A Systematic Study of Fuzzy PID Controllers—Function-Based Evaluation Approach

Bao-Gang Hu, *Senior Member, IEEE*, George K. I. Mann, and Raymond G. Gosine

***Abstract*—A function-based evaluation approach is proposed for a systematic study of fuzzy proportional-integral-derivative (PID)-like controllers. This approach is applied for deriving process-independent design guidelines from addressing two issues: *simplicity* and *nonlinearity*. To examine the simplicity of fuzzy PID controllers, we conclude that direct-action controllers exhibit simpler design properties than gain-scheduling controllers. Then, we evaluate the inference structures of direct-action controllers in five criteria: *control-action composition, input coupling, gain dependency, gain-role change,* and *rule/parameter growth*. Three types of fuzzy PID controllers, using one-, two- and three-input inference structures, are analyzed. The results, according to the criteria, demonstrate some shortcomings in the Mamdani's two-input controllers. For keeping the simplicity feature like a linear PID controller, a one-input fuzzy PID controller with "one-to-three" mapping inference engine is recommended. We discuss three evaluation approaches in nonlinear approximation study: *function-estimation-based, generalization-capability-based* and *nonlinearity-variation-based* approximations. The present study focuses on the last approach. A nonlinearity evaluation is then performed for several one-input fuzzy PID controllers based on two measures: *nonlinearity variation index (NVI)* and *linearity approximation index (LAI)*. Using these quantitative indices, one can make a reasonable selection of fuzzy reasoning mechanisms and membership functions without requiring any process information. From the study we observed that the Zadeh-Mamdani's "max-min-gravity" (MMG) scheme produces the highest score in terms of nonlinearity variations, which is superior to other schemes, such as Mizumoto's "product-sum-gravity" (PSG) and "Takagi–Sugeno–Kang" (TSK) schemes.**

***Index Terms*—Approximation capability, function-based evaluation, fuzzy control, nonlinearity variation analysis, PID controller, systematic study.**

I. INTRODUCTION

FUZZY logic control (**FLC**) technique has been successfully applied in many engineering areas and consumer products since the pioneer work of Mamdani in 1974 [1]. However, a systematic design of fuzzy controllers is still of great concern due to the following facts. First, there is lack of sufficient theories to show that why fuzzy control, either sometimes or most of times, is able to outperform over the conventional control. Second, there is limited knowledge or design guideline available regarding implementation aspects. For this reason, fuzzy control design usually requires a quite amount of "*trial and error*" procedures based on computer simulation or process test. Third, the final tuning of a fuzzy controller for improved plant performance is still a complex task as compared with the straightforward tuning procedures of conventional PID controllers. All these weaknesses have greatly hindered the extensive applications of fuzzy controllers in industries.

In the past some researchers have taken initiatives to investigate the design aspects of different fuzzy systems systematically for efficient fuzzy control. In 1988, Mizumoto compared twelve different inference schemes based on the closed-loop performance of a first-order process with time delay [2]. This pioneer work is important since it demonstrated a systematic approach for selection of valid inference schemes. As the evaluation method is process dependent the conclusions may lose generalization for other higher-order process systems. Recognizing the limitations of Mizumoto's approach, Ying conducted an analytically-based method to assess the different fuzzy inference systems [3]. He derived the closed-form solutions of four-rule fuzzy PI-like controllers using four different inference schemes. By analyzing the desired properties of typical control actions, Ying was able to eliminate one reasoning scheme called "Bounded Product Inference". This evaluation method, being process-independent, derives the generalized conclusions for scheme selections. However, this work did not rank different valid fuzzy reasoning schemes for the best selection. Selections and evaluations of reasoning methods and defuzzification methods later became a key issue for a systematic study of fuzzy control [4]–[6].

In this work, we propose a function-based evaluation approach for a systematic study of fuzzy proportional-integral-derivative (PID) controllers. While a performance-based evaluation approach is more common in assessing control technique, the function-based evaluation approach receives less attention. In fact, each approach emphasizes different aspects in control that usually cannot be derived from the other approach. Table I lists two sets of evaluation issues addressed by two approaches respectively in control theory and applications. As shown in the table, the performance-based evaluation approach is to examine the controller in terms of performance characteristics of response to the external process being controlled. As a result, the findings from the performance-based evaluation approach are basically process-dependent. In contrast, the

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B.-G. Hu is with National Laboratory of Pattern Recognition Institute of Automation, Beijing 100080, China (e-mail: hubg@nlpr.ia.ac.cn).

G. K. I. Mann is with Department of Mechanical Engineering, Queen's University, Kingston, Ontario K7L 3N6, Canada (e-mail: mann@me.queensu.ca).

R. G. Gosine is with C-CORE and Faculty of Engineering and Applied Science, Memorial University of Newfoundland, St. John's, A1B 3X5 Canada (e-mail: rgosine@engr.mun.ca).

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TABLE I
COMPARISONS OF EVALUATION ISSUES FOR TWO EVALUATION APPROACHES OF CONTROLLERS

Evaluation Approach	Performance-based Approach	Function-based Approach
	Response error	Control mechanism
	Stability	Continuous/discrete
	Robustness	Sensitivity
	Observability	Transparency
Evaluation Issues	Controllability	Nonlinearity
	Optimality	Soft/hard reasoning
	Adaptation	Self-tuning/organizing
	Anti-perturbation	Fault tolerance
	Frequency response (Closed-loop)	Frequency response (Open-loop)
	Static response	Parallelism
	Dynamic response	Static/dynamic
	Model accuracy	Complexity
	Real-time	Interactivity
	Others	Others

function-based evaluation approach is to reveal the intrinsic properties of controllers. This approach does not require any process information, and the conclusions from this approach are more general regardless of the process type. A complete systematic study should include both performance-based evaluation as well as function-based one.

One of the objectives of this work is to demonstrate that the function-based evaluation approach is a unique tool for deriving process-independent guidelines in a systematic design. We address two basic issues in the design of fuzzy PID controllers, that is, the *simplicity* related to the fuzzy inference structures, and the *nonlinearity* related to the reasoning schemes and membership functions. Systematic design procedure will be given to the fuzzy PID controllers according to the two issues of functionality. The rest of this paper is organized as follows. In Section II, we classify the fuzzy controllers according to the PID principles. An initial selection of fuzzy PID structures is made in Section III with respect to simplicity. In Section IV, we propose five simple evaluation criteria in assessment of the Mamdani's two-input controllers. Using the same criteria, a similar analysis is made to three- and one-input fuzzy PID controllers in Sections V and VI, respectively. In Section VII, the functional behaviors and the linguistic representations are discussed. We apply two numerical indices for an indepth nonlinearity evaluation of the fuzzy controllers proposed in Section VIII. Eight examples are studied as evaluation examples in Section IX. Finally, a summary and discussions are presented in Section X.

II. CLASSIFICATION OF FUZZY CONTROLLERS

Considering the extensive applications of PID technique in industry [7], this study is primarily on fuzzy PID-like

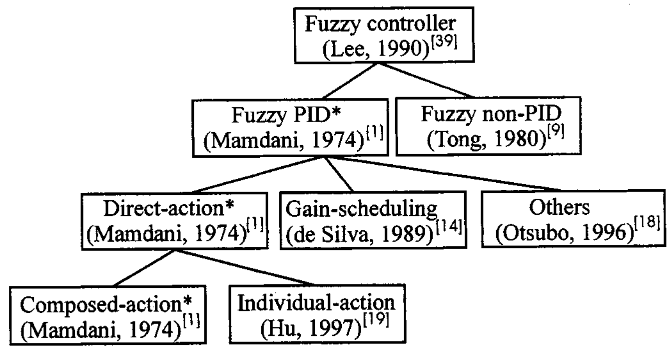

Fig. 1. Classification of fuzzy controllers in view of control actions. (The main reference source is given in the parentheses. The asterisk indicates the commonly-used type of controllers compared with its counterparts.)

controllers. Let an absolute expression of three-term PID controllers be given by

$$u_{\mathrm{PID}}(r) = K_P e(r) + K_I \sum_{i=0}^{r} e(i)\Delta t + K_D \frac{\Delta e(r)}{\Delta t} \qquad r = 0, 1, 2, \ldots \quad (1)$$

where e and Δe are the error and the change of error during the sampling interval Δt; and K_P, K_I, K_D are the proportional, integral and derivative gains, respectively. Equation (1) can be rewritten in a form

$$u_{\mathrm{PID}} = u_P + u_I + u_D \quad (2)$$

where u_P, u_I and u_D are proportional, integral, and derivative actions, respectively. The total, or composed, output of three actions is denoted by u_{PID}.

