Axiom I′. $i(X \cap Y) = i(X) \cap i(Y)$.
Axiom II′. $i(X) \subseteq X$.
Axiom III′. $i(U) = U$.
Axiom IV′. $i(i(X)) = i(X)$.

Proposition 2. *Let U be a finite set, the domain of discourse, and R an equivalence relation on U. The lower and upper approximations generated by R are the interior and closure operators, respectively.*

3 First Type of Covering Generalized Rough Sets

In this section, we study the first type of covering-based rough sets. We will address the following issues. First, we present the similarity and difference between the properties of this type of rough sets and those of classical rough sets. Then, we explore the conditions under which two coverings generate the same lower or upper approximations. We also consider the interdependency of the lower and upper approximations. The axiomization of lower and upper approximations is the next problem we investigate in this paper. Last, we study the relationships between the lower approximation operation and the interior operator and the relationships between the upper approximation operation and the closure operator.

3.1 Concepts and Properties

Definition 10 (CL and FH). *Let $\mathbf{C}$ be a covering of U. The operations CL and $FH : P(U) \to P(U)$ are defined as follows: $\forall X \in P(U)$,*

$$CL(X) = \cup\{K \in \mathbf{C} | K \subseteq X\},$$
$$FH(X) = CL(X) \cup \{Md(x) | x \in X - CL(X)\}.$$

We call CL the covering lower approximation operation and FH the first type of covering upper approximation operation.

Proposition 3 [4]. *$CL(X) = X$ if and only if X is a union of some elements of $\mathbf{C}$.*

Proposition 4 [4]. *$FH(X) = X$ if and only if X is a union of some elements of $\mathbf{C}$.*

Corresponding to the properties of Pawlak's rough sets listed in Section 2.1, we have the following results.

Proposition 5 [67], [69]. *CL has properties (1L), (2L), (3L), (5L), (7L), and (9L) in Proposition 1.*

Proposition 6 [67], [69]. *FH has properties (1H), (2H), (3H), (5H), and (9H) in Proposition 1.*

Remark 1. Properties (4L), (6), and (8L) in Proposition 1 do not hold for CL. A counterexample is in [67] and [69].

Remark 2. Properties (4H), (6), (7H), and (8H) in Proposition 1 do not hold for FH. A counterexample is in [67] and [69].

The following theorems from [75] give the conditions under which CL satisfies (4L), (6), and (8L) in Proposition 1, and FH satisfies (4H), (6), (7H), and (8H) in Proposition 1.

Theorem 1. *CL satisfies (4L) in Proposition 1, that is*

$$CL(X \cap Y) = CL(X) \cap CL(Y)$$

if and only if $\mathbf{C}$ *satisfies the following:* $\forall K_1, K_2 \in \mathbf{C}$, $K_1 \cap K_2$ *is a union of finite elements in* $\mathbf{C}$.

Proof ($\Rightarrow$). Since $K_1 \cap K_2 = CL(K_1) \cap CL(K_2) = CL(K_1 \cap K_2)$ and $CL(K_1 \cap K_2)$ is a union of finite elements in $\mathbf{C}$ by definition, $K_1 \cap K_2$ is a union of finite elements in $\mathbf{C}$.

($\Leftarrow$). By (7L) in Proposition 5, it is easy to see that

$$CL(X \cap Y) \subseteq CL(X) \cap CL(Y).$$

On the other hand, by Proposition 3, let $CL(X) = K_1 \cup \ldots \cup K_m$ and $CL(Y) = K'_1 \cup \ldots \cup K'_n$, where

$$K_i, K'_j \in \mathbf{C}, 1 \le i \le m, \text{ and } 1 \le j \le n.$$

For any $1 \le i \le m$ and $1 \le j \le n$, $K_i \cap K'_j \subseteq X \cap Y$. By the assumption in this theorem, $K_i \cap K'_j$ is a union of finite elements in $\mathbf{C}$. Let us say $K_i \cap K'_j = W_1 \cup \ldots \cup W_l$, where $W_h \in \mathbf{C}$, $1 \le h \le l$. For $1 \le h \le l$, since $W_h \subseteq K_i \cap K'_j \subseteq X \cap Y$, we have $W_h \subseteq CL(X \cap Y)$. Thus, $K_i \cap K'_j \subseteq CL(X \cap Y)$ for $1 \le i \le m$ and $1 \le j \le n$. From

$$CL(X) \cap CL(Y) = \cup_{i=1}^{m}{}_{j=1}^{n}(K_i \cap K'_j),$$

we prove that $CL(X) \cap CL(Y) \subseteq CL(X \cap Y)$. Therefore, $CL(X \cap Y) = CL(X) \cap CL(Y)$. □

Theorem 2. *FH satisfies (7H) in Proposition 1, that is,*

$$X \subseteq Y \Rightarrow FH(X) \subseteq FH(Y)$$

if and only if $\mathbf{C}$ *satisfies the following:* $\forall K_1$, $K_2 \in \mathbf{C}$, *and* $K_1 \cap K_2$ *is a union of finite elements in* $\mathbf{C}$.

Proof ($\Rightarrow$).

$$FH(K_1 \cap K_2) \subseteq FH(K_1) = K_1 \text{ and}$$
$$FH(K_1 \cap K_2) \subseteq FH(K_2) = K_2,$$

so $FH(K_1 \cap K_2) \subseteq K_1 \cap K_2$. By property (3H) in Proposition 6,

$$K_1 \cap K_2 \subseteq FH(K_1 \cap K_2),$$

so $K_1 \cap K_2 = FH(K_1 \cap K_2)$. By Proposition 4, $K_1 \cap K_2$ is the union of finite elements in $\mathbf{C}$.

($\Leftarrow$). By the definition of FH, there exist $y_1, y_2, \ldots y_m \in X - CL(X)$ and $K_1, K_2, \ldots K_m \in \mathbf{C}$ such that

$$K_i \in Md(y_i), 1 \le i \le m,$$

and $FH(X)$ is expressed as $CL(X) \cup K_1 \ldots \cup K_m$. It is obvious that $y_i \in Y$. For $1 \le i \le m$, if $y_i \in Y - CL(Y)$, it is easy to see that $K_i \subseteq FH(Y)$. If $y_i \notin Y - CL(Y)$, then $y_i \in CL(Y)$. Thus, there exists a $K'_i \in \mathbf{C}$ such that $y_i \in K'_i \subseteq CL(Y)$. By the assumption of this theorem, $K_i \cap K'_i$ is a union of finite elements in $\mathbf{C}$. Let us say $K_i \cap K'_i = W_1 \cup \ldots \cup W_l$, where $W_h \in \mathbf{C}$, $1 \le h \le l$. Since $y_i \in K_i \cap K'_i$, there exists $1 \le j \le l$ such that $y_i \in W_j$. By $K_i \in Md(y_i)$ and $W_j \subseteq K_i$, we have $K_i = W_j$. Thus, $K_i \subseteq K'_i$. Therefore,

$$K_i \subseteq CL(Y) \subseteq FH(Y), 1 \le i \le m.$$

From property (7L) in Proposition 5,

$$CL(X) \subseteq CL(Y) \subseteq FH(Y),$$

so $FH(X) \subseteq FH(Y)$. □

Before we investigate the issue of property (4H) in Proposition 1, we prove a lemma to show that property (4H) is equivalent to property (7H) in Proposition 1 for the first type of upper approximation operations.

