The Exact and Unique Solution for Phase-Lead and Phase-Lag Compensation

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***Abstract*—Phase-lead and -lag compensation is one of the most commonly used techniques for designing control systems in the frequency domain, especially when the Bode diagram or root locus is used. In most cases, the graphic-based approximation or trial-and-error approach has been utilized in the design process. This paper presents the exact and unique solution to the design of phase-lead and phase-lag compensation when the desired gains in the magnitude and phase are known at a given frequency. It also gives the concise condition for determining the existence of single-stage lead or lag compensation.**

***Index Terms*—Bode diagram, control systems, frequency design methods, lag compensation, lead compensation, phase compensation.**

I. Introduction

In spite of the advances in modern control system design techniques, such as state-based optimal and frequency-based H^∞ controls, much design, especially industrial design is still conducted using classical frequency-domain procedures. Because of its simplicity and easy implementation, cascade or series compensation is the most popular method in the frequency-domain design of feedback control systems. This method is particularly valuable when the plant to be controlled is unknown and only experimental data are available. Two of the most commonly used series compensation strategies are PID (proportional-integral-derivative) control and phase lead/lag compensation. Since cascade compensation was first introduced, the determination of the lead or lag compensator has been taught as a trial-and-error procedure based on graphic and approximated information. Usually, an appropriate compensator comes only after trial and error.

In 1976, Wakeland [1] found an analytical solution to the design of single-stage phase-lead compensation. His solution is of the quadratic form in terms of compensation gain. In the following year, Mitchell [2] improved Wakeland's solution and pointed out that the improved solution can also be used to solve the design problem of phase-lag compensation.

In this paper, we present a simple and unique solution to both phase lead/lag compensation problems. The solution has been discovered independently of both Wakeland and Mitchell's work. Its procedure is much simpler and uniform for both lead and lag design. It should be useful for developing analytical procedures for other frequency-based techniques, especially optimal designs. An extensive search of the relevant literature has not found similar results (see major control textbooks [3]–[9]).

II. The Solution

Assume that a single-stage compensator is expressed as [3],

$$G_c(s) = \frac{1+\alpha\tau s}{1+\tau s} \tag{1}$$

where $\alpha < 1$ yields a phase-lag compensator, and $\alpha > 1$ yields a phase-lead compensator.

Let M (in dB) and p (in rad) be the desired gain in the magnitude and phase, to be contributed by G_c, at a given crossover frequency (or any given frequency) ω_c. Then

$$|G_c(j\omega_c)| = 10^{M/20} \equiv c, \quad \angle G_c(j\omega_c) = p \tag{2}$$

which leads to

$$\frac{1+(\alpha\tau\omega_c)^2}{1+(\tau\omega_c)^2} = c^2, \quad \tan^{-1}(\alpha\tau\omega_c) - \tan^{-1}(\tau\omega_c) = p. \tag{3}$$

Clearly, $-\pi/2 \leqq p \leqq \pi/2$ can be assumed here. Introducing new variable and parameter

$$\sigma = \alpha\tau\omega_c, \quad \delta = \tan(p). \tag{4}$$

Then from (3) and

$$\tan(x-y) = \frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$$

the following two equations are found in terms α of and σ

$$\frac{1+\sigma^2}{1+(\frac{\sigma}{\alpha})^2} = c^2, \quad \frac{\sigma}{\alpha} = \frac{\sigma-\delta}{1+\delta\sigma}. \tag{5}$$

The second equation in (5) is obtained by taking the tangent of both sides of the second equation in (3). Eliminating σ/α from the first equation using the second equation, one gets an equation expressed in terms of the unknown variable σ only

$$(1+\sigma^2)(1+\delta\sigma)^2 = c^2\left\lfloor(1+\delta\sigma)^2 + (\sigma-\delta)^2\right\rfloor.$$

The above equation can be further simplified

$$(1+\sigma^2)\left\lfloor(1+\delta\sigma)^2 - c^2(1+\delta^2)\right\rfloor = 0. \tag{6}$$

Since α, τ, and ω_c are all real and positive (as required by a phase-lag or phase-lead compensator), σ is always real and positive, therefore, the only possible valid solution to (6) is

$$\sigma = \frac{\pm c\sqrt{1+\delta^2}-1}{\delta}. \tag{7}$$

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The above equation presents two possible solutions for σ (+ sign and − sign, respectively). Now one can prove that there is only one possible solution, i.e., only + sign is allowed in (7).

