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Localization of dual periodic scaling and wavelet functions

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In 1996, we constructed periodic interpolatory scaling functions φ_j, wavelet functions L_j and their dual basis $\tilde{\varphi}_j$ and $\widetilde{L}_j$ with properties such as symmetry, biorthogonality, any order of smoothness, real-valuedness, explicit expressions and interpolatory. We proved the localization of φ_j in 1997, and in 1998 with Li proved the localization of L_j. In this paper we shall give a detailed proof of the localization for the dual functions $\tilde{\varphi}_j$ and $\widetilde{L}_j$.

Keywords: localization, dual function, periodic wavelet

1. Introduction

Periodic wavelets were first studied in the book of Meyer [22] and Daubechies [14]. Subsequently, many researchers contributed to the development of this subject, which we have documented in our reference list.

It is well known that we cannot have a univariate wavelet which is simultaneously orthogonal, compactly supported, symmetric and continuous. To overcome this difficulty, some efforts have been devoted to constructing multiwavelets [24,34,35] or wavelets with dilation m, $m > 2$ [23,36]. These wavelets usually increase the computational cost or lack other desirable properties. For this reason, our interest turned to periodic wavelets.

In 1996, we constructed periodic interpolatory wavelets [12] and proved that they have the following properties: explicit representation, symmetry, any order of smoothness, biorthogonality, real-valued and interpolatory. In 1997, we proved the localization of the periodic scaling function by two different approaches [11]. Following the method in [11], in 1998, we proved the localization of the periodic interpolating wavelet [19]. In this paper, we shall give a detailed proof of the localization of the dual functions.

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We consider the localization of function in one of the following two ways:

$D1$. The functions f_j decays exponentially in one period of f_j as j tends to infinity, where j is the level of scaling.

$D2$. The circular variance of f_j, denoted by $Var(f_j)$, tends to zero as j tends to infinity.

Recall that $Var(f)$ is defined as follows: For a T-periodic continuous differentiable function f whose L^2 norm is one, we set

$$\tau(f) := \int_0^T e^{i(2\pi/T)t} \left| f(t) \right|^2 dt$$

and

$$Var(f) := 1 - \left| \tau(f) \right|.$$

The size of $1 - |\tau(f)|$ is a good measure of how localized $|f|^2$ is about $\tau(f)$. For example, if $|f(t)|^2$ approaches a point mass located at t_0, then $\tau(f)$ approaches e^{it_0}, and $1 - |\tau(f)|$ approaches zero. Conversely, if $1 - |\tau(f)| = 0$, then $|f|^2$ is the distribution corresponding to a point mass located at $\tau(f)$ (see [5,19,25]). In this paper we prove the localization of the dual functions in the sense of $D2$. In the next section, we review their construction.

2. The properties of $\tilde{\varphi}_j$

Before we construct the PISF we shall first recall the definition of the generators for periodic multi-resolution analysis. Let j be a nonnegative integer, k a positive integer, and $k_j = 2^j k$, $h_j = 2\pi/k_j$. Let G be a 2π-periodic, continuous differentiable function whose Fourier coefficients are positive, i.e.,

$$G(x) = \sum_{n \in \mathbb{Z}} d_n e^{inx}, \quad d_n > 0, \ \forall n \in \mathbb{Z}.$$

For $l = 0, 1, \ldots, k_j - 1$, we define

$$\mathcal{C}_l^j(x) := \sum_{k=0}^{k_j - 1} G(x - kh_j) e^{iklh_j},$$

from which it follows that

$$\mathcal{C}_l^j(x) = k_j \sum_{n \in \mathbb{Z}} d_{nk_j - l} e^{i(nk_j - l)x}.$$

Let

$$\widetilde{\mathcal{C}}_l^j(x) := \frac{\mathcal{C}_l^j(x)}{\|\mathcal{C}_l^j\|},$$

where we use the norm $\|f\|^2 = (1/2\pi)\int_0^{2\pi} |f(x)|^2\,\mathrm{d}x$ for a periodic function $f \in L^2([0, 2\pi])$. Since $\mathcal{C}_l^j(0) > 0$, we can define the following periodic cardinal interpolatory scaling function (PISF)

$$\varphi_j(x) := \frac{1}{k_j}\sum_{l=0}^{k_j-1}\frac{\mathcal{C}_l^j(x)}{\mathcal{C}_l^j(0)}, \quad j > 0,\ j \in \mathbb{Z}.$$

The dual scaling function is defined by the equation

$$\tilde{\varphi}_j(x) := \sum_{\nu=0}^{k_j-1} c_j^\nu \varphi_j(x - \nu h_j), \tag{1}$$

where

$$c_j^\nu = \sum_{l=0}^{k_j-1}(a_{j,l})^{-1}\omega^{-\nu l} \tag{2}$$

and

$$a_{j,l} = \frac{(\sum_{n\in\mathbb{Z}} d_{nk_j-l}^2)}{(\sum_{m\in\mathbb{Z}} d_{mk_j-l})^2}. \tag{3}$$

Using the methods introduced in [5, p. 172] it follows that

$$\langle\tilde{\varphi}_j(\cdot - kh_j),\ \varphi(\cdot - lh_j)\rangle = \delta_{k,l}$$

for $k, l = 0, 1, \ldots, k_j - 1$.