Lemma 1. *FH satisfies*

$$\text{(7H)} \;\; X \subseteq Y \Rightarrow FH(X) \subseteq FH(Y)$$

if and only if FH satisfies

$$\text{(4H)} \;\; FH(X \cup Y) = FH(X) \cup FH(Y).$$

Proof ($\Rightarrow$). By (7H),

$$FH(X) \subseteq FH(X \cup Y) \text{ and}$$
$$FH(Y) \subseteq FH(X \cup Y)$$

so $FH(X) \cup FH(Y) \subseteq FH(X \cup Y)$. On the other hand, by property (3H) in Proposition 6,

$$X \cup Y \subseteq FH(X) \cup FH(Y).$$

By (7H), $FH(X \cup Y) \subseteq FH(FH(X) \cup FH(Y))$. By Proposition 4, $FH(FH(X) \cup FH(Y)) = FH(X) \cup FH(Y)$, so $FH(X \cup Y) \subseteq FH(X) \cup FH(Y)$. Therefore,

$$FH(X \cup Y) = FH(X) \cup FH(Y).$$

($\Leftarrow$). If $X \subseteq Y$,

$$FH(Y) = FH(X \cup Y) = FH(X) \cup FH(Y),$$

so $FH(X) \subseteq FH(Y)$. □

Theorem 3. *FH satisfies (4H) in Proposition 1, that is,*

$$FH(X \cup Y) = FH(X) \cup FH(Y)$$

if and only if $\mathbf{C}$ *satisfies the following:* $\forall K_1$, $K_2 \in \mathbf{C}$, *and* $K_1 \cap K_2$ *is a union of finite elements in* $\mathbf{C}$.

Proof. It comes from Theorem 2 and Lemma 1. □

Combining Theorem 1, Theorem 2, and Theorem 3, we have the following result.

Corollary 1. *FH satisfies*

$$\text{(4H)} \;\; FH(X \cup Y) = FH(X) \cup FH(Y)$$

if and only if it satisfies

$$\text{(7H)} \;\; X \subseteq Y \Rightarrow FH(X) \subseteq FH(Y)$$

and if and only if CL satisfies

$$\text{(4L)} \;\; CL(X \cap Y) = CL(X) \cap CL(Y).$$

The following two theorems are about properties (8L) and (8H) in Proposition 1 for the first type of upper approximation operations.

Theorem 4. *CL satisfies (8L) in Proposition 1, that is,*

$$CL(-CL(X)) = -CL(X)$$

if and only if $\forall K_1, \ldots, K_m \in \mathbf{C}$, $-(K_1 \cup \ldots \cup K_m)$ is a union of finite elements in $\mathbf{C}$.

Proof ($\Rightarrow$). $\forall K_1, \ldots, K_m \in \mathbf{C}$,

$$CL(-(K_1 \cup \ldots \cup K_m)) = CL(-CL(K_1 \cup \ldots \cup K_m)) = - CL(K_1 \cup \ldots \cup K_m) = -(K_1 \cup \ldots \cup K_m),$$

so $-(K_1 \cup \ldots \cup K_m)$ is a union of finite elements in $\mathbf{C}$.

($\Leftarrow$). By Proposition 3, $CL(X)$ is a union of finite elements in $\mathbf{C}$, say, $CL(X) = K_1 \cup \ldots \cup K_m$, where $K_1, \ldots, K_m \in \mathbf{C}$. $CL(-CL(X)) = CL(-(K_1 \cup \ldots \cup K_m))$. By Proposition 3 and the assumption of this theorem,

$$CL(-(K_1 \cup \ldots \cup K_m)) = -(K_1 \cup \ldots \cup K_m),$$

so $CL(-CL(X)) = -CL(X)$. □

Theorem 5. *FH satisfies (8H) in Proposition 1, that is,*

$$FH(-FH(X)) = -FH(X)$$

if and only if $\forall K_1, \ldots, K_m \in \mathbf{C}$, $-(K_1 \cup \ldots \cup K_m)$ is a union of finite elements in $\mathbf{C}$.

Proof. It is similar to that of the above theorem. □

Corollary 2. *CL satisfies*

$$\text{(8L)} \;\; CL(-CL(X)) = -CL(X)$$

if and only if FH satisfies

$$\text{(8H)} \;\; FH(-FH(X)) = -FH(X).$$

Proof. It comes directly from Theorem 4 and Theorem 5. □

As for property (6) in Proposition 1, we get only a partial solution: a necessary condition so that property (6) in Proposition 1 holds.

Theorem 6. *If CL and FH satisfy (6) in Proposition 1, that is,*

$$FH(-X) = -CL(X),$$

then $\forall K_1, \ldots, K_m \in \mathbf{C}$, $-(K_1 \cup \ldots \cup K_m)$ is a union of finite elements in $\mathbf{C}$.

Proof. $\forall K_1, \ldots, K_m \in \mathbf{C}$,

$$\begin{aligned} FH(-(K_1 \cup \ldots \cup K_m)) &= -CL(K_1 \cup \ldots \cup K_m) \\ &= -(K_1 \cup \ldots \cup K_m), \end{aligned}$$

so by Proposition 3, $-(K_1 \cup \ldots \cup K_m)$ is a union of finite elements in $\mathbf{C}$. □

3.2 Interdependency of the Lower and Upper Approximation Operations

In the classical rough set theory, two different equivalence relations will certainly generate two different lower and upper approximation operations. However, in the covering-based rough set model, two different coverings might generate the same lower and upper covering-based approximation operations as shown in the following example:

Example 1 (Two different coverings generate the same CL and FH). Let

$$U = \{a, b, c, d\}, K_1 = \{a\}, K_2 = \{b, c\}, K_4 = \{d\},$$
$$K_4 = \{a, d\}, \mathbf{C} = \{K_1, K_2, K_3, K_4\}, \text{ and } \mathbf{C}' = \{K_1, K_2 K_3\}.$$

Then, $\mathbf{C}$ and $\mathbf{C}'$ generate the same lower covering-based approximation operation and the same first type of upper covering-based approximation operation.

Thus, we want to investigate the conditions under which two coverings generate the same covering lower and upper approximation operations. Furthermore, in Pawlak's rough set theory, lower and upper approximation operations are dual, so they are dependent on each other. As we can see from Remarks 1 and 2, the covering-based operations CL and FH on U are not dual. Now, we want to ask whether CL and FH are dependent on each other. In this section, we address these questions mainly based on the results obtained in [67], [69].