From the definitions of phase-lag ($\alpha < 1$) or phase–lead ($\alpha > 1$) compensation and the first equation in (3), one can show that a phase-lead or phase-lag compensator must satisfy the following conditions.

A. Phase-Lead Compensation

Since

$$1 < c^2 = \alpha^2 - \frac{\alpha^2 - 1}{1 + (\tau\omega_c)^2} < \alpha^2, \quad 0 < p < \frac{\pi}{2}$$

therefore

$$\alpha > c > 1, \delta > 0$$

for phase-lead compensation.

B. Phase-Lag Compensation

Since

$$1 > c^2 = \alpha^2 + \frac{1 - \alpha^2}{1 + (\tau\omega_c)^2} > \alpha^2, -\frac{\pi}{2} < p < 0$$

therefore

$$\alpha < c < 1, \delta < 0$$

for phase-lag compensation.

Thus, only the positive sign is allowed for phase-lead compensation in (7) (otherwise there would be a negative value for σ). In the case of phase-lag compensation, the negative sign in (7) seems to be valid, however, a further analysis using (5) and (7) shows that the negative sign will lead to the following conclusions

$$\sigma - \delta = -\,\delta\left[\frac{c\sqrt{1+\delta^2}+1}{\delta^2} + 1\right] > 0$$
$$\alpha = \frac{c^2(1+\delta^2) + c\sqrt{1+\delta^2}}{\delta(\sigma - \delta)} < 0$$

which is invalid since α must be positive. Therefore, only the positive sign is valid in (7) for both phase-lead and phase-lag compensation. Hence, from the second equation of (5) and the first equation in definition (4), one has

$$\alpha = \frac{c\left(c\sqrt{1+\delta^2} - 1\right)}{c - \sqrt{1+\delta^2}}, \quad \tau = \frac{c - \sqrt{1+\delta^2}}{c\delta\omega_c} \tag{8}$$

for both phase-lead and phase-lag compensators.

The equations in (8) present the only possible solutions to phase compensation; those solutions are unique, but may not be valid. However, based on (8), it is easy to show the following theorem.

The Lead-Lag Compensation Theorem:

a) A single-stage phase-lead compensation exists if and only if

$$\sqrt{1+\delta^2} < c \text{ and } \delta > 0. \tag{9}$$

b) A single-stage phase-lag compensation exists if and only if

$$\sqrt{1+\delta^2} < \frac{1}{c} \text{ and } \delta < 0. \tag{10}$$

In both cases, the compensation solution is unique and given by (8).

Proof: Clearly, for any $c > 0$ and $\delta \neq 0$, one has

$$\left(\sqrt{1+\delta^2} - 1\right)c > 0 > \left(1 - \sqrt{1+\delta^2}\right)$$

and the first equation of (8) can be rewritten as

$$\alpha = c\left[1 + \frac{(c+1)(\sqrt{1+\delta^2} - 1)}{c - \sqrt{1+\delta^2}}\right].$$

Then when (9) is true, one finds $c > 1$, and $\alpha > c > 1$, and a phase-lead compensation exists and is unique. It is easy to show from (8) that, if a phase-lead compensation exists, then condition (9) must be true since $c > 1$ and α must be positive.

Similarly, when (10) is true, one has $c < 1$, and $\alpha < c < 1$ from the two inequalities above, and a phase-lag compensation exists and is unique. One can easily show from (8) that if a phase-lag compensation exists then condition (10) must be true since $c < 1$ and α must be positive.