In view of PID control principles, we suggest a classification of the existing fuzzy controllers as shown in Fig. 1. If a fuzzy controller is designed (or implied equivalently) to generate the control actions within the proportional/integral/derivative (P/I/D) concept(s) like a conventional PID controller, we call it a fuzzy PID-like (or fuzzy PID) controller. The first fuzzy controller developed by Mamdani [1], [8] is a fuzzy PI controller. The type of fuzzy non-PID controllers can be found in model-based fuzzy controllers [9], such as the MIMO models [10] using the T-S-K consequent representations [11], [12]. We term the direct-action (DA) type for the Mamdani's controller [13], since its fuzzy inference deduces the control-action output directly to drive the process. The fuzzy gain-scheduling (GS) type [14]–[16] is most similar to the conventional GS controller [17] in changing the gains for varied operating conditions or process dynamics. The other class of fuzzy PID controllers can be found like the "hybrid" controller [18], which employs the fuzzy PID for the coarse tuning and linear PID for fine tuning. We also define the Mamdani's controller to be composed-action (CA) type since its fuzzy inference output is a composed force of proportional and integral actions. In [19], we proposed a single-input fuzzy controller and termed it individual-action (IA) type. In each level of the classification (Fig. 1), the Mamdani's type is the most common in applications.

Another type of classification of fuzzy controllers can be made based on the dimensionality (defined as the total number of input variables) of the inference engines. Up to now, most researchers have adopted the Mamdani's two-input fuzzy inference structure [20]–[22]. They used e and Δe as the input variables to the fuzzy inference. Three-input fuzzy PID controllers have been reported in [23], [24]. The general algorithms for n-dimensional fuzzy controller design are given in [25], and [26].

III. Initial Selection of Fuzzy PID Structures With Respect to Simplicity

While many investigations demonstrated the performance of each fuzzy PID controller, there is a lack of systematic study of an optimal selection from various design structures. In this section, we will show how to make an initial selection of fuzzy PID structures if simplicity is concerned in applications. Therefore, we suppose *a prior* knowledge to form every control design is sufficient and its performance of control process is not an issue for each design.

The initial selection of fuzzy PID controllers is made between the DA type and GS type. The basic linguistic representation for each type of controllers is

DA: If (``process error" is ...),
then (``control action" is ...)
GS: If (``process error" is ...),
then (``control gain" is ...).

From these representations one can see that the difference between two types of controllers will be the corresponding forms of nonlinear relationships, $f(\mathbf{e})$, from the fuzzy inference mapping, where $\mathbf{e}$ is an error vector. While the nonlinear function of DA-type fuzzy controllers benefits the property of "zero control action for zero error," or $f(\mathbf{e}=0)=0$, one cannot expect such property for the GS-type fuzzy controllers. In addition, DA-type fuzzy controllers also have the property of "maximum control action for maximum error" from control principle. However, there are no such common rules for the GS-type fuzzy controllers. All these process-independent properties will simplify the design of nonlinear functions and will be helpful in reducing some free parameters of fuzzy systems. Therefore, it is reasonable to conclude that DA-type fuzzy controllers will have a simpler nonlinear function than that of GS-type fuzzy controllers. In the following sections, we will discuss DA-type fuzzy controllers further for an optimal selection of design structures with respect to some criteria of simplicity.

IV. Two-Input Fuzzy PID Controllers

In this section, we analyze two-input fuzzy PID controllers. Although this type of controllers is the most common in fuzzy control applications, their functional behaviors and the associated weakness in controller tuning have not been well investigated and recognized. For comparing with the functional behaviors of the conventional PID controllers, we proposed five evaluation criteria as follows to examine this type of controllers:

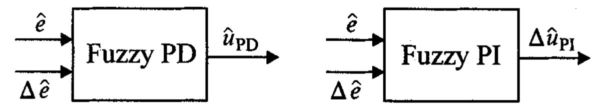

Fig. 2. Two-input fuzzy PID controllers. (a) Absolute output. (b) Incremental output.

1) Control-Action Composition: Most fuzzy PID controllers are two-input structures as shown in Fig. 2. The linguistic representations are as follows:
incremental output

$$\begin{aligned}&\text{Rule } r\text{: If } \hat{e} \text{ is } E_i \text{ and } \Delta\hat{e} \text{ is } \Delta E_j,\\&\qquad \text{then } \Delta\hat{u}_{\mathrm{PI}} \text{ is } (\Delta U_{\mathrm{PI}})_m\end{aligned} \tag{3}$$

absolute output

$$\begin{aligned}&\text{Rule } r\text{: If } \hat{e} \text{ is } E_i \text{ and } \Delta\hat{e} \text{ is } \Delta E_j,\\&\qquad \text{then } \Delta\hat{u}_{\mathrm{PD}} \text{ is } (\Delta U_{\mathrm{PD}})_m\end{aligned} \tag{4}$$

where $\hat{e}$ and $\Delta\hat{e}$ are normalized error and change of error; $E_i, \Delta E_j, (\Delta U_{\mathrm{PI}})_m$ and $(U_{\mathrm{PD}})_m$ are fuzzy variables. The first fuzzy controller developed by Mamdani is a fuzzy PI-like controller using (3), since it has the similar relation to the incremental expression of a conventional PI controller

$$\Delta u_{\mathrm{PI}}(r) = K_I e(r) + K_P \frac{\Delta e(r)}{\Delta t}, \quad r = 0, 1, 2, \ldots. \tag{5}$$

If an absolute output is used in (4), the system becomes a fuzzy PD-type controller. Comparing (5) with (1), one can see the associations of "gain-with-variable" are changed with the form of controller output. For example, K_P associated with e in (1) changes to associate with Δe in (5). These associations are one-to-one for a linear PID controller, but become invalid for the two-input fuzzy PID controllers. The two-input fuzzy controllers represented by (3) and (4) produce the composed Δu_{PI} [Fig. 2(a)] and composed u_{PD} [Fig. 2(b)] outputs, respectively. Due to a single output directly from the "two-to-one" mapping in the fuzzy inference, one is unable to decompose the output for the exact component of each action. This behavior is called "control-action composition."

According to the classical control theory, the effects of individual P/I/D actions (or gains) of a controller has been summarized by de Silva [27] as follows:

u_P: speed up response, decrease rise time, and increase overshoot.
u_I: reduce the steady-state error.
u_D: increase the system damping, decrease settling time.

The feature of independent tuning of each control action is very useful in control engineering practice. However, the present two-input fuzzy PID controllers do not share this feature due to the control-action composition.

2) Input Coupling: To explain the input-coupling behavior, we use the derivation results from Ying [3] for fuzzy PI-type controllers. The conclusions based on this analysis are also valid for fuzzy PD-type controllers since (3) and (4) are basically the same except for the form of output. Four types of inference schemes (including Mamdani's) have been investigated by Ying, but a uniform equation for the composed fuzzy output was derived

$$\Delta u_{\mathrm{PI}}(r) = \tilde{K}_I e(r) + \tilde{K}_P \Delta e(r), \quad r = 0, 1, 2, \ldots \tag{6-a}$$

and

$$\tilde{K}_I = \beta \cdot S_e \quad \tilde{K}_P = \beta \cdot S_{\Delta e} \tag{6-b}$$

where S_e and $S_{\Delta e}$ are the scaling factors to error and error change signals, respectively. Note that some notations are different from Ying's, where we call $\tilde{K}_P$ and $\tilde{K}_I$ the "apparent proportional gain" and "apparent integral gain," respectively. The term of "apparent" is used because the exact value for each control action (or equivalent gain) can never be obtained from the composed output [13]. Equation (6-a) shows the decomposed two terms only in an apparent sense based on (5) for approximations of two control actions. The value of β in (6-b) can be represented by a general form of functions [3, Table III]

$$\beta = f(e, \Delta e). \tag{6-c}$$

Therefore, the two apparent control actions will be

$$\tilde{u}_P = f_1(e, \Delta e), \quad \tilde{u}_I = f_2(e, \Delta e). \tag{7}$$

These equations indicate that each control action will be a function of both error and error-rate signals. This behavior is called "input coupling." In general, this coupling presents negative effects on control tuning operations as well as on control performance. First, if an input scalar of fuzzy inference is adjusted, each control action will be changed at the same time. This makes the independent tuning of each apparent control action quite difficult if not impossible. Second, the original meaning of each control action is changed due to the input coupling. For example, the proportional action, being proportional to error signal, is also a function of its error-rate signal. This controller may become more sensitive to noisy data than a conventional PID controller.

3) Gain Dependency: As each P/I/D action has different effects in the response characteristics, it is always desired to have decoupled PID gains for independent tuning. However, the present two-input fuzzy PID controller shows "gain dependency" behavior from (6-b)

$$\tilde{K}_I = \frac{S_e}{S_{\Delta e}} \tilde{K}_P. \tag{8}$$

This equation suggests that the apparent integral gain is partially given by the apparent proportional gain. This behavior of gain dependency, causing decreased tuning range of the equivalent gains, will limit the controller to possess a broad range of controllability.