We start to find the conditions under which two coverings generate the same lower approximation operation or the same first type of upper approximation operation.

Definition 11. *Let $\mathbf{C}$ be a covering of a domain U and $K \in \mathbf{C}$. If K is a union of some sets in $\mathbf{C} - \{K\}$, we say K is reducible in $\mathbf{C}$; otherwise, K is irreducible.*

Definition 12. *Let $\mathbf{C}$ be a covering of U. If every element in $\mathbf{C}$ is irreducible, we say $\mathbf{C}$ is irreducible; otherwise, $\mathbf{C}$ is reducible.*

Proposition 7. *Let $\mathbf{C}$ be a covering of a domain U. If K is reducible in $\mathbf{C}$, $\mathbf{C} - \{K\}$ is still a covering of U, and $\forall x \in U$, $Md(x)$ in $\mathbf{C} - \{K\}$ is the same as it is in $\mathbf{C}$.*

Proposition 8. *Let $\mathbf{C}$ be a covering of U, $K \in \mathbf{C}$, K be reducible in $\mathbf{C}$, and $K_1 \in \mathbf{C} - \{K\}$. K_1 is reducible in $\mathbf{C}$ if and only if it is reducible in $\mathbf{C} - \{K\}$.*

Proposition 7 guarantees that after deleting a reducible element from a covering, it is still a covering. Proposition 8 shows that deleting a reducible element in a covering will not generate any new reducible elements or make other previous reducible elements irreducible. Consequently, we can compute the reduct of a covering of a domain by deleting all reducible elements or by deleting one reducible element in a step. The remainder still consists of a covering of the domain, and it is irreducible.

Definition 13. *For a covering $\mathbf{C}$ of a domain U, the new irreducible covering through the above reduction is called the reduct of $\mathbf{C}$ and denoted by $reduct(\mathbf{C})$.*

Proposition 8 guarantees that a covering has only one reduct.

Theorem 7. *Let $\mathbf{C_1}$ and $\mathbf{C_2}$ be two coverings of U. $\mathbf{C_1}$ and $\mathbf{C_2}$ generate the same covering lower approximation operation if and only if $reduct(\mathbf{C_1}) = reduct(\mathbf{C_2})$.*

Theorem 8. *Let $\mathbf{C_1}$ and $\mathbf{C_2}$ be two coverings of U. $\mathbf{C_1}$ and $\mathbf{C_2}$ generate the same first type of covering upper approximation operation if and only if $reduct(\mathbf{C_1}) = reduct(\mathbf{C_2})$.*

```
set reductOfCovering(set C){
   set reduct(C) = C;
   for(i = 1; i ≤ n; i++) {
      set temp = φ;
      for(j = 1; j ≤ n; j++){
         if(K_j ⊂ K_i){
            temp = temp ∪ K_j;
         }
      }
      if( temp == K_i){
         reduct(C) = reduct(C) − {K_i};
      }
   }
   return reduct(C);
}
```

Fig. 1. Algorithm to compute the reduct of a covering.

Theorem 9. *Let $\mathbf{C_1}$ and $\mathbf{C_2}$ be two coverings of U. $\mathbf{C_1}$ and $\mathbf{C_2}$ generate the same covering lower approximation if and only if they generate the same first type of covering upper approximation.*

Theorem 9 shows that the upper approximation operation and lower approximation operation uniquely determine each other.

3.3 An Algorithm for Computing the Reduct

As you can see from the last section, reduct is one of the core concepts in the first type of covering-based rough set theory. We present an algorithm for computing the reduct $reduct(\mathbf{C})$ of a covering $\mathbf{C}$ in Fig. 1. The basic idea for this algorithm is based on the following observation. For any member K_i in a covering $\mathbf{C} = \{K_1, K_2, \ldots, K_n\}$ of U, we select all other members in $\mathbf{C}$ that are proper subsets of K_i. K_i is reducible if and only if the union of these members is equal to K_i.

Example 2 (Reduct of a covering). Let $U = \{a, b, c, d\}$, $K_1 = \{a\}$, $K_2 = \{b, c\}$, $K_4 = \{d\}$, $K_4 = \{a, d\}$, and $\mathbf{C} = \{K_1, K_2, K_3, K_4\}$. Since $K_4 = K_1 \cup K_3$, K_4 is a reducible element of covering $\mathbf{C}$. K_4 is the only reducible element in $\mathbf{C}$; thus, $reduct(\mathbf{C}) = \{K_1, K_2, K_3\}$.

3.4 Axiomization of Lower Approximation Operations

As shown in Section 2.1, Pawlak's lower and upper approximation operations have been axiomatized. Now, we need to know which are the characteristic properties for the lower approximation operation and the first type of covering upper approximation operation. We present an axiomatic system of covering lower approximation operations as follows.

Theorem 10 [67], [69]. *Let U be a nonempty set. If an operation $L : P(U) \to P(U)$ satisfies the following properties: For any $X, Y \subseteq U$,*

$$\begin{array}{lll} (1L) & L(U) = U & (Conormality) \\ (3L) & L(X) \subseteq X & (Contraction) \\ (5L) & L(L(X)) = L(X) & (Idempotency) \\ (7L) & X \subseteq Y \Rightarrow L(X) \subseteq L(Y) & (Monotone), \end{array}$$

then there exists a covering $\mathbf{C}$ of U such that the covering lower approximation operation CL generated by $\mathbf{C}$ is equal to L. Furthermore, the above four properties for the covering lower approximation operation are independent.

However, the axiomization of the first type of covering upper approximation operations is still an open problem, and, as we can see in the next section, this might not be an easy task.

3.5 Axiomization of the First Type of Upper Approximation Operations

The properties from (1L) to (9H) listed in Proposition 1 are essential or sufficient for the lower and upper approximation operations of Pawlak's rough sets. Theorem 10 in Section 3.4 shows that properties (1L), (2L), (3L), (5L), and (7L) listed in Proposition 5 are enough for covering lower approximation operations. However, as shown in the following example, properties (1H), (2H), (3H), and (5H) listed in Proposition 6 are not sufficient to characterize the first type of upper approximation operations.

Example 3. An operation $H : P(U) \to P(U)$ satisfies properties (1H), (2H), (3H), and (5H) in Proposition 5, but there is no covering $\mathbf{C}$ on U such that H is the first type of upper approximation operation generated by $\mathbf{C}$.

Specifically, let $U = \{a, b, c\}$. Define $H : P(U) \to P(U)$ as follows:

$H(\phi) = \phi$, $H(\{a\}) = \{a\}$, $H(\{b\}) = \{b\}$, for all other $X \subseteq U$, $H(X) = U$.