Thus, the proof is completed for the lead-lag compensation theorem.

Note that none of the major textbooks in control theory (see [3]–[10] for example), nor any of the literature on compensator design known to the author, explicitly mentioned condition (9) or (10) as the necessary and sufficient conditions for lead or lag solutions. However, Mitchell [2] did mention the two conditions as the sufficient conditions.

Figs. 1 and 2 show the two conditions in normal and logarithmic scales, respectively. The absolute value of the required phase contribution (p) is used in both figures. Note that phase lead and phase lag are symmetric in decibel.

III. Relation to Wakeland Solution

In 1976, Wakeland [1] solved the compensation problem for lead networks in the following form:

$$G_c(s) = \frac{1 + \tau' s}{1 + \frac{\tau' s}{\alpha'}}.$$

For achieving a gain (dB) and phase contribution (Φ) at the desired frequency ($\omega = \omega_c$), Wakeland's solution is

$$(p'^2 - c' + 1)\alpha'^2 + 2p'^2 c'\alpha' + p'^2 c'^2 + c'^2 - c' = 0 \tag{11}$$

where

$$p' = \tan\Phi = \frac{\omega\tau' - \left(\frac{\omega\tau'}{\alpha'}\right)}{1 + \frac{(\omega\tau')^2}{\alpha'}},$$
$$dB = 10\log\left[\frac{1 + (\omega\tau')^2}{1 + \left(\frac{\omega\tau'}{\alpha'}\right)^2}\right]; \text{ or}$$
$$c' = 10^{dB/10} = \frac{(1 + (\omega\tau')^2)}{\left(1 + \left(\frac{\omega\tau'}{\alpha'}\right)^2\right)}.$$

Wakeland did not go further with (11), but one can show that it will lead to the results given in (8), although the derivation is different and simpler than Wakeland's. To show this fact, first, in terms of the notation defined in this paper

$$\alpha' = \alpha, \tau' = \alpha\tau, p' = \delta, c' = c^2.$$

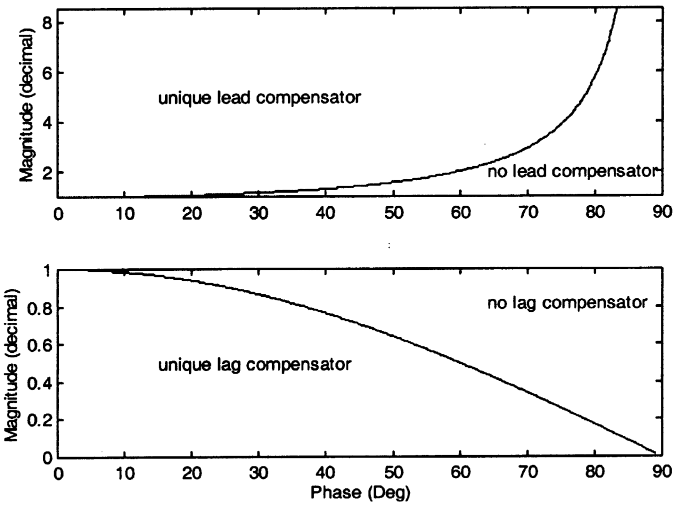

Fig. 1. Admissible magnitude-phase relationship in decimal.