Gain dependency and input coupling are two different concepts. Mathematically, input coupling can be interpreted by using the following matrix expression for the fuzzy PI controller

$$\begin{Bmatrix} \Delta\tilde{u}_P \\ \Delta\tilde{u}_I \end{Bmatrix} = \begin{bmatrix} K_P & K_{\mathrm{PI}} \\ K_{\mathrm{IP}} & K_I \end{bmatrix} \begin{Bmatrix} \Delta e \\ e \end{Bmatrix} \quad \text{or} \quad \Delta\tilde{\mathbf{u}} = \mathbf{K}\mathbf{e} \tag{9}$$

where $\mathbf{K}$ is a gain matrix, $\mathbf{e}$ and $\Delta\tilde{\mathbf{u}}$ are the error and apparent incremental control action vectors, respectively. The nondiagonal terms in $\mathbf{K}$ cause the input coupling from $\mathbf{e}$ to $\Delta\tilde{\mathbf{u}}$ (9). Note that the actual output of the controller is the sum of two control actions, $\Delta u_{\mathrm{PI}} = \mathrm{sum}\{\Delta\tilde{u}_i\}$. The decomposition of the output is made apparently. Further, the apparent gains in (6-a) form a diagonal matrix

$$\tilde{\mathbf{K}} = \begin{bmatrix} \tilde{K}_P & 0 \\ 0 & \tilde{K}_I \end{bmatrix} \tag{10}$$

where we call $\tilde{\mathbf{K}}$ apparent gain matrix, which shows apparently decoupling of the control actions. Two diagonal terms in $\tilde{\mathbf{K}}$ are actually inter-related (8). From a viewpoint of control engineering, a coupling effect indicates that one control action is caused by two or more input variables. Gain dependency means that tuning of an individual gain influences to the other gain(s).

4) Gain-Role Change: Most industrial process control systems contain transportation delay of signal and therefore it is reasonable to expect a certain degree of time delay in the process of control. The Mamdani's fuzzy PI controllers may suffer a problem due to this effect. In a set-point control process having a time delay, the error-rate signal is always presented to be zero or negligible ($\Delta e = 0$) during the initial time-lag period: $0 \leq t \leq t_d$, where t_d is the time delay. According to (6-a), the output of the two-input fuzzy PI controller is then uniquely produced by a component of the apparent integral action, $\Delta u = \tilde{\mathbf{K}}_I e + \tilde{\mathbf{K}}_P 0$. The role of the apparent integral gain is actually corresponding to a proportional action, but is changed back when $t > t_d$.

The tuning principles of a linear PID technique suggest that: 1) if the error is maximum, the component of the proportional action requires to be great for a fast response of the process to reach the set-point and 2) when the error is near zero, the component of the integral action should be dominant for reducing the steady-state error. The apparent integral action represented in (7) is included with the proportional component, the "*gain-role change*" of the apparent integral gain will result in negative effects in applications. First, it is more difficult to adjust the gain for the compromise of performance within two periods. Second, the overall performance of the process may be decreased from the compromise. This functional problem, however, is related to the incremental form of fuzzy inference output. This discussion is also valid for a linear PID controller.

5) Rule/Parameter Growth: Suppose the rule numbers in each input variable are the same, denoted by n. The total number of rules is "n^2" for the two-input fuzzy controllers. Since the parameters of membership functions (**MF**s) are associated with rules, the parameter growth will be increased with the rule growth.

V. Three-Input Fuzzy PID Controllers

Fig. 3 shows the three-input fuzzy PID controllers. Based on the discussions in the previous section, it is understandable that these controllers exhibit control-action composition and coupling inference due to their "three-to-one" mapping. The controller using absolute output [Fig. 3(a)] usually has the difficulty in formulating the fuzzy rules for the variable $\Sigma\hat{e}$. The

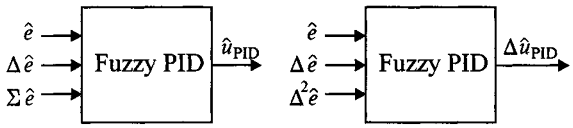

Fig. 3. Three-input fuzzy PID controllers. (a) Absolute output. (b) Incremental output.

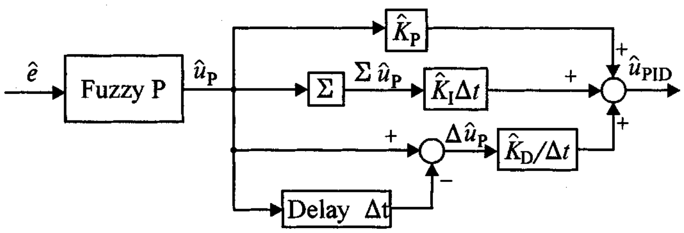

Fig. 4. One-input fuzzy PID controller with a "one-to-one" inference mapping.

incremental-form controller [Fig. 3(b)] presents a functional problem of gain-role change for time-delayed processes. Using the closed-form solution derived in [13], we find the controller in Fig. 3(a) also possesses gain dependency. The rule growth for three-input controllers is "n^3."

VI. One-Input Fuzzy PID Controllers

We have developed a one-input fuzzy PID controller [19], which used a "one-to-one" fuzzy inference mapping (Fig. 4). The output of the inference is the fuzzy proportional action, $\hat{u}_P$. The other two actions, the integral and derivative, are deducted from $\hat{u}_P$. This controller is more analogous to a conventional PID controller

$$\hat{u}_{\text{PID}}(r) = \hat{K}_P \hat{u}_P(r) + \hat{K}_I \sum_{i=0}^{r} \hat{u}_P(i)\Delta t + \hat{K}_D \frac{\Delta \hat{u}_P(r)}{\Delta t} \qquad r = 0, 1, 2, \ldots \quad (11)$$

where $\hat{K}_P$, $\hat{K}_I$ and $\hat{K}_D$ are the normalized gains. The three control actions are calculated individually, and as a result the problems of control-action composition and input coupling are eliminated. The equivalent gains of the fuzzy PID controller can be obtained by comparing (4) and (11):

$$(\hat{K}_P)_{\text{eq}} = \hat{K}_P \frac{\hat{u}_P}{\hat{e}}, \quad (\hat{K}_I)_{\text{eq}} = \hat{K}_I \frac{\Sigma \hat{u}_P}{\Sigma \hat{e}}, \quad (\hat{K}_D)_{\text{eq}} = \hat{K}_D \frac{\Delta \hat{u}_P}{\Delta \hat{e}}. \quad (12)$$

The association of "gain-with-variable" in the conventional PID controllers is remained for this controller in a general sense.

Examining (12), we can obtain the following relationship:

$$(\hat{K}_D)_{\text{eq}} = \frac{\hat{K}_D}{\hat{K}_P} \left[(\hat{K}_P)_{\text{eq}} + \hat{e} \frac{d(\hat{K}_P)_{\text{eq}}}{d\hat{e}} \right]. \quad (13)$$

This equation indicates that the equivalent derivative gain is dependent on the equivalent proportional gain. Actually, three equivalent gains are calculated from the same fuzzy proportional action (12). Therefore, all the equivalent gains are dependent [13]. In order to achieve an independent tuning property of the gains, we proposed a "one-to-three" mapping structure [13] for the single-input fuzzy controller (Fig. 5). Three independent fuzzy proportional actions are generated. Substituting three individual $\hat{u}_{Pj} (j = 1, 2, 3)$ into three terms of (11), respectively, this controller will produce three independent equivalent gains. Using this fuzzy mapping strategy, the rule growth is "$3n$" for this controller. The absolute form of the fuzzy inference output will keep the PID principles even in a time-delayed process.

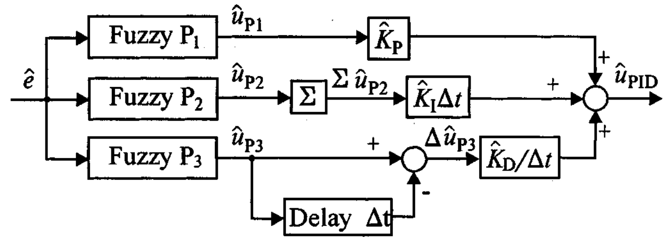

Fig. 5. One-input fuzzy PID controller with a "one-to-three" inference mapping.