It is obvious that H satisfies (1H), (2H), (3H), and (5H), but there is no covering $\mathbf{C}$ on U such that H is its first type of upper approximation operation FH since such a covering $\mathbf{C}$ should have elements $\{a\}$ and $\{b\}$. Clearly, in this case,

$$FH(\{a, b\}) = \{a, b\} \neq \{a, b, c\} = H(\{a, b\});$$

thus, H is not the upper approximation operation generated by $\mathbf{C}$.

It is still an open problem to find an axiomatic system for the first type of covering upper approximation operations.

3.6 Relationships with Closure and Interior Operators

Yao studied Pawlak's rough set through the topological properties of lower and upper approximation operations [44]. Pomykala studied the topological properties for a class of generalized rough sets [32]. Lin and Liu investigated axioms for approximation operations by the topological method [17]. Now, we examine the topological properties of the lower approximation operation and the first type of upper approximation operation for covering generalized rough sets.

Interior and closure operators are two core concepts in topology, and for Pawlak's rough sets, the lower and upper approximation operations on a set are also the interior and closure operators on this set, respectively. In this section, we investigate the conditions under which the covering lower and upper approximation operations are also the interior and closure operators, respectively.

From Remarks 1 and 2, we know that, generally, FH is not a closure operator, for Axiom I is not generally valid for

FH, and, generally, CL is not an interior operator, for Axiom I$'$ is not generally valid for CL.

Now, we present the condition under which CL is an interior operator and the condition under which FH is a closure operator. We have the following results.

Theorem 11 (Condition under which CL is an interior operator). *Operation CL is an interior operator if and only if $\mathbf{C}$ satisfies the following: $\forall K_1, K_2 \in \mathbf{C}$, $K_1 \cap K_2$ are unions of finite elements in $\mathbf{C}$.*

Proof. It comes from Proposition 5 and Theorem 1. □

Theorem 12 (Condition under which FH is a closure operator). *Operation FH is a closure operator if and only if $\mathbf{C}$ satisfies the following: $\forall K_1, K_2 \in \mathbf{C}$, $K_1 \cap K_2$ are unions of finite elements in $\mathbf{C}$.*

Proof. It comes from Proposition 6 and Theorem 3. □

Combining the above two theorems, we have the following result.

Corollary 3. *CL is an interior operator if and only if FH is a closure operator.*

4 Second Type of Covering Generalized Rough Sets

The second type of covering-based rough sets were already studied in [18], [19], [30], [67]. In this section, we start to address the similar issues for the second type of covering-based rough sets as in the first type one in Section 3.

4.1 Concepts and Properties

For the second type of covering-based rough set model, the lower approximation operation is the same as that in the first type of covering-based rough set model; thus, we define only the second type of covering upper approximation operation as follows:

Definition 14 (SH)**.** *Let $\mathbf{C}$ be a covering of U. The second type of covering upper approximation operation SH is defined as follows: $\forall X \subseteq U$, $SH(X) = \cup\{K | K \in \mathbf{C}, K \cap X \neq \phi\}$.*

Proposition 9. *If $\mathbf{C}$ is a partition of U, $SH(X)$ is the upper approximation operation as specified by Pawlak's original definitions.*

Proposition 10. *The second type of covering upper approximation has properties (1H), (2H), (3H), (4H), and (7H) in Proposition 1.*

Proof. Properties (1H), (2H), (3H), and (7H) are evident from the definition of $SH(X)$.

For property (4H), by (7H), we have $SH(X) \subseteq SH(X \cup Y)$ and $SH(Y) \subseteq SH(X \cup Y)$, so

$$SH(X) \cup SH(Y) \subseteq SH(X \cup Y).$$

As for

$$SH(X \cup Y) \subseteq SH(X) \cup SH(Y),$$

or any $x \in SH(X \cup Y)$, by the definition, there exists a $K \in \mathbf{C}$ such that $x \in K$ and $K \cap (X \cup Y) \neq \phi$. From $K \cap (X \cup Y) = (K \cap X) \cup (K \cap Y)$, $K \cap X$ and $K \cap Y$ cannot be both empty. If $K \cap X \neq \phi$, we have $x \in SH(X)$. Otherwise, we have $x \in SH(Y)$. Therefore, $x \in SH(X) \cup SH(Y)$. Now, we have proved that $SH(X \cup Y) \subseteq SH(X) \cup SH(Y)$. □

Remark 3. Generally, properties (5H), (6), (8H), and (9H) in Proposition 1 do not hold for the second type of covering upper approximations. A counterexample is given as follows:

Example 4. Let $U = \{a, b, c, d\}$, $K_1 = \{a, b\}$, $K_2 = \{a, b, c\}$, $K_3 = \{c, d\}$, and $\mathbf{C} = \{K_1, K_2, K_3\}$. $\mathbf{C}$ is a covering of U.

(5H) If $X = \{a, b\}$, we have $SH(X) = \{a, b, c\}$. However, $SH(\{a, b, c\}) = \{a, b, c, d\}$, so

$$SH(SH(X)) \neq SH(X).$$

(6) If $X = \{a, b\}$, we have $CL(X) = \{a, b\}$. However, $SH((-X)) = SH(\{c, d\}) = \{a, b, c, d\}$, so

$$CL(X) \neq -SH((-X)).$$

(8H) For $X = \{a\}$,

$$SH(X) = \{a, b\} \text{ and}$$
$$SH((-SH(X))) = SH(\{c, d\}) = \{a, b, c, d\},$$

so, $SH((-SH(X))) \neq -SH(X)$.

(9H) Let $K = K_2 = \{a, b, c\}$, we have

$$SH(K) = U \neq K.$$

Now that properties (5H), (6), (8H), and (9H) in Proposition 1 are not generally valid for SH, what are the conditions that guarantee that any of them holds for SH? We have the following conclusions [76].

Theorem 13. *SH satisfies*

$$\text{(5H)} \quad SH(SH(X)) = SH(X) \qquad \text{(Idempotency)}$$

if and only if $\mathbf{C}$ satisfies the following:

$$\forall K, K_1, K_2, \ldots, K_m \in \mathbf{C},$$

if $K_1 \cap K_2 \cap \ldots, K_m \neq \phi$ and

$$K \cap (K_1 \cup K_2 \cup \ldots \cup K_m) \neq \phi,$$

then $K \subseteq (K_1 \cup K_2 \cup \ldots \cup K_m)$.

Proof ($\Rightarrow$). $\forall K, K_1, K_2, \ldots, K_m \in \mathbf{C}$, if $K_1 \cap K_2 \cap \ldots, K_m \neq \phi$ and $K \cap (K_1 \cup K_2 \cup \ldots \cup K_m) \neq \phi$, then there exists an x such that $x \in K_1 \cap K_2 \cap \ldots, K_m$. By the definition of SH,

$$K_1 \cup K_2 \cup \ldots \cup K_m \subseteq SH(\{x\}) \text{ and}$$
$$K \subseteq SH(SH(\{x\})).$$

Since $\forall X \subseteq U$, $SH(SH(X)) = SH(X)$, $K \subseteq SH(\{x\})$; thus, $K \subseteq (K_1 \cup K_2 \cup \ldots \cup K_m)$.