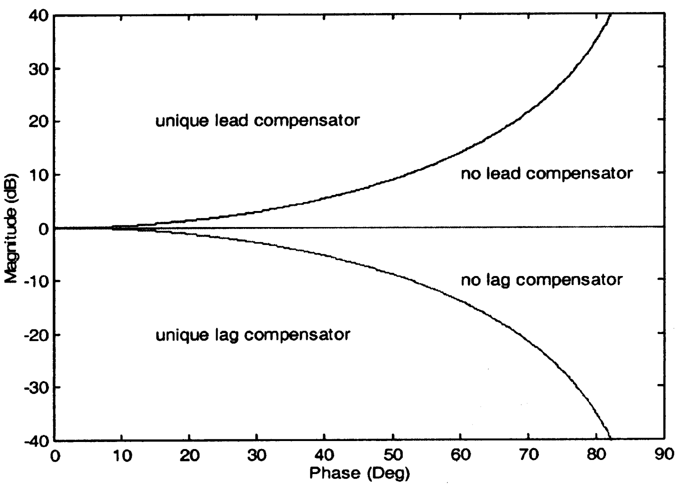

Fig. 2. Admissible magnitude-phase relationship in decibel.

Equation (11) becomes

$$(\delta^2 - c^2 + 1)\alpha^2 + 2\delta^2 c^2 \alpha + \delta^2 c^4 + c^4 - c^2 = 0. \qquad (12)$$

After some manipulations, its determinant can be found as

$$\Delta = 4\delta^4 c^4 - 4(\delta^2 - c^2 + 1)(\delta^2 c^4 + c^4 - c^2) = 4c^2(1+\delta^2)(c^2-1)^2$$

therefore, the two solutions of (12) are

$$\alpha_{1,2} = \frac{-\delta^2 c^2 \pm c\sqrt{1+\delta^2}\,\left|c^2-1\right|}{\delta^2+1-c}$$

and thus for $c > 1$

$$\alpha_{1,2} = \frac{\pm c\sqrt{1+\delta^2}(c^2-1) - c^2\delta^2}{\delta^2+1-c}.$$

One can show easily that the positive sign will lead to a negative α, thus invalid since α must be positive, while the minus sign leads to (8).

For $c < 1$

$$\alpha_{1,2} = \frac{\pm c\sqrt{1+\delta^2}(1-c^2) - c^2\delta^2}{\delta^2+1-c}.$$

In this case, the positive sign leads to (8), while the minus sign leads to a negative α, hence, an invalid solution.

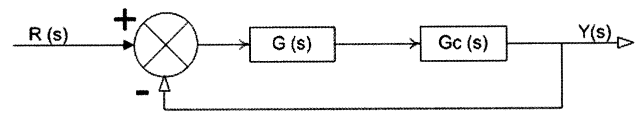

Fig. 3. A position control system in [10].

Therefore, the author has demonstrated that the only valid design of compensation from Wakeland's equation is the unique solution given in (8). Wakeland's solution is just one step away from the author's solution. However, the derivation here is more straightforward and simple than Wakeland's.

IV. Numerical Design Examples

Fig. 3 shows a simplified position control system used in design examples for lag and lead compensation in a control text book [10], where $G(s) = 100K/[s(s+36)(s+100)]$ is the uncompensated system, and $G_c(s)$ is the compensator to be designed. (For details, see [10, Fig. 11.2 and pp. 687 to 701].)

For phase-lag compensation, a four-step iterative design procedure, based on the Bode diagram, is outlined in [10]. In [10, Example 11.2], $K = 583.9$ and the required phase margin $PM = 59.2°$ for a 9.5% overshoot. Starting with $PM = 69.2°$ (10° is added to compensate for the phase angle contribution of the lag compensator, guesswork) and $\omega_c = 9.8$ rad/s, the iterative procedure leads to the following final solution:

$$G_c(s) = 0.063\frac{s+0.98}{s+0.062} = \frac{1+0.063\times 16.129s}{1+16.129s}$$

which results in a $PM = 62°$ and $\omega_c = 11$ rad/s and a response of 10% overshoot and 0.25 s peak time in the time domain.

To use the results given in this paper, one first finds all possible ω_c under a given PM for lag compensation using the lead-lag compensation theorem in Section II. As shown in Fig. 4(a) (p versus ω_c) and Fig. 4(b) ($c\sqrt{1+\delta^2}$ vs. ω_c), ω_c must be less than 19.795 rad/s for $PM = 50°$ and less than 9.395 rad/s for $PM = 70°$. (In this example, Fig. 4(a) has to be used to determine the admissible ω_c.)