VII. Linguistic Representations and Fuctional Behaviors

The five-functional behaviors proposed above are determined by the linguistic representations of fuzzy knowledge. The pioneer work on the relationship between linguistic representations and functional behaviors was discussed by de Silva [28]. The Mamdani form for fuzzy rules is represented by

$$\text{Rule } r\text{: } [(\text{if } \hat{e}_1 \text{ is } E_{1,i}) \text{ and} \ldots \text{and } (\text{if } \hat{e}_p \text{ is } E_{p,a}), \text{ then } (\hat{u}_1 \text{ is } U_{1,j}) \text{ and} \ldots \text{ and } (\hat{u}_q \text{ is } U_{q,b})] \quad (14)$$

where $\hat{e} \triangleq [\hat{e}_1, \ldots, \hat{e}_p]$ is the input variable vector, and $\hat{\boldsymbol{u}} \triangleq [\hat{u}_1, \ldots, \hat{u}_q]$ is the output variable vector. This equation represents a "p-to-q" mapping of fuzzy reasoning. We suggest a simple criterion for examining coupled rules of the conventional fuzzy system

$$p > 1.$$

In this case, the controller generally exhibits behaviors of control-action composition and input coupling. This criterion is very useful when constructing a fuzzy-knowledge base. Although knowledge rules are usually extracted from the experts of controllers, it is suggested to apply uncoupled rules if possible. When the convenient tuning features, like the conventional PID controller's, are the most concern in the design, a single-input fuzzy PID controller with a "one-to-three" mapping is considered to be the optimal structure (Table II). The rules for this controller are

$$\text{Rule } r\text{: [if } \hat{e} \text{ is } E_i \text{ then, } (\hat{u}_{P1} \text{ is } U_{p1,j}) \text{ and } (\hat{u}_{P2} \text{ is } U_{p2,k}) \text{ and } (\hat{u}_{P3} \text{ is } U_{p3,l})]. \quad (15)$$

VIII. Nonlinearity Evaluations

The function-based evaluation presents the most importance in fuzzy control designs due to the nonlinearity characterized

TABLE II
COMPARISONS BETWEEN LINEAR AND FUZZY PID CONTROLLERS FROM THE TUNING ASPECTS. (N = THE TOTAL NUMBER OF FUZZY SETS IN EACH INPUT VARIABLE)

Controller types		Individual Control-action Calculation	Uncoupled Input	Independent Gain Tuning	Total Rules
Linear PID controllers		Yes.	Yes.	Yes.	0
Fuzzy PID	three-to-one	No.	No.	No.	N^3
	two-to-one	No.	No.	No.	N^2
	one-to-one	Yes.	Yes.	No.	N
	one-to-three	Yes.	Yes.	Yes.	$3N$

by the controllers. For this reason, a nonlinearity evaluation of fuzzy PID controllers is taken as the second issue concerned in this work. Considering fuzzy systems as universal approximators, we believe nonlinear approximation capability will be a basic content in the nonlinearity evaluation. Three concepts have been applied in the study of nonlinear approximation capability:

1) *function-estimation-based* approximation;
2) *generalization-capability-based* approximation [29];
3) *nonlinearity-variation-based* approximation [30], [31].

Significant differences exist among the three concepts. The first approximation is the conventional concept in nonlinear function approximation. The estimation error, say, in regression, will be a measure in the evaluation. The second concept has been adopted mostly in the study of neural networks or statistical learning theory. While an approximator is constructed based on a set of training data, its approximation evaluation will be made based on a set of testing data. The performance of the approximator in obtaining the correct estimation for testing data is called generalization capability. While a generalization error was given as a measure in evaluation of neural networks [29], Vapnik presented a more generalized form for the approximator evaluation, called risk function in statistical learning [32]. On the contrary, the concept of nonlinearity variation [30], [31] is used to evaluate a controller according to its capability of generating a group of nonlinear functions, rather than to its approximation accuracy to a specific function. This concept is appropriate in control applications since the nonlinear functions for approximation are usually unknown. In addition, the space of nonlinear functions spanned by an approximator for a given number of free parameters corresponds to the nonlinear adaptation, or optimization, space in control processes. In this sense, the larger the space of nonlinearity variations, the greater the possibility of a high performance controller.

We consider that the nonlinearity-variation-based approximation will be proper in evaluation of fuzzy PID controllers. The concept of the nonlinearity variations can be used as a measure to evaluate "*the 'goodness' of the transformation from knowledge base to nonlinear mapping*" as stated by Wang in [33 pp. 204] for fuzzy systems. The analysis based on nonlinearity variations can provide the process-independent design guidelines for selection of fuzzy inference schemes, defuzzifications, or even membership functions. The study of nonlinearity variations is quite novel at present. In this work, we adopt two quantitative indices in the previous study [30], [31] for demonstrating how to implement a systematic design in this regard. For a complete understanding of the analysis procedures, we rewrite two indices below with a minor modification.

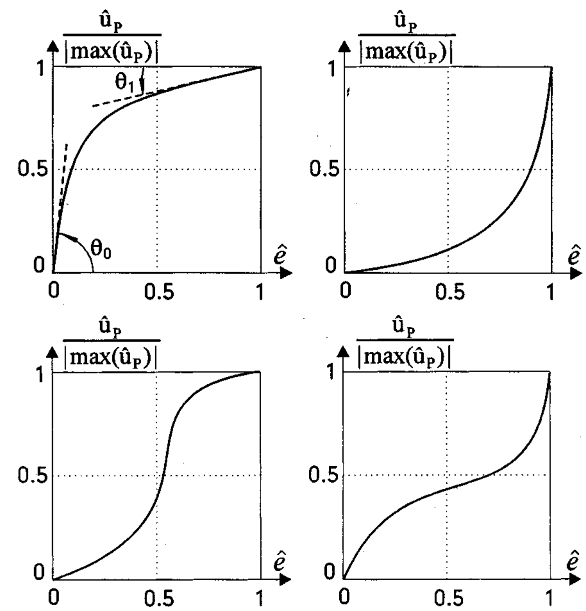

Fig. 6. Four types of simple nonlinear curves.

1) Nonlinearity Variation Index (NVI): Suppose any design parameters related to fuzzy structures are called *nonlinear tuning parameters*. We can understand that increasing the number of these parameters will increase the nonlinearity variations, but this will make the nonlinearity evaluation difficult. For simplicity and without losing generality, we consider the simplest rules (say, two or three in this work) and two nonlinear tuning parameters for the comparative study.

Since a one-input fuzzy controller only involves a control curve design, the nonlinearity analysis will be based on a two-dimensional space. In the previous study [31], we have found that the four types of simple nonlinear curves (Fig. 6) are basic for such nonlinearity analysis. For this reason, a quantitative measure is proposed. Let θ_0 and θ_1 be the angles in radians corresponding to the curve slops $(\partial \hat{u}_P / \partial \hat{e})$ at $\hat{e} = 0$ and $\hat{e} = 1$

(Fig. 6), respectively. The change of nonlinear tuning parameters result in different types of curves with different θ_0 and θ_1.

To examine the nonlinearity variations approximately, we define the admissible area (or curve) of the nonlinearity diagram on the "θ_0 and θ_1" plane. Fig. 9 shows the admissible area for a controller called MMG-I, which we will introduce later. Here we call θ_0 and θ_1 *nonlinearity examination parameters*. The point within the admissible area (or on the admissible curve) means that a control curve associated with these θ_0 and θ_1 can be produced by the controller. The larger the admissible area, the greater the flexibility of the system in generating the nonlinear functions. The NVI is defined in a dimensionless form

$$\mathrm{NVI}(N_v, N_t N_e) = \frac{\text{admissible region in } N_e \text{ dimensional space}}{\text{whole region in } N_e \text{ dimensional space}} \quad (16)$$

where N_v, N_t, N_e are the total numbers of input variables, nonlinear tuning parameters and nonlinear examination parameters, respectively. In this work, $\mathrm{NVI}(1,2,2)$ is calculated for the controllers.

2) Linearity Approximation Index (LAI): We propose a conservative design strategy for a fuzzy PID controller [31]: "*A fuzzy PID controller should be able to perform a linear, or approximately linear, PID function such that the system performance is no worse than its conventional counterpart*." If the controller is able to generate a perfect linear function, $\hat{u}_P = \hat{e}$, we call it a guaranteed-PID-performance (GPP) system. Along the line of this strategy, a safe performance bound is produced for the fuzzy PID system from the performance analysis of its counterpart that has the same PID connective structure. For examining the system on this aspect, an LAI is given

$$\mathrm{LAI} = 1 - \frac{\max|\hat{u}_P(\hat{e}) - \bar{u}_P(\hat{e})|}{\max|\bar{u}_P(\hat{e})|} \quad (17)$$

where $\bar{u}_P$ is a linear function which is imposed to pass through the origin point, $\hat{u}_P(\hat{e}=0)=0$ and the ending point $\hat{u}_P(\hat{e}=1)$. This index, representing the most linearity approximation that can be produced by the controller, is normalized within a range of $[0,1]$. When "$\mathrm{LAI}=1$," it corresponds to a perfect linear PID controller. The larger the value of LAI, the higher degree of linear approximation the fuzzy PID controller produces. This index is a quantitative measure of confidence in using a GPP bound calculated from the linear PID controller.

IX. Evaluation Examples

In this work, eight differently designed controllers are studied as evaluation examples to demonstrate how to conduct an in-depth analysis of the nonlinear systems. Four design schemes, including the three commonly-used fuzzy reasoning schemes, are selected and given in details as follows. Some schemes have been investigated in either numerical [2] or analytical [3] ways. We will show the nonlinearity analysis from a different viewpoint. In the present evaluations, the one-input fuzzy controllers with two nonlinear tuning parameters are considered. All controllers are designed to be compatible with the heuristic properties proposed in [30].

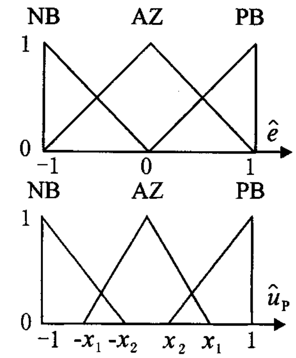

Fig. 7. Membership functions for "MMG-I," "MMG-IV," and "PSG-I" controllers.