($\Leftarrow$). It is obvious that $SH(X) \subseteq SH(SH(X))$. Now, we prove that $SH(SH(X)) \subseteq SH(X)$. First, we prove that $SH(SH(\{x\})) \subseteq SH(\{x\})$.

According to the definition of SH, there exists $K_1, K_2, \ldots, K_m \in \mathbf{C}$ such that

$$SH(\{x\}) = K_1 \cup K_2 \cup \ldots \cup K_m.$$

$\forall y \in SH(SH(\{x\}))$, there exists a $K \in \mathbf{C}$ such that $y \in K$, and $K \cap SH(\{x\}) \neq \phi$, so $K \cap (K_1 \cup K_2 \cup \ldots \cup K_m) \neq \phi$. By the property, $K \subseteq (K_1 \cup K_2 \cup \ldots \cup K_m) = SH(\{x\})$, so $y \in SH(\{x\})$. Now, we have proved that

$$SH(SH(\{x\})) \subseteq SH(\{x\}).$$

By property (4H) listed in Proposition 10, $\forall X \subseteq U$, $SH(SH(X)) = SH(X)$. □

Theorem 14. *SH satisfies*

$$\text{(8H)} \quad SH(-SH(X)) = -SH(X)$$

if and only if $\{SH(\{x\}) | x \in U\}$ is a partition.

Proof ($\Rightarrow$). Since SH also satisfies (3H) and (4H), by [17], [65], and [66], SH is a classical upper approximation operation, so $\{SH(\{x\}) | x \in U\}$ is a partition.

($\Leftarrow$). If $\{SH(\{x\}) | x \in U\}$ is a partition, by property (4H) listed in Proposition 10, SH is the classical upper approximation operation generated by the partition $\mathbf{P} = \{SH(\{x\}) | x \in U\}$, so $SH(-SH(X)) = -SH(X)$. □

Corollary 4. *If SH satisfies*

$$\text{(8H)} \quad SH(-SH(X)) = -SH(X),$$

then it also satisfies

$$\text{(5H)} \quad SH(SH(X)) = SH(X).$$

Proof. It comes directly from Theorem 13 and Theorem 14.□

As for property (6), we get the following result.

Theorem 15. *CL and SH satisfy*

$$\text{(6)} \quad SH(-X) = -CL(X)$$

if and only if $\mathbf{C}$ is a partition.

Proof ($\Rightarrow$). $\forall K_1, K_2 \in \mathbf{C}$, if $K_1 \cap K_2 \neq \phi$, let us assume that $x \in K_1 \cap K_2$. By $CL(K_1) = K_1$ and property (6), we have $SH(-K_1) = -K_1$, so $K_2 \subseteq K_1$. Otherwise, there exists $y \in K_2$ such that $y \in -K_1$. By the definition of SH, $x \in SH(-K_1)$. This contradicts to $SH(-K_1) = -K_1$. In the same way, we can prove that $K_1 \subseteq K_2$; thus, $K_1 = K_2$. Therefore, $\mathbf{C}$ is a partition.

($\Leftarrow$). It is evident. □

Theorem 16. *SH satisfies*

$$\text{(9H)} \quad \forall K \in \mathbf{C}, SH(K) = K$$

if and only if $\mathbf{C}$ is a partition.

Proof ($\Rightarrow$). $\forall K$, $K' \in \mathbf{C}$, and $K' \neq K$, since $SH(K) = K$, by the definition of SH, we have $K' \cap K = \phi$ or $K' \subset K$. If $K' \subset K$, then, again by the definition of SH, $K \subseteq SH(K') = K'$. This is a contradiction. Thus, $K' \cap K = \phi$. Therefore, $\mathbf{C}$ is a partition.

($\Leftarrow$). It is evident. □

Corollary 5. *SH satisfies property (9H) if and only if it satisfies property (8H).*

Corollary 6. *If SH satisfies property (9H), then it also satisfies property (5H).*

4.2 Independence of the Covering Lower and Upper Approximation Operations

In this section, we investigate whether the lower approximation operation uniquely determines the second type of covering upper approximation operation and whether the second type of covering upper approximation operation uniquely determines the lower approximation operation.

The concept of reduct is a powerful tool for dealing with lower and upper approximation operations for the first type of covering generalized rough sets, but it is not so useful for the second type of upper approximation operations. In the next two examples, we prove the independence of the second type of lower and upper approximation operations.

Example 5. $reduct(\mathbf{C})$ and $\mathbf{C}$ do not generate the same second type of covering upper approximation operations.

Let $U = \{a, b\}$, $K_1 = \{a\}$, $K_2 = \{b\}$, and $K_3 = \{a, b\}$. For covering $\mathbf{C} = \{K_1, K_2, K_3\}$, the corresponding second type of covering upper approximation operation SH_1 is

$$SH_1(\phi) = \phi \text{ and } SH_1(X) = U, \forall \phi \subset X \subseteq U.$$

For covering $reduct(\mathbf{C}) = \{K_1, K_2\}$, the corresponding second type of covering upper approximation operation is

$$SH_2(X) = X, \forall X \subseteq U.$$

These two second type of covering upper approximation operations are not the same. On the other hand, by Theorem 1 in [69], $\mathbf{C}$ and $reduct(\mathbf{C})$ generate the same covering lower approximation operation. As a consequence, we reach at an important conclusion that for two coverings, the same covering lower approximation operations do not imply the same second type of covering upper approximation operation.

Example 6. Two different coverings can generate the same second type of covering upper approximation operation but different covering lower approximation operations.

Let $U = \{a, b\}$, $K_1 = \{a\}$, and $K_2 = \{a, b\}$. For coverings $\mathbf{C_1} = \{K_1, K_2\}$ and $\mathbf{C_2} = \{K_2\}$, the corresponding second type of covering upper approximation operations are the same:

$$SH(\phi) = \phi \text{ and } SH(X) = U \text{ for others.}$$

On the other hand, the corresponding second type of covering lower approximation operations are different. For covering $\mathbf{C_1}$, $CL_1(\phi) = CL_1(\{b\}) = \phi$, $CL_1(\{a\}) = \{a\}$, and $CL_1(U) = U$.

For covering $\mathbf{C_2}$, $CL_2(\phi) = CL_2(\{a\}) = CL_2(\{b\}) = \phi$, and $CL_2(U) = U$.

Now, we draw a conclusion about interdependency of the lower and the second type of upper approximation operations.

Theorem 17. *The covering lower approximation operation and the second type of upper approximation operation cannot uniquely determine each other.*

Proof. From Example 5 above, the covering lower approximation operation does not uniquely determine the second type of upper approximation operations. From

Example 6 above, the second type of upper approximation operation does not uniquely determine the covering lower approximation operations. □

4.3 Exclusions of Coverings

For the upper approximation operation in the second type of covering generalized rough sets, the concept of reduct is no longer a very useful tool, so we propose an alternative concept, *exclusion*, as a partial solution to the problem under which conditions two coverings generate an identical second type of upper approximation operations.