We begin by studying the localization of $\tilde{\varphi}_j$.

Theorem 2.1. Suppose that

$$\left\{\frac{d_n}{d_{n+1}} - 1\colon\ n \in \mathbb{Z}\right\} \in l^2 \tag{4}$$

and there is a positive constant c such that

$$\inf\left\{\frac{d_r}{s_r^j}\colon\ |r| \leqslant k_{j-1},\ j \in \mathbb{Z}_+\right\} \geqslant c, \tag{5}$$

where $s_r^j := \sum_{n\in\mathbb{Z}} d_{nk_j-r}$, $j \in \mathbb{Z}_+$, then we have that

$$Var(\tilde{\varphi}_j) = \mathrm{O}\left(\frac{1}{\sqrt{k_j}}\right), \quad j \to \infty.$$

For the proof of theorem 2.1, we have to establish some lemmas.

Lemma 2.1. If $\tilde{\xi}_j$ is defined as

$$\tilde{\xi}_j := \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}x}\left|\tilde{\varphi}_j(x)\right|^2 \mathrm{d}x,$$

then

$$\tilde{\xi}_j = \sum_{\mu=0}^{k_j-1} \frac{s_\mu^j s_{\mu+1}^j v_{\mu,\mu+1}^j}{q_\mu^j q_{\mu+1}^j},$$

where

$$s_r^j = \sum_{n\in\mathbb{Z}} d_{nk_j-r}, \qquad q_r^j = \sum_{n\in\mathbb{Z}} d_{nk_j-r}^2, \qquad v_{r,s}^j = \sum_{n\in\mathbb{Z}} d_{nk_j-r} d_{nk_j-s}. \tag{6}$$

Proof. The proof follows by a computation with equations (1)–(3). We leave the details to the reader. □

Lemma 2.2. If $\tilde{\eta}_j$ is defined as

$$\tilde{\eta}_j := \|\tilde{\varphi}_j\|^2$$

then

$$\tilde{\eta}_j = \sum_{r=0}^{k_j-1} \frac{(s_r^j)^2}{q_r^j}.$$

Proof. From (1), we have that

$$\tilde{\eta}_j = \left(\sum_{\nu,l=0}^{k_j-1} c_j^\nu \overline{c_j^l}\right) I, \tag{7}$$

where

$$I := \frac{1}{2\pi}\int_0^{2\pi} \varphi(x-\nu h_j)\overline{\varphi_j(x-lh_j)}\,\mathrm{d}x.$$

A direct calculation leads to the formula

$$I = \sum_{\lambda,\mu=0}^{k_j-1} \frac{\mathrm{e}^{\mathrm{i}(\lambda\nu-\mu l)h_j}}{\mathcal{C}_\lambda^j(0)\mathcal{C}_\mu^j(0)} \sum_{m,n\in\mathbb{Z}} d_{nk_j-\lambda} d_{mk_j-\mu} \delta_{n,m}\delta_{\mu,\lambda},$$

that is,

$$I = \sum_{\lambda=0}^{k_j-1} \left(\frac{1}{\mathcal{C}_\lambda^j(0)}\right)^2 \mathrm{e}^{\mathrm{i}\lambda(\nu-l)h_j} q_\lambda^j. \tag{8}$$

Now, we substitute (8) into (7) to obtain the desired result. □

Proof of theorem 2.1. By the definition of variance, we conclude that $Var(\tilde{\varphi}_j) = (1/\tilde{\eta}_j)(\tilde{\eta}_j - \tilde{\xi}_j)$. From lemmas 2.1 and 2.2, we obtain that

$$Var(\tilde{\varphi}_j) = \frac{1}{\tilde{\eta}_j}\sum_{r=0}^{k_j-1}\frac{(s_r^j)^2}{q_r^j q_{r+1}^j}\left(q_{r+1}^j - \frac{s_{r+1}^j v_{r,r+1}^j}{s_r^j}\right)$$
$$= \frac{1}{\tilde{\eta}_j}\sum_{r=0}^{k_j-1}\frac{(s_r^j)^2}{q_r^j q_{r+1}^j}\left[\left(1 - \frac{s_{r+1}^j}{s_r^j}\right)v_{r,r+1}^j - v_{r,r+1}^j + q_{r+1}^j\right],$$