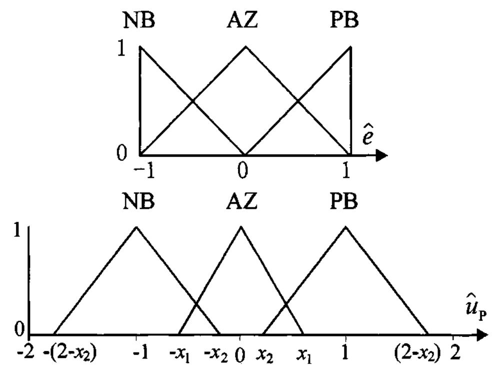

Fig. 8. Membership functions for "MMG-II " "MMG-III," and "PSG-II" controllers.

A. Zadeh-Mamdani's "Max-Min-Gravity (MMG)" Scheme

This scheme uses the standard Zadeh-Mamdani "max-min" reasoning mechanism and "center of area" for the defuzzification. A term of max-min-gravity (MMG) is used to denote this type of controllers. Four controllers are designed using this scheme. They all use triangular membership functions with three rules as

$$\begin{aligned} &\text{Rule 1:} \quad \text{If } (\hat{e} \text{ is NB}), \text{ then } (\hat{u}_P \text{ is NB}) \\ &\text{Rule 2:} \quad \text{If } (\hat{e} \text{ is PB}), \text{ then } (\hat{u}_P \text{ is PB}) \\ &\text{Rule 3:} \quad \text{If } (\hat{e} \text{ is AZ}), \text{ then } (\hat{u}_P \text{ is AZ}) \end{aligned} \quad (18)$$

where "NB," "PB" and "AZ" stand for "negative big," "positive big" and "approximate zero," respectively. The main differences of each controller are presented below.

1) "MMG-I" Controller: Fig. 7 shows the membership functions of this controller. Two nonlinear tuning parameters, $x_1 \in (0,1]$ and $x_2 \in [0,1)$, are used to change the nonlinearity of the control curves. And the membership functions of $\hat{u}_P$ are distributed within the range of $[-1,1]$. The closed form solution of $\hat{u}_P(\hat{e})$ was derived in [31], but is also given in Appendix A, since some results are used later by other controllers. The output of this controller is not fully normalized since it has $\max(|\hat{u}_P|) = (2+x_2)/3 \le 1$.

2) "MMG-II" Controller: This controller is designed to be the same as "MMG-I" except that the range of the membership functions of $\hat{u}_P$ are extended to $[-2+x_2, 2-x_2]$ (Fig. 8). The controller has the symmetrical membership functions for

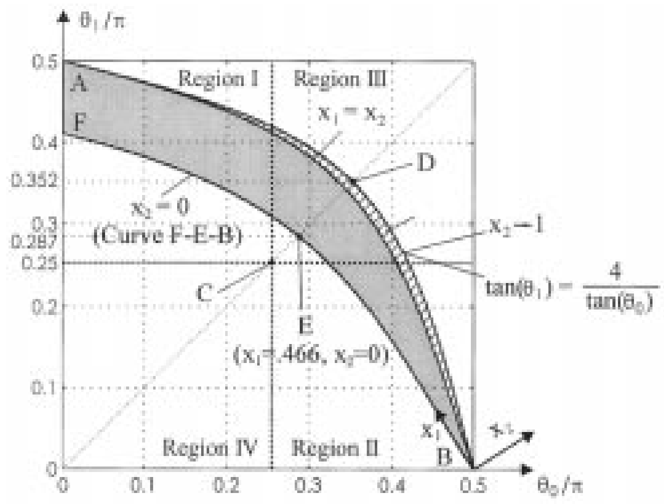

Fig. 9. Admissible area of nonlinearity diagram for "MMG-I" controller. Hatched area: for nonoverlapping case. Grey area: for overlapping case. E: point for approximation of a linear PID.

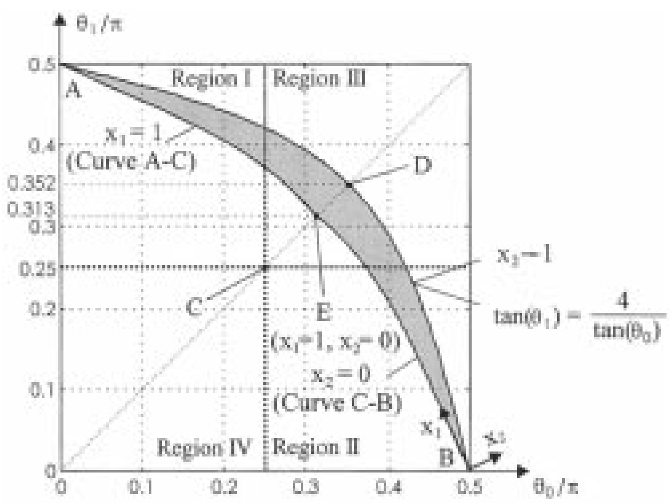

Fig. 10. Admissible area/curve of nonlinearity diagram for "MMG-II" controller. Curve A-D-B: admissible curve for nonoverlapping case. Grey area: admissible area for overlapping case. E: point for approximation of a linear PID.

the fuzzy consequent sets. This change leads the $\hat{u}_P$ to be fully normalized, or $\max(|\hat{u}_P|) = 1$. Appendix B presents the closed-form solution of this fuzzy inference.

3) "MMG-III" Controller: Based on the design guidelines in the previous work [30], we find the parameter, x_2, can be extended to a larger range but also satisfies the property of $\mathrm{NS} = \partial\hat{u}_P/\partial\hat{e} \geq 0$. This modification is able to enlarge the NVI value of the "MMG-II" controller. A new controller, named "MMG-III", is designed in which the parameter, x_2, is changed to a new range of $x_2 \in [-x_1, 1)$, while the others are kept the same as the "MMG-II" controller. The closed-form solution of $\hat{u}_P(\hat{e})$ of this controller is discussed in Appendix C.

4) "MMG-IV" Controller: Using the improving strategy of the enlarged x_2 on the "MMG-I" controller, we design the "MMG-IV" where the parameter, x_2, is changed to a new range of $x_2 \in [-x_1, 1)$. The other parts of design are kept the same as the "MMG-I" controller. The closed-form solution of $\hat{u}_p(\hat{e})$ is discussed in Appendix C.

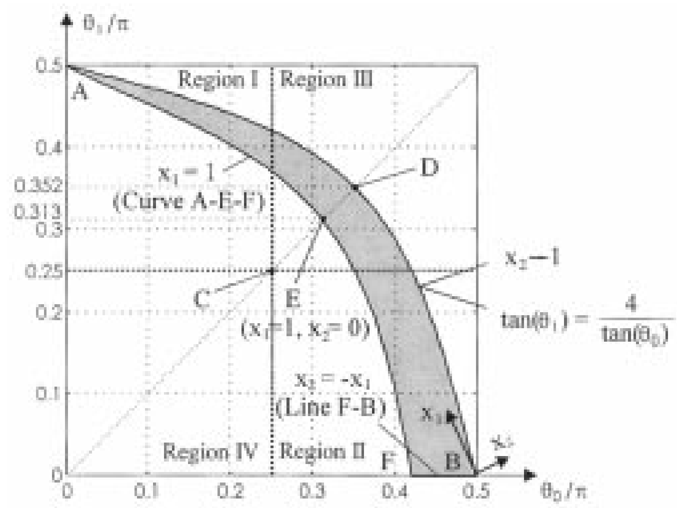

Fig. 11. Admissible area/curve of nonlinearity diagram for "MMG-III" controller. Curve A-D-B: admissible curve for nonoverlapping case. Gray area: admissible area for overlapping case. E: point for approximation of a linear PID.

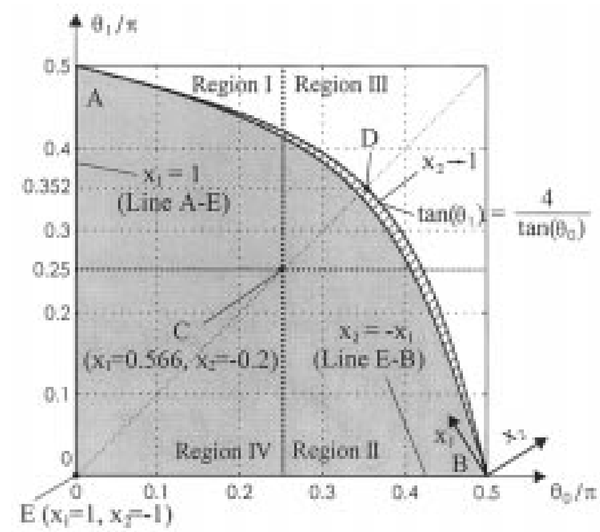

Fig. 12. Admissible area of nonlinearity diagram for "MMG-IV" controller. Hatched area: for nonoverlapping case. Gray area: for overlapping case. C: point for approximation of a linear PID.