Definition 15. *Let $\mathbf{C}$ be a covering of U and K an element of $\mathbf{C}$. If there exists another element K' of $\mathbf{C}$ such that $K \subset K'$, we say that K is an immured element of covering $\mathbf{C}$.*

Definition 16 (semireduced [18]). *Let $\mathbf{C}$ be a covering of U. $\mathbf{C}$ is called semireduced or semi-irredundant if it satisfies the following condition:*

$$\forall K_1, K_2 \in \mathbf{C}, \text{ and } K_1 \subseteq K_2 \Rightarrow K_1 = K_2.$$

In other words, there is no $K_1, K_2 \in \mathbf{C}$ such that $K_1 \subset K_2$.

Proposition 11. *A covering $\mathbf{C}$ of a set U is semireduced if and only if it has no immured elements.*

Proposition 12. *A covering $\mathbf{C}$ of a set U is semireduced if and only if $MaximalD(\mathbf{C}) = \mathbf{C}$.*

Proposition 13. *A covering $\mathbf{C}$ of a set U is semireduced if and only if it satisfies the following condition: $\forall x \in U$ and $\forall K \in \mathbf{C}$:*

$$x \in K \Leftrightarrow K \in Md(x).$$

Proposition 14. *Let $\mathbf{C}$ be a covering of U and K an immured element of $\mathbf{C}$, then $\mathbf{C} - \{K\}$ is also a covering of U.*

Proposition 15. *Let $\mathbf{C}$ be a covering of U and K an immured element of $\mathbf{C}$, then $\mathbf{C}$ and $\mathbf{C} - \{K\}$ generate identical second type of upper covering approximation operations.*

Proposition 16. *Let $\mathbf{C}$ be a covering of U, K and K' two elements of $\mathbf{C}$, and K an immured element of $\mathbf{C}$. K' is an immured element of $\mathbf{C}$ if and only if K' is an immured element of the covering $\mathbf{C} - \{K\}$.*

Definition 17 (Exclusion). *Let $\mathbf{C}$ be a covering of U. When we remove all immured elements from $\mathbf{C}$, the set of all remaining elements is still a covering of U, and this new covering has no immured element. We call this new covering an exclusion of $\mathbf{C}$, and it is denoted by $exclusion(\mathbf{C})$.*

Proposition 17. *Let $\mathbf{C}$ be a covering of U,*

$$MaximalD(\mathbf{C}) = exclusion(\mathbf{C}).$$

Proposition 18. *Let $\mathbf{C}$ be a covering of U. $\mathbf{C}$ and $exclusion(\mathbf{C})$ generate an identical second type of upper covering approximation operations.*

Proof. It comes from Proposition 16. □

Proposition 19. *Let $\mathbf{C}$ and $\mathbf{C}'$ be two coverings of U. If $exclusion(\mathbf{C}) = exclusion(\mathbf{C}')$, $\mathbf{C}$ and $\mathbf{C}'$ generate an identical second type of upper covering approximation operation.*

Example 7 (Two different coverings generate identical second type of lower and upper approximation operations). Let

$$U = \{a, b, c, d\}, K_1 = \{a\}, K_2 = \{b\},$$
$$K_3 = \{a, b\}, \text{ and } K_4 = \{a, b, c, d\}.$$

$\mathbf{C} = \{K_1, K_2, K_4\}$ and $\mathbf{C}' = \{K_1, K_2, K_3, K_4\}$ are two coverings of U. It is easy to see that $reduct(\mathbf{C}) = reduct(\mathbf{C}')$ and

$$exclusion(\mathbf{C}) = exclusion(\mathbf{C}');$$

thus, they generate the same lower and upper approximation operations.

4.4 Axiomization of the Second Type of Upper Approximation Operations

As in Section 3.5, the properties in Proposition 10 are not enough for the upper approximation operations in the second type of covering generalized rough sets.

Example 8. An operation $H : P(U) \to P(U)$ satisfies properties (1H), (2H), (3H), (4H), and (7H) in Proposition 5; however, there is no covering $\mathbf{C}$ on U such that H is the second type of upper approximation operation generated by $\mathbf{C}$.

Let $U = \{a, b\}$. Define $H : P(U) \to P(U)$ as follows: $H(\phi) = \phi$, $H(\{a\}) = \{a, b\}$, $H(\{a, b\}) = \{a, b\}$, and $H(\{b\}) = \{b\}$. U has four possible coverings, given as follows:

1. $\{a, b\}$,
2. $\{a\}, \{b\}$,
3. $\{a\}, \{a, b\}$, and
4. $\{b\}, \{a, b\}$,

but none of them can generate H as their second type of covering upper approximation operations.

It is still an open problem to find an axiomatic system for the second type of covering upper approximation operations.

4.5 Relationships between the Upper Approximation Operation and the Closure Operator

From Remark 3 in Section 4.1, we know that, generally, SH is not a closure operator, since Axiom IV is not generally valid for SH. In this section, we investigate the conditions under which SH is a closure operator.

Theorem 18 (Condition under which SH is a closure operator). *Operation SH is a closure operator if and only if $\forall x \in U$ and $K \in \mathbf{C}$, either $K \subseteq Friends(x)$ or*

$$K \cap Friends(x) = \phi.$$

Proof. If $\forall x \in U$ and $K \in \mathbf{C}$, either $K \subseteq Friends(x)$ or $K \cap Friends(x) = \phi$, then $\forall x \in U$, $SH(SH(x)) = SH(x)$. Thus, by property (4H) in Proposition 10, $\forall X \subseteq U$, $SH(SH(X)) = SH(X)$.

On the other hand, if there exists an $x \in U$ and $K \in \mathbf{C}$ such that $K \not\subseteq Friends(x)$ and $K \cap Friends(x) \neq \phi$, then there is a $y \in K$ such that $y \notin Friends(x)$. As a result,

$SH(x) = Friends(x)$, and $y \in K \subseteq SH(SH(x))$. Thus, $SH(SH(x)) \neq SH(x)$. □

5 Third Type of Covering Generalized Rough Sets

A third type of covering generalized rough sets has been introduced in [37]. The proposed third definition of the upper approximation operation is considered to be more reasonable than those of the first and second types, but no properties of this new class of covering generalized rough sets have been discussed. In this section, we investigate the corresponding issues for this type of upper approximation operations.

5.1 Concepts and Properties

Definition 18 (TH)**.** *Let $\mathbf{C}$ be a covering of U. The third type of covering upper approximation operation TH is defined as follows: $\forall X \subseteq U$,*

$$TH(X) = \cup\{Md(x)|x \in X\}.$$

Proposition 20. *If $\mathbf{C}$ is a partition of U, $TH(X)$ is the upper approximation operation as specified by Pawlak's original definitions.*

Proposition 21. *The third type of covering upper approximations have properties (1H), (2H), (3H), (4H), and (7H) in Proposition 1.*

Remark 4. Properties (5H), (6), (8H), and (9H) in Proposition 1 do not hold for the third type of covering upper approximations. The following is a counterexample.