and using the Cauchy–Schwarz inequality we have that

$$Var(\tilde{\varphi}_j) \leqslant \frac{1}{\tilde{\eta}_j}\left\{\sum_{r=0}^{k_j-1}\left[\frac{(s_r^j)^2}{q_r^j q_{r+1}^j}v_{r,r+1}^j\right]^2\right\}^{1/2}\left\{\sum_{r}^{k_j-1}\left(1 - \frac{s_{r+1}^j}{s_r^j}\right)^2\right\}^{1/2}$$
$$+ \frac{1}{\tilde{\eta}_j}\left\{\sum_{r=0}^{k_j-1}\frac{(s_r^j)^4}{(q_r^j)^2}\right\}^{1/2}\left\{\sum_{r=0}^{k_j-1}\left(\frac{1}{q_{r+1}^j}\right)^2\left[\sum_n d_{nk_j-r-1}(d_{nk_j-r-1} - d_{nk_j-r})\right]^2\right\}^{1/2}$$
$$= \frac{1}{\tilde{\eta}_j}(T_1T_2 + T_3T_4).$$

We estimate T_1, T_2, T_3 and T_4 separately. By using (6) we obtain the estimate

$$T_1 = \left\{\sum_{r=0}^{k_j-1}\left[\frac{(s_r^j)^2}{q_r^j q_{r+1}^j}v_{r,r+1}^j\right]^2\right\}^{1/2} \leqslant \left\{\sum_{r=0}^{k_j-1}\frac{(s_r^j)^4}{q_r^j q_{r+1}^j}\right\}^{1/2}$$

and also use the notation in (6) to obtain that

$$T_1 \leqslant \left\{\sum_{r=-k_{j-1}}^{k_{j-1}-1}\frac{(s_r^j)^4}{d_{-r}^2 d_{-r-1}^2}\right\}^{1/2} = \left\{\sum_{r=-k_{j-1}}^{k_{j-1}-1}\left(\frac{s_r^j}{d_{-r}}\right)^4\left(\frac{d_{-r}}{d_{-r-1}}\right)^2\right\}^{1/2}.$$

By using the condition of the theorem, we provide the conclusion that

$$T_1 \leqslant ck_j^{1/2}. \tag{9}$$

We estimate T_2 in the following way. By using its definition and the Cauchy–Schwarz inequality, we have that

$$T_2 \leqslant \left\{\sum_{r=0}^{k_j-1}\frac{\sum_{n\in\mathbb{Z}} d_{nk_j-r}^2}{(\sum_{n\in\mathbb{Z}} d_{nk_j-r})^2}\sum_{n\in\mathbb{Z}}\left(\frac{d_{nk_j-r-1}}{d_{nk_j-r}} - 1\right)^2\right\}^{1/2},$$

and using the fact that all d's are positive, we obtain the inequality

$$T_2 \leqslant \left\{\sum_{r=0}^{k_j-1}\sum_{n\in\mathbb{Z}}\left(\frac{d_{nk_j-r-1}}{d_{nk_j-r}} - 1\right)^2\right\}^{1/2}.$$

Hence from (4), we conclude that there is a positive constant c such that

$$T_2 \leqslant c, \tag{10}$$

while from (5), we obtain that

$$T_3 = \left\{ \sum_{r=0}^{k_j-1} \frac{(s_r^j)^4}{(q_r^j)^2} \right\}^{1/2} \leqslant c k_j^{1/2}. \tag{11}$$

Similarly, we have that

$$T_4^2 \leqslant \sum_{r=0}^{k_j-1} \frac{1}{q_{r+1}^j} \sum_{n\in\mathbb{Z}} (d_{nk_j-r-1} - d_{nk_j-r})^2 \leqslant \sum_{m\in\mathbb{Z}} \left(1 - \frac{d_{m+1}}{d_m}\right)^2,$$

and therefore from the condition of the theorem, there is a positive constant $c > 0$ such that

$$T_4 \leqslant c. \tag{12}$$

This provides us, by (9)–(12), with the inequality

$$Var(\tilde{\varphi}_j) \leqslant c \frac{1}{\tilde{\eta}_j} \sqrt{k_j}, \tag{13}$$

where c is a constant independent of j. From lemma 2.2 we have that $\tilde{\eta}_j \geqslant k_j$, and combined (13), we derive the desired bound and prove the result. □