Figs. 9–12 shows the admissible areas/curves of nonlinearity diagrams for four controllers, respectively. These diagrams provide the valuable information to guide the design. Note that all diagrams are based on a normalized sense, $\hat{u}_p(\hat{e})/\max|\hat{u}_P|$, to derive θ_0 and θ_1. This normalization will eliminate nonlinearity redundancy if the $\hat{u}_p(\hat{e})$ is directly used for the nonlinearity diagrams [31]. In addition, Point C, corresponding to the most approximately linear function, will always locate at $\theta_0 = \theta_1 = \pi/4$. Four regions are divided in the diagram which correspond to the four types of curves in Fig. 6 accordingly. Table III lists the comparisons of four fuzzy PID controllers using the MMG scheme. It is interesting to see that, while a minor change is made to each controller, its associated NVI and LAI are changed significantly. The best design is considered the "MMG-IV" controller due to its largest NVI and LAI values. The admissible areas of other controllers are fully covered by the area of this

TABLE III
COMPARISONS OF FUZZY PID CONTROLLERS USING MAX-MIN-GRAVITY SCHEME

Inference and defuzzification	Max-Min-Gravity scheme			
Controller name	MMG-I	MMG-II	MMG-III	MMG-IV
Parameter range For x_I	$x_1 \in (0,1]$ $x_2 \in [0,1)$	$x_1 \in (0,1]$ $x_2 \in [0,1)$	$x_1 \in (0,1]$ $x_2 \in [-x_1,1)$	$x_1 \in (0,1]$ $x_2 \in [-x_1,1)$
Range of MFs for $\hat{u}_P$	[-1, 1]	[-2+x_2, 2-x_2]	[-2+x_2, 2-x_2]	[-1, 1]
Max$\lvert\hat{u}_P\rvert$	$(2+x_2)/3$	1.0	1.0	$(2+x_2)/3$
NVI(1,2,2)	0.205	0.0908	0.144	**0.755**
LAI	0.963	0.959	0.959	**0.974**
Type of nonlinear curves	I, II, III	I, II, III	I, II, III	**I, II, III, IV**

controller. Moreover, this controller is able to generate all four types of simple nonlinear curves.

B. Mizumoto's "Product-Sum-Gravity (PSG)" Scheme

This scheme is proposed by Mizumoto [2] and has also been accepted by many researchers due to its simpler inference results than Zadeh-Mamdani's reasoning scheme. Two controllers are designed using this scheme. They all use triangular membership functions with three rules as (18). The main differences of each controller are on the membership functions of the $\hat{u}_P$.

1) "PSG-I" Controller: This controller applies the membership functions of $\hat{u}_P$ in a range of $\hat{u}_P \in [-1, 1]$ (Fig. 7). The controller does not have the normalized output. The closed-form solution of $\hat{u}_P(\hat{e})$ is given in Appendix D.

2) "PSG-II" Controller: This controller has the membership functions shown as Fig. 8. A normalized output is realized by this change. The closed-form solution of the inference results is discussed in Appendix D.

We observe from Fig. 13 that both controllers receive the identical admissible line for the NVI analysis. This is attributed to that the inference engine is actually governed by a single independent parameter, $z = (1 - x_2)/2x_1$, even though the controllers are designed with two tuning parameters. Both controllers can realize a perfect linear function, or LAI $= 1$ (Table IV). This means that the controllers are considered to be the GPP systems. .

C. "Takagi-Sugeno-Kang (TSK)" Scheme

A fuzzy PID controller can be realized by using the TSK model [11], [12] for the consequent parts. For this scheme, only two rules are used

$$\begin{aligned} &\text{Rule 1:} \quad \text{If } (|\hat{e}| \text{ is BG}), \text{ then } (\hat{u}_P = f_1(\hat{e})) \\ &\text{Rule 2:} \quad \text{If } (|\hat{e}| \text{ is AZ}), \text{ then } (\hat{u}_P = f_2(\hat{e})) \end{aligned} \tag{19}$$

where "BG" stands for "big," and $f_1(\hat{e})$ and $f_2(\hat{e})$ are the given functions. We apply the membership functions in the positive domain of error in Fig. 7 for $|\hat{e}|$. The total number of two tuning parameters is used for the functions. The "gravity" defuzzification is used. Two controllers are designed below.

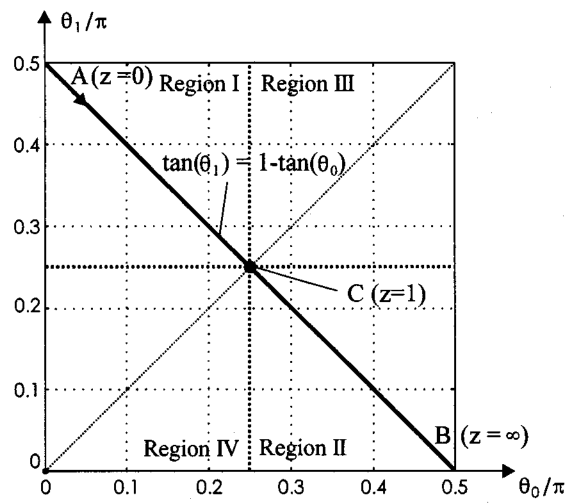

Fig. 13. Admissible line of nonlinearity diagram for "PSG-I" and "PSG-II" controllers. Line A–B: admissible line. C: point of a linear PID, where $z = (1 - x_1)/2x_2 = 1$.

1) "TSK-I" Controller: This controller is designed using the following functions in (19):

$$f_1 = x_1\hat{e}; \quad f_2 = x_2\hat{e}. \tag{20}$$

The closed-form solution of $\hat{u}_P$, (E-1, Appendix E), shows a two-term second-order polynomial function. However, a single-independent-parameter $(z = x_2/x_1)$ is deduced to the two angles. For this reason, an admissible curve is generated which also includes a perfect linear function on point C (Fig. 14).

2) "TSK-II" Controller: In order to preserve the feature of two-independent-parameter inference, we applies the following functions to (19):

$$f_1 = (1 - x_1)\frac{\hat{e}}{|\hat{e}|} + (x_1 + x_2)\hat{e} - x_2\hat{e}^2\frac{\hat{e}}{|\hat{e}|}, \quad f_2 = 0. \tag{21}$$

The closed-form solution of $\hat{u}_P$, (F–1, Appendix F), shows a three-term third-order polynomial function. The nonlinearity diagram is shown in Fig. 15. The admissible area of this controller fully covers the admissible curve produced by "TSK-I" controller. This example shows the significance of proper selections

TABLE IV
COMPARISONS OF FUZZY PID CONTROLLERS USING OTHER SCHEMES. (*R_1–R_4 ARE BOUNDARIES FOR x_i)

Inference and defuzzification	Product-Sum-Gravity scheme		T-S-K scheme	
Controller name	PSG-I	PSG-II	TSK-I	TSK-II
Parameter range for x_i	$(1-x_2)/2\ x_1 \in [0, \infty)$	$(1-x_2)/2\ x_1 \in [0, \infty)$	$x_2/x_1 \in [0,2]$	$^*R_1$- R_4
Range of MFs for $\hat{u}_P$	[-1, 1]	$[-2+x_2, 2-x_2]$	MFs. are not used.	MFs. are not used.
$\max\lvert\hat{u}_P\rvert$	$(2+x_2)/3$	1.0	x_1	1.0
NVI(1,2,2)	Admissible line	Admissible line	Admissible curve	0.695
LAI	1.0	1.0	1.0	1.0
Type of nonlinear curves	I, II	I, II	I, II	I, II, III, IV

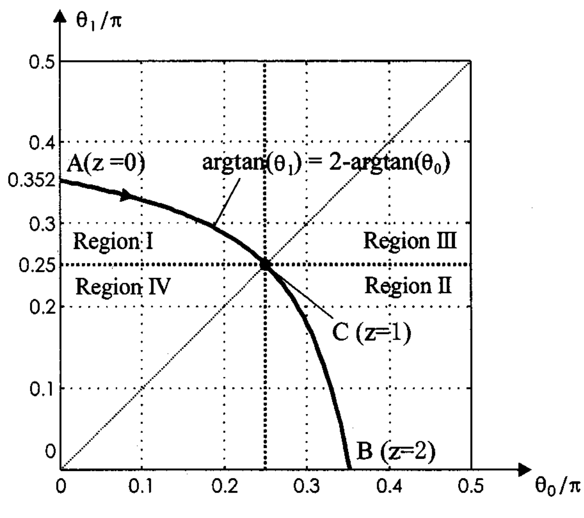

Fig. 14. Admissible curve of nonlinearity diagram for "TSK-I" controller. Curve A-C-B: admissible curve. C: point of a linear PID, where $z = x_2/x_1 = 1$.