Example 9. $U = \{a, b, c\}$, $\mathbf{C} = \{\{a, b\}, \{b, c\}\}$,

(5H) $X = \{a\}, TH(X) = \{a, b\}$,
$TH(TH(X)) = \{a, b, c\}$.

(6) $X = \{a\}, CL(-X) = \{b, c\}, TH(X) = \{a, b\}$,
$-TH(X) = \{c\}$.

(8H) $X = \{a\}, -TH(X) = \{a, b\}$,
$TH(-TH(X)) = U$.

(9H) $K = \{a, b\} \in \mathbf{C}, TH(K) = U$.

5.2 Interdependency of Lower and Upper Approximation Operations

As in Section 3.2, reduct is also a useful concept for upper approximation operations in the third type of covering generalized rough sets. We use this concept to achieve results partially similar to those in Section 3.2.

First, we prove that the lower approximation uniquely determines the upper approximation.

Proposition 22. *$K \in \mathbf{C}$ is reducible if and only if $\forall x \in U, K \notin Md(x)$.*

Proof. If $K \in \mathbf{C}$ is reducible, then there exist $K_1, \ldots, K_m \in \mathbf{C}$ such that $K = K_1 \cup \ldots \cup K_m$ and $K_i \subset K$ for $i = 1, \ldots, m$. $\forall x \in U$, if $x \notin K$, it is obvious that $K \notin Md(x)$. If $x \in K$, there must be an i such that $1 \leq i \leq m$ and $x \in K_i$; thus, $K \notin Md(x)$.

On the other hand, if $\forall x \in U, K \notin Md(x)$, by the definition of $Md(x)$, $\forall x \in K$, there exists a $K_x \in \mathbf{C}$ such that $K_x \subset K$. As a result, $K \subseteq \cup\{K_x|x \in K\} \subseteq K$. That proves that $K = \cup\{K_x\}$; thus, K is reducible. □

Proposition 23. *Let $\mathbf{C}$ be a covering of U. $\mathbf{C}$ and $reduct(\mathbf{C})$ generate the same third type of covering upper approximation operation.*

Proof. The conclusion about the third type of covering upper approximation operation comes from its definition and Proposition 22. □

Theorem 19. *Let $\mathbf{C_1}$ and $\mathbf{C_2}$ be two coverings of U. If $reduct(\mathbf{C_1}) = reduct(\mathbf{C_2})$, $\mathbf{C_1}$ and $\mathbf{C_2}$ generate the same third type of covering upper approximation.*

Proof. It comes from Proposition 23. □

Theorem 20. *Let $\mathbf{C_1}$ and $\mathbf{C_2}$ be two coverings of U. If $\mathbf{C_1}$ and $\mathbf{C_2}$ generate the same covering lower approximation operation, then they also generate the same third type of covering upper approximation.*

Proof. If $\mathbf{C_1}$ and $\mathbf{C_2}$ generate the same covering lower approximation operation, by Theorem 7,

$$reduct(\mathbf{C_1}) = reduct(\mathbf{C_2}).$$

By Theorem 19, $\mathbf{C_1}$ and $\mathbf{C_2}$ generate the same third type of covering upper approximation. □

Theorem 20 shows that the lower approximation operation uniquely determines the third type of upper approximation operation. Now, we will prove that the upper approximation operation does not uniquely determine the lower one by the following example:

Example 10 (Two coverings yield the same upper approximation operation, but their reducts are not equal [70]). Let $U = \{a, b, c\}$, $K_1 = \{a, b\}$, $K_2 = \{b, c\}$, $K_2 = \{a, c\}$, $K_4 = \{a, b, c\}$, $\mathbf{C_1} = \{K_1, K_2, K_3\}$, and $\mathbf{C_2} = \{K_4\}$. Then, $\forall X \neq \phi$, $TH(X) = \{a, b, c\}$ in both coverings. On the other hand, $reduct(\mathbf{C_1}) = \mathbf{C_1}$, and $reduct(\mathbf{C_2}) = \mathbf{C_2}$. From Theorem 7, it also means that $\mathbf{C_1}$ and $\mathbf{C_2}$ generate different covering lower approximation operations.

In the end of this section, we present the condition under which two coverings generate an identical third type of upper approximations.

Theorem 21. *Let $\mathbf{C_1}$ and $\mathbf{C_2}$ be two coverings of U. $\mathbf{C_1}$ and $\mathbf{C_2}$ generate the same third type of covering upper approximations if and only if $\forall x \in U$, $CFriends(x)$ is the same for these two coverings.*

Proof. It is easy to see that $TH(x) = CFriends(x)$. By property (4H) in Proposition 21, we prove this theorem.□

5.3 Axiomization of the Third Type of Upper Approximation Operations

As in Sections 3.5 and 4.4, the properties in Proposition 21 are not enough to characterize the upper approximation operation in the third type of covering generalized rough sets. This conclusion comes from the following example:

Example 11. An operation satisfies (1H), (2H), (3H), (4H), and (7H), but it is not a third type of covering upper approximation operation. Let $U = \{a, b, c\}$. Define $H : P(U) \to P(U)$ as follows:

$$H(\phi) = \phi, H(\{a\}) = \{a, b\}, H(\{b\}) = \{b, c\}, H(\{c\}) = \{c\},$$
$$H(\{a, b\}) = U, H(\{b, c\}) = \{b, c\}, H(\{a, c\}) = U, H(U) = U.$$

It is obvious that H satisfies properties (1H), (2H), (3H), (4H), and (7H), but it is easy to show that there is no covering $\mathbf{C}$ of U such that H is its third type of covering upper approximation operation.

It is still an open problem to find an axiomatic system for the third type of covering upper approximation operations.

5.4 Relationships with Closure Operations

From Remark 4 in Section 5.1, we know that, generally, TH is not a closure operation, since Axiom IV is not generally valid for TH. We start to investigate the conditions under which the third type of covering upper approximation operation is a closure operator.

Definition 19 (Nonexpansive). *Let $\mathbf{C}$ be a covering of U. $\mathbf{C}$ is called a nonexpansive covering if $\forall x \in U$, $K \in \mathbf{C}$, and $K \cap CFriends(x)$ is a union of elements of $\mathbf{C}$.*

Theorem 22 (Condition under which TH is a closure operator). *Operation TH is a closure operator if and only if $\mathbf{C}$ is a nonexpansive covering of U.*

Proof. It is easy to see that $TH(\{x\}) = CFriends(x)$. If $\mathbf{C}$ is nonexpansive, by the definition,

$$TH(TH(\{x\})) = TH(\{x\}).$$

By property (4H) in Proposition 21, $\forall X \subseteq U$ and $TH(TH(X)) = TH(X)$.