For $PM = 69.2°$ and $\omega_c = 9.8$ rad/s, there is no lag compensator since $p = 0.0253° > 0$ in this case (therefore, strictly speaking, the starting point in [10, Example 11.2] is incorrect; ω_c must be less than 9.78 rad/s in this case). For any ω_c with $p < 0$ and $c\sqrt{1+\delta^2} < 1$, a lag compensator can be found easily using (8). For example, when $PM = 59.2°$ and $\omega_c = 9.8$ rad/s, one has

$$G_{c1}(s) = 0.0618\frac{s+1.8412}{s+0.1139} = \frac{1+0.0618\times 8.7826s}{1+8.7826s}$$

resulting in a 15% overshoot and 0.30 s peak time for the compensated system. When $PM = 62°$ and $\omega_c = 11$ rad/s, one has

$$G_{c2}(s) = 0.0711\frac{s+0.9807}{s+0.0697} = \frac{1+0.0711\times 14.3472s}{1+14.3472s}$$

resulting in a 9.8% overshoot and 0.26 s peak time. Note that $G_c(s)$ and $G_{c2}(s)$ should be identical. The difference is a result of the graphic approximation introduced in the iterative design procedure based on the Bode diagram.

For phase-lead compensation, a twelve-step iterative design procedure based on the Bode diagram is outlined in [10]. In

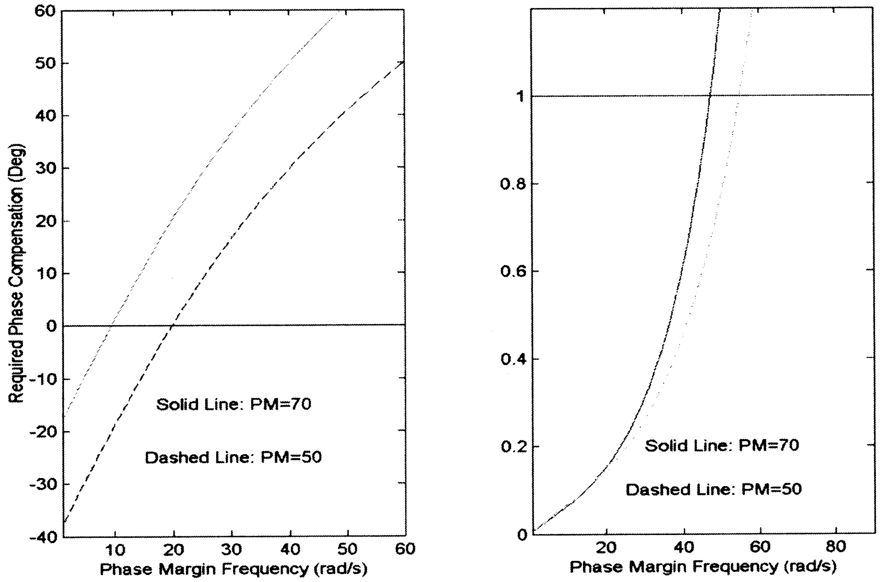

Fig. 4. Admissible phase margin frequency for [10, Example 11.2].