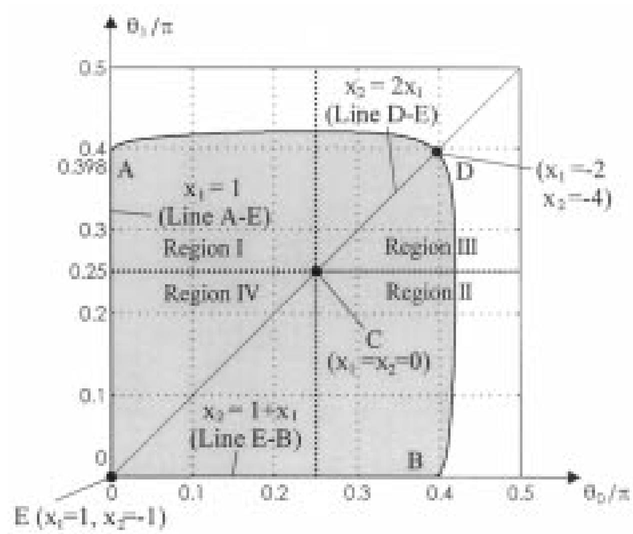

Fig. 15. Admissible area of nonlinearity diagram for "TSK-II" controller. Gray area: admissible area. C: point of a linear PID, where $x_1 = x_2 = 0$.

of the given functions in the TSK scheme. Table IV shows the comparison results when using "PSG" and "TSK" schemes.

X. SUMMARY AND DISCUSSIONS

Although fuzzy control lies to its strength to deal with high-level, task-orient problems, further development has shown that another perspective of fuzzy control technique is for fuzzy PID controllers to evolve into general control elements [34], like the conventional PID regulators applicable for various processes. In this work, we addressed a systematic study of fuzzy PID controllers by using a function-based evaluation approach. Two basic issues, *simplicity* and *nonlinearity*, related to the selection of fuzzy inference structures, reasoning schemes and membership functions, are investigated to demonstrate the applicability of the function-based evaluation approach. For the first issue, we have discussed the basic difference between direct-action type and gain-scheduling type of fuzzy PID controllers in design of nonlinear functions. Due to the simpler features of nonlinear properties of DA-type controllers, we suggest this type of controllers will be a better selection than GS type. Next, five process-independent evaluation criteria are proposed for examining the functional behaviors of DA-type fuzzy PID controllers from the viewpoints of simple operations of gain tuning. These criteria include control-action composition, input coupling, gain dependency, gain-role change, and rule/parameter growth. Using the criteria, we find that the Mamdani's two-input fuzzy PID controllers have lack of many preferred features that usually exist in linear PID controllers. A one-input fuzzy controller which consists of a "one-to-three mapping" fuzzy inference to generate three independent (proportional, integral and derivative) control actions, has shown to be the optimal structure with respect to the five criteria in comparison with two- and three-input fuzzy controllers.

For the second issue, we summarize three concepts in nonlinear approximator evaluation, namely, function-estimation-based approximation; generalization-capability-based approximation; and nonlinearity-variation-based approximation. Comparing with the two former evaluation approaches, we consider the last approach is the most proper to evaluate fuzzy inference schemes, defuzzifications and membership functions. An indepth, nonlinearity evaluation is made on the one-input fuzzy controllers. The NVI and LAI are the basic measures for such evaluations. While the NVI aims to assess a controller in regards to its nonlinearity freedom or limitation, the LAI is used to examine the controller's ability of realizing a perfect linear function which, we consider, is an important property for a controller to achieve a GPP system. Eight controllers are studied when using different inference schemes or membership functions. From the given examples, it is found that the Zadeh-Mamdani's MMG is the best reasoning scheme compared with Mizumoto's PSG and TSK schemes if the nonlinearity variation is a main concern. The advantage in using the NVI and LAI is well demonstrated from the evaluation examples of the controllers. Without requiring any computer simulation or controller testing to a specific process, a designer is able to select or improve the design from the nonlinearity analysis of the controllers.

It has been a great concern of fuzzy control versus conventional control within both control communities [35]–[37]. One of the arguments is related to the performance issue: "*which control technique is better*?" The challenge posed from the conventional control community to the statement that "*fuzzy control outperformances over the conventional control*" does make a good, but critical, point. Control engineering practice seeks the answers to these questions: *Is the statement true for any process*? *If not, what is the condition of realizing the statement*? Although the questions are performance related, a rigorous proof using a performance-based approach will be difficult, if not impossible. However, a function-based evaluation approach may provide an effective solution, since the functionality gap between the two control techniques may explain the reasons of the performance differences. While fuzzy control is successfully used as "*an addition to conventional control*" as stated by Zadeh [37], this technique is also necessarily to be enhanced by augmenting the preferred functionality of conventional control. In this perspective, much work needs to be addressed on the integration of the functionality between the different control techniques.

Finally, we recognize that the present function-based evaluation is not complete for a systematic study of fuzzy controllers. A final selection of the inference structure and reasoning schemes should also be based on a performance-based evaluation. For example, a two-input fuzzy controller may show a sliding-mode controller for robust control [38]. Sometimes, a compromise between function and performance criteria may have to be considered for the final design. The main point raised by authors in this work is to stress the significance of the function-based evaluation for a systematic study of fuzzy controllers. However, much work remains in this regard. For example, a rigorous analysis is needed for an evaluation of nonlinearity variations of fuzzy controllers with any inference structures.

APPENDIX A
CLOSED-FORM SOLUTION OF "MMG-I" CONTROLLER

Case 1 (Nonoverlapping): $x_1 \le x_2$

$$\hat{u}_P = \frac{y_2\hat{e}[3x_2(2-|\hat{e}|)+y_2(3-\hat{e}^2)]}{3[2x_1(1-\hat{e}^2)+y_2(2|\hat{e}|-\hat{e}^2)]}. \tag{A-1}$$

Case 2 (Overlapping): $x_1 > x_2$

Range A: $0 \le |\hat{e}| \le \hat{e}_d$

$$\hat{u}_P = \frac{\hat{e}\left[3\left(1-x_1^2\right)+3x_1^2|\hat{e}|-x_1^2\hat{e}^2\right]}{3[2x_1+2(1-x_1)|\hat{e}|-x_1\hat{e}^2)]}. \tag{A-2}$$

Range B: $\hat{e}_d < |\hat{e}| \le 1-\hat{e}_d$
[see (A-3) at the bottom of page].

Range C: $1-\hat{e}_d < |\hat{e}| \le 1$
[see (A-4) at the bottom of page].

in which the intermediate variables are defined as

$$z_1 = 1-|\hat{e}|, \quad z_2 = 1+|\hat{e}|, \quad z_3 = 1-2|\hat{e}|,$$
$$y_1 = x_1 - x_2, \quad y_2 = 1-x_2, \quad \hat{e}_d = y_1/(1+y_1).$$

The two specific angles, θ_0 and θ_1, are calculated by [see (A-5) and (A-6), at the bottom of the next page]

APPENDIX B
CLOSED-FORM SOLUTION OF "MMG-II" CONTROLLER

Case 1 (Nonoverlapping): $x_1 \le x_2$

$$\hat{u}_P = \frac{\hat{e}(2-|\hat{e}|)y_2}{x_1(1-\hat{e}^2)+y_2(2|\hat{e}|-\hat{e}^2)]}. \tag{B-1}$$

Case 2 (Overlapping): $x_1 > x_2$

Range A: $0 \le |\hat{e}| \le \hat{e}_d$

$$|\hat{u}|_P = \frac{\hat{e}}{3}\left[\frac{\left(y_2^2-x_1^2\right)(3-3|\hat{e}|+\hat{e}^2)+3(1+2y_2-y_2|\hat{e}|)}{2x_1+2(1-x_1+y_2)|\hat{e}|-(x_1+y_2)\hat{e}^2}\right]. \tag{B-2}$$

Range B: $\hat{e}_d < |\hat{e}| \le 1-\hat{e}_d$
[see (B-3) at the bottom of the next page]

Range C: $1-\hat{e}_d < |\hat{e}| \le 1$
[see (B-4), at the bottom of the next page].

$$\hat{u}_p = \frac{\hat{e}\{y_2\hat{e}[6-3|\hat{e}|-y_2(3-3|\hat{e}|+\hat{e}^2)]-y_1\hat{e}_d[3x_1(1-\hat{e}_d)-2\hat{e}_d y_2+y_1]\}}{3|\hat{e}|[2x_1(1-\hat{e}^2)+y_2(2|\hat{e}|-\hat{e}^2)-y_1\hat{e}_d]}. \tag{A-3}$$

$$\hat{u}_p = \frac{\hat{e}\left\{z_1\left[3-x_1^2(1+|\hat{e}|+\hat{e}^2)\right]-y_2z_3[3-y_2(1-|\hat{e}|+\hat{e}^2)]\right\}}{3|\hat{e}|\left[z_1(2+x_1z_2)-y_2z_3\right]}. \tag{A-4}$$