On the other hand, if $\mathbf{C}$ is not nonexpansive, there exists some $x \in U$ and $K \in \mathbf{C}$ such that $K \not\subseteq CFriends(x)$, and $Y = K \cap CFriends(x)$ is not a union of elements of $\mathbf{C}$. Thus, there exists a $w \in Y$ and $K_w \in \mathbf{C}$ such that $K_w \in Md(w)$, and $K_w \not\subseteq Y$. As a result, $K_w \not\subseteq TH(\{x\})$. On the other hand, $K_w \subseteq TH(TH(X))$; thus,

$$TH(TH(\{x\})) \neq TH(\{x\}).$$

We have proved that TH is not a closure operator. □

Corollary 7. *If $\mathbf{C}$ is pointwise covered, then TH is a closure operator.*

6 Relationships among the Three Types of Covering Rough Sets

For a covering $\mathbf{C}$ of U, we have presented three types of covering upper approximation operations in this paper. In this section, we will study the relationships among these three types of covering upper approximation operations.

Since

$$TH(X) = \cup\{Md(x) | x \in X\}$$
$$= \cup\{Md(x) | x \in CL(X)\} \cup \{Md(x) | x \in X - CL(X)\}$$

and $\forall x \in X$, $Md(x) \cap X \neq \phi$, from the definitions of three types of upper approximation operations, it is easy to see that the following rules hold: For a covering $\mathbf{C}$ of U and $X \subseteq U$,

$$FH(X) \subseteq TH(X) \subseteq SH(X). \quad (1)$$

But, generally, the corresponding equalities do not hold. We have the following results about the conditions that guarantee that these equalities hold [70].

6.1 Conditions under Which FH and TH Are Identical

From the above discussion, for a covering $\mathbf{C}$ of U and $X \subseteq U$, we have $FH(X) \subseteq TH(X)$, and the equality does not hold generally. In the following theorem, we present a sufficient and necessary condition under which the equality holds.

Theorem 23. *$FH = TH$ if and only if $\mathbf{C}$ is unary.*

Proof. If $\mathbf{C}$ is unary, from $FH(X) = CL(X) \cup \{Md(x) | x \in X - CL(X)\}$ and

$$TH(X) = \cup\{Md(x) | x \in X\} = \cup\{Md(x) | x \in CL(X)\}$$
$$\cup \{Md(x) | x \in X - CL(X)\},$$

we need to prove only $\cup\{Md(x) | x \in CL(X)\} \subseteq CL(X)$. In fact, $\forall x \in CL(X)$, by the definition of $CL(X)$, there exists a $K \in \mathbf{C}$ such that $x \in K \subseteq X$. Let K' be the only element in $Md(x)$. If $K' \subseteq K$, it is obvious that $K' \subseteq CL(X)$. Otherwise, $\exists y \in K'$ such that $y \notin K$. By the definition of $Md(x)$, there is some K'' such that $K'' \subseteq K$ and $K'' \in Md(x)$. Since $y \notin K$, $y \notin K''$. It concludes $K'' \neq K'$; therefore, $Md(x)$ has at least two elements. That contradicts the assumption that $\mathbf{C}$ is unary.

If $\mathbf{C}$ is not unary, $\exists x \in U$ such that $Md(x)$ has at least two different elements K and K'. By the definition of the first type and third type of upper covering approximations, we have $FH(K) = K$ and $TH(K) = \cup\{Md(x) | x \in K\}$; thus, $K' \subseteq TH(K)$. That means $FH(K) \neq TH(K)$. Otherwise, $K' \subset K$; that contradicts the minimum of K. □

6.2 Conditions under Which SH and TH Are Identical

From the above discussion, for a covering $\mathbf{C}$ of U and $X \subseteq U$, we have $TH(X) \subseteq SH(X)$, and the equality does not hold generally. In the following theorem, we present a sufficient and necessary condition under which the equality holds.

Theorem 24. *$TH = SH$ if and only if covering $\mathbf{C}$ is pointwise covered.*

Proof. If $TH = SH$, $\forall x \in U, K \in \mathbf{C}$ such that $x \in K$, we have $K \subseteq SH(\{x\}) = TH(\{x\}) = \cup Md(x)$; thus, $\mathbf{C}$ is pointwise covered.

On the other hand, if $\mathbf{C}$ is pointwise covered, $\forall X \subseteq U$, $SH(X) = \cup\{K | K \in \mathbf{C} \wedge K \cap X \neq \phi\}$. $\forall K \in \mathbf{C}$ and $K \cap X \neq \phi$. Let $x \in K \cap X$, we have $x \in K$, $x \in X$. By the definition of a pointwise-covered covering, $K \subseteq \cup Md(x)$; thus, $K \subseteq TH(X)$, which proves that $SH(X) \subseteq TH(X)$. $TH(X) \subseteq SH(X)$ is obvious from their definitions; therefore, $SH(X) = TH(X)$. □

6.3 Conditions under Which FH and SH Are Identical

Theorem 25. *$FH = SH$ if and only if covering $\mathbf{C}$ is a partition.*

Proof. If $\mathbf{C}$ is a partition, it is easy to see that $FH = SH$.

On the other hand, if $FH = SH$, by inequality (1) in this section, we have $FH = TH = SH$. From $FH = TH$ and Theorem 23, $\mathbf{C}$ is unary. From $SH = TH$ and Theorem 24, $\mathbf{C}$ is pointwise covered. Then, by Theorem 9 in [70], $\mathbf{C}$ is a partition. □

7 Conclusions

Rough set is one of the useful tools for data mining. The covering-based rough set model is an extension to the classical rough sets. It is more flexible in dealing with uncertainty and granularity in information systems. In this paper, we have studied three types of covering-based rough sets. First, a key concept, $reduct$, is proposed to establish the interdependency between the lower and the upper approximation operations in the first and third types of covering generalized rough sets. We also propose another key concept, $exclusion$, to establish the interdependency between the lower and the upper approximation operations in the second types of covering-based rough sets. Then, we discuss the axiomatic systems for lower and upper approximation operations. Furthermore, we study the relationships between the lower approximation operation and the interior operator and the relationships between the three types of upper approximation operations and the closure operator. In the end, the relationships among the three types of covering upper approximation operations are explored.

There are several issues in covering-based rough sets deserving further investigation. For example, finding a set of axioms for the three types of covering upper approximation is still an open problem. Topological properties of the covering-based rough set are also a potential topic for future research. In addition, the generalization of covering-based rough sets in fuzzy setting is also a promising topic. On the other hand, the applications of covering-based rough set methods to data mining and social computing [14] are exciting areas to be explored. We will study these issues in our future works.

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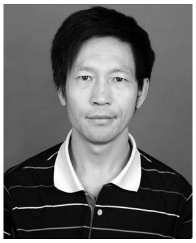

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