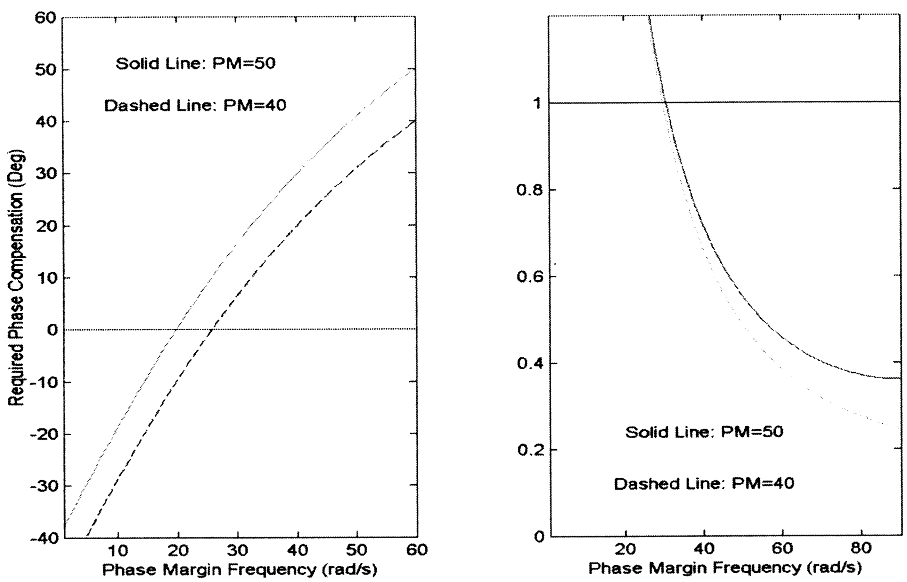

Fig. 5. Admissible phase margin frequency for [10, Example 11.3].

[10, Example 11.3], $K = 1440$, and the required phase margin $PM = 48.1^\circ$ for a 20% overshoot. Starting with $PM = 58.1^\circ$ (10° is added to compensate for the phase angle contribution of the lead compensator, guesswork again) and $\omega_c = 29.7$ rad/s, the iterative procedure leads to the following final solution:

$$G_c(s) = 2.38\frac{s + 25.3}{s + 60.2} = \frac{1 + 2.38 \times 0.0166s}{1 + 0.0166s}$$

which results in a $PM = 45.5^\circ$ and $\omega_c = 39$ rad/s, and a response of 21% overshoot and 0.075 s peak time in the time domain.

As shown in Fig. 5(a) (p versus ω_c) and Fig. 5(b) ($\sqrt{1+\delta^2}/c$ vs. ω_c), ω_c must be larger than 29.743 rad/s for $PM = 40^\circ$ and larger than 30.557 rad/s for $PM = 50^\circ$. (In this example, Fig. 5(b) has to be used to determine the admissible ω_c.) For $PM = 58.1^\circ$ and $\omega_c = 29.7$ rad/s, there is no lead compensator since $\sqrt{1+\delta^2}/c = 1.0915 > 1$ (ω_c must be larger than 32.086 rad/s in this case). For any ω_c with $p > 0$ and $\sqrt{1+\delta^2}/c < 1$, a lead compensator can be found easily using (8). For example, when $PM = 45.5^\circ$ and $\omega_c = 39$ rad/s, one has

$$G_{c3}(s) = 2.3799\frac{s + 25.2720}{s + 60.1458} = \frac{1 + 2.3799 \times 0.0166s}{1 + 0.0166s}$$

resulting in a 22.6% overshoot and 0.072 s peak time for the compensated system. Note that $G_c(s)$ and $G_{c3}(s)$ should be identical. As for Example 11.2, the difference is the result of the graphic approximation in the iterative design procedure.

The advantage of using the current method is obvious, no guesswork is needed; and all possible solutions can be obtained analytically.

V. Conclusion

This paper presents a one-step analytic and unique solution to the design of phase-lead and phase-lag compensation when the desired gains in the magnitude and phase are known at a given frequency. A simple condition for determining the existence and uniqueness of single-stage lead or lag compensation has also been found. Therefore, no trial-and-error or other guesswork is needed. Note that in most frequency-based designs, gain and phase margin as criteria for performance are only approximate; therefore, it seems that a unique solution is not all that meaningful, other than making the teaching of lag/lead compensation possibly a little more straightforward and appealing. However, the uniqueness is very useful in a computer-aided design process for control systems. The solution should also be useful for developing analytical procedures for other frequency-based techniques, especially optimal designs [11].

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