The intermediate variables are the same as those in Appendix A. The two specific angles, θ_0 and θ_1, are calculated by

$$\tan(\theta_0) = \frac{\frac{\partial \hat{u}_P}{\partial \hat{e}}\big|_{\hat{e}=0}}{\max(\hat{u}_P)} = \begin{cases} \dfrac{2(1-x_2)}{x_1} & x_1 \le x_2 \\ \dfrac{(2-x_1-x_2)(2+x_1-x_2)}{2x_1} & x_1 > x_2 \end{cases} \quad \text{(B-5)}$$

and

$$\tan(\theta_1) = \frac{\frac{\partial \hat{u}_P}{\partial \hat{e}}\big|_{\hat{e}=1}}{\max(\hat{u}_P)} = \begin{cases} \dfrac{2x_1}{1-x_2} & x_1 \le x_2 \\ \dfrac{(x_1+x_2)(2+x_1-x_2)}{2(1-x_2)} & x_1 > x_2 \end{cases} \quad \text{(B-6)}$$

APPENDIX C
CLOSED-FORM SOLUTIONS OF "MMG-III" AND "MMG-IV" CONTROLLERS

The closed-form solutions of "MMG-III" and "MMG-IV" controllers [see (C-1) and (C-2), at the bottom of page] are given respectively by where the extended ranges for using (B-2) to (B-4) and (A-2) to (A-4) are given by

Range A: $[(x_1 - x_2) \le 1$ AND $0 \le \hat{e} \le |\hat{e}_d]$ OR $[(x_1 - x_2) > 1$ AND $0 \le |\hat{e}| \le 0.5]$
Range B: $[(x_1 - x_2) \le 1$ AND $\hat{e}_d < |\hat{e}| \le 1 - \hat{e}_d]$
Range C: $[(x_1 - x_2) \le 1$ AND $1 - \hat{e}_d < |\hat{e}| \le 1]$ OR $[(x_1 - x_2) > 1$ AND $0.5 < |\hat{e}| \le 1]$

APPENDIX D
CLOSED-FORM SOLUTIONS OF "PSG-I" AND "PSG-II" CONTROLLERS

The fuzzy proportional actions of "PSG-I" and "PSG-II" controllers are

$$\text{``PSG-I''} \quad \hat{u}_P = \frac{\hat{e}}{3}\frac{2-x_1-x_2^2}{(1-2x_1-x_2)|\hat{e}|+2x_1}. \quad \text{(D-1)}$$

and

$$\text{``PSG-II''} \quad \hat{u}_P = \frac{(1-x_2)\hat{e}}{(1-2x_1-x_2)|\hat{e}|+2x_1}. \quad \text{(D-2)}$$

Both controllers have the same equations for θ_0 and θ_1:

$$\tan(\theta_0) = \frac{\frac{\partial \hat{u}_P}{\partial \hat{e}}|_{\hat{e}=0}}{\max(\hat{u}_P)} = \frac{1-x_2}{2x_1} \quad \text{(D-3)}$$

and

$$\tan(\theta_1) = \frac{\frac{\partial \hat{u}_P}{\partial \hat{e}}|_{\hat{e}=1}}{\max(\hat{u}_P)} = \frac{2x_1}{1-x_2} \quad \text{(D-4)}$$

$$\tan(\theta_0) = \frac{\frac{\partial \hat{u}_P}{\partial \hat{e}}\big|_{\hat{e}=0}}{\max(\hat{u}_P)} = \begin{cases} \dfrac{3\left(1-x_2^2\right)}{2x_1(2+x_2)} & x_1 \le x_2 \\ \dfrac{3\left(1-x_1^2\right)}{2x_1(2+x_2)} & x_1 > x_2 \end{cases} \quad \text{(A-5)}$$

and

$$\tan(\theta_1) = \frac{\frac{\partial \hat{u}_P}{\partial \hat{e}}\big|_{\hat{e}=1}}{\max(\hat{u}_P)} = \begin{cases} \dfrac{4x_1}{1-x_2} & x_1 \le x_2 \\ \dfrac{(x_1+x_2)(3x_1-x_2+4)}{(1-x_2)(2+x_2)} & x_1 > x_2. \end{cases} \quad \text{(A-6)}$$

$$\hat{u}_P = \frac{\hat{e}}{3|\hat{e}|}\left[\frac{6y_2(2|\hat{e}|-\hat{e}^2) - y_1\hat{e}_d[3x_1+y_1-\hat{e}_d(3x_1+2y_2)]}{2x_1(1-\hat{e}^2)+2y_2(2|\hat{e}|-\hat{e}^2)-y_1\hat{e}_d}\right]. \quad \text{(B-3)}$$

$$\hat{u}_P = \frac{\hat{e}}{3|\hat{e}|}\left[\frac{\left(y_2^2-x_1^2\right)(1-|\hat{e}|^3)+3(1-|\hat{e}|)-3y_2(1-4|\hat{e}|+\hat{e}^2)}{x_1(1-\hat{e}^2)-y_2(1-4|\hat{e}|+\hat{e}^2)+2(1-|\hat{e}|)}\right]. \quad \text{(B-4)}$$

``MMG-III" (B-1) to (B-6), subject to the extended ranges (C-1)

and

``MMG-IV" (A-1) to (A-6), subject to the extended ranges (C-2)

APPENDIX E
CLOSED-FORM SOLUTION OF "TSK-I" CONTROLLER

The fuzzy proportional actions of this controllers is

$$\hat{u}_P = \hat{e}[x_2 + (x_1 - x_2)|\hat{e}|]. \tag{E-1}$$

Two angles, θ_0 and θ_1, are calculated by

$$\tan(\theta_0) = \frac{\frac{\partial \hat{u}_P}{\partial \hat{e}}|_{\hat{e}=0}}{\max(\hat{u}_P)} = \frac{x_2}{x_1} \tag{E-2}$$

and

$$\tan(\theta_1) = \frac{\frac{\partial \hat{u}_P}{\partial \hat{e}}|_{\hat{e}=1}}{\max(\hat{u}_P)} = 2 - \frac{x_2}{x_1} \tag{E-3}$$

APPENDIX F
CLOSED-FORM SOLUTION OF "TSK-II" CONTROLLER

The fuzzy proportional actions of this controllers is:

$$\hat{u}_P = \hat{e}[1 - x_1 + (x_1 + x_2)|\hat{e}| - x_2\hat{e}^2]. \tag{F-1}$$

Two angles, θ_0 and θ_1, are calculated by:

$$\tan(\theta_0) = \frac{\frac{\partial \hat{u}_P}{\partial \hat{e}}|_{\hat{e}=0}}{\max(\hat{u}_P)} = 1 - x_1 \tag{F-2}$$

and

$$\tan(\theta_1) = \frac{\frac{\partial \hat{u}_P}{\partial \hat{e}}|_{\hat{e}=1}}{\max(\hat{u}_P)} = 1 + x_1 - x_2. \tag{F-3}$$

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Bao-Gang Hu (M'94–SM'99) received the M.S. degree from the University of Science and Technology, Beijing, China , and the Ph.D. degree from McMaster University, Ontario, Canada, both in Mechanical Engineering, in 1983 and 1993, respectively.

From 1983 to 1987, he was a Lecturer with the University of Science and Technology, Beijing, China. In 1993, he was with NEFAB Inc., Canada. From 1994 to 1997, he was a Research Engineer and Senior Research Engineer at C-CORE, Memorial University of Newfoundland, Canada. He is presently a Professor with National Laboratory of Pattern Recognition, Institute of Automation, Chinese Academy of Sciences, Beijing, China. Since 2000, he has served as the Chinese Director of the Chinese-French Joint Laboratory for Computer Science, Control and Applied Mathematics (**LIAMA**). His current research interests are in the areas of intelligent systems, pattern recognition, and plant growth modeling.

George K. I. Mann received the B.Sc. (honors) degree in engineering from the University of Moratuwa, Sri Lanka. He received the M.Sc. degree in computer integrated manufacture from Loughborough University of Technology, U.K. and the Ph.D. degree from Memorial University of Newfoundland, St. James, Canada, in 1989 and 1999, respectively.

From 1999 to 2001, he was a Research Engineer with the Intelligent Systems group at C-CORE, Memorial University of Newfoundland. Recently, he joined the Mechanical Engineering Department at Queen's University, Canada as recipient of the prestigious NSERC Postdoctoral Fellowship Award. Prior to this position, he was a Lecturer with the Department of Mechanical Engineering, University of Moratuwa, Sri Lanka. His research interests include classical/intelligent control, machine vision, and automation.

Raymond G. Gosine received the B.Eng. degree in electrical engineering from Memorial University of Newfoundland, Canada, and the Ph.D. degree in engineering from the University of Cambridge, U.K., in 1986 and 1990, respectively.

From 1988 to 1991, he was a Research Fellow at Selwyn College, University of Cambridge. From 1991 to 1993, he was an Assistant Professor of Mechanical Engineering and NSERC and Junior Chair in Industrial Automation at the University of British Columbia, Vancouver, BC, Canada. Since 1994, he has been a faculty member with Memorial University of Newfoundland, where he is Professor of Electrical and Computer Engineering. He also holds the J. I. Clark Chair in Intelligent Systems, which is supported by C-CORE and INCO. His research interests include industrial automation, telerobotics, and machine vision.