3. The properties of $\widetilde{L}_j$

In this section, we study the localization of the function $\widetilde{L}_j$. To this end, we define the functions

$$\mathcal{D}_l^j(x) := \left\{ c_l^{j+1} \widetilde{\mathcal{C}}_l^{j+1}(x) - c_{k_j+l}^{j+1} \widetilde{\mathcal{C}}_{k_j+l}^{j+1}(x) \right\} \mathrm{e}^{\mathrm{i} l h_{j+1}} \tag{14}$$

for $l = 0, 1, \ldots, k_j - 1$, where $c_l^{j+1} = \|\mathcal{C}_{k_j+l}^{j+1}\| / \|\mathcal{C}_l^j\|$ which is the same as in equation (2).

$$L_j(x) := \frac{1}{k_j} \sum_{l=0}^{k_j-1} \frac{\mathcal{D}_l^j(x)}{\mathcal{D}_l^j(h_{j+1})}, \tag{15}$$

and introduce the matrix

$$M := \mathrm{diag}\left\{ Q_j(1), Q_j(\omega), \ldots, Q_j\left(\omega^{k_j-1}\right) \right\},$$

where

$$Q_j(z) := \sum_{k=0}^{k_j-1} \left\langle L_j(\cdot), L_j(\cdot - k h_j) \right\rangle z^k. \tag{16}$$

The dual function of L_j is defined by the equation

$$\widetilde{L}_j(x) := \sum_{\nu=0}^{k_j-1} d_{j,\nu} L_j(x - \nu h_j), \tag{17}$$

where

$$d_{j,\nu} := \frac{1}{k_j} \sum_{l=0}^{k_j-1} \big(Q_j(\omega^l)\big)^{-1} \omega^{-l\nu}. \tag{18}$$

As for the localization of $\widetilde{L}_j$, we have the following theorem.

Theorem 3.1. Under the conditions (4) and (5), we have that

$$Var\big(\widetilde{L}_j\big) = \mathrm{O}\bigg(\frac{1}{\sqrt{k_j}}\bigg), \quad j \to \infty.$$

To estimate the localization of $\widetilde{L}_j$, we have to establish some lemmas.

Lemma 3.1. Let $\widetilde{L}_j$ be defined as in (17), then

$$\|\widetilde{L}_j\|^2 = \frac{1}{k_j} \sum_{l=0}^{k_j-1} \overline{\big(Q_j(\omega^l)\big)}^{-1}. \tag{19}$$

Proof. This result follows by a direct computation with equations (16)–(18), we leave the details to the reader. □

In our next result we give an alternative expression for $\|\widetilde{L}_j\|$.

Lemma 3.2.

$$\|\widetilde{L}_j\| = \left\{ \sum_{l=0}^{k_j-1} \frac{(s_l^{j+1} q_{k_j+l}^{j+1} + s_{k_j+l}^{j+1} q_l^{j+1})^2}{q_l^{j+1} q_{k_j+l}^{j+1} (q_l^{j+1} + q_{k_j+l}^{j+1})} \right\}^{1/2}, \tag{20}$$

where s_r and q_r are given in equation (6).

Proof. Since $Q_j(z) = \sum_{\nu=0}^{k_j-1} q_{j,\nu} z^\nu$, we calculate $q_{j,\nu} := \langle L_j(\cdot), L_j(\cdot - \nu h_j)\rangle$, by using the formula (15). Specifically, we have that

$$q_{j,\nu} = \frac{1}{k_j^2} \sum_{l,k=0}^{k_j-1} \frac{1}{\mathcal{D}_l^j(h_{j+1}) \overline{\mathcal{D}_k^j(h_{j+1})}} I_{l,k}^\nu,$$

where $I_{l,k}^\nu$ is defined to be

$$I_{l,k}^\nu := \big\langle \mathcal{D}_l^j(\cdot), \mathcal{D}_k^j(\cdot - \nu h_j)\big\rangle.$$

From definition (14) a direct computation leads to the formula

$$I_{l,k}^{\nu} = \mathrm{e}^{-\mathrm{i}l\nu h_j}\left(\frac{\|\mathcal{C}_{k_j+l}^{j+1}\|^2}{\|\mathcal{C}_l^j\|^2} + \frac{\|\mathcal{C}_l^{j+1}\|^2}{\|\mathcal{C}_l^j\|^2}\right)\delta_{l,k},$$

which we substitute into $q_{j,\nu}$ to obtain

$$q_{j,\nu} = \frac{1}{k_j^2}\sum_{\tilde{l}=0}^{k_j-1}\frac{\|\mathcal{C}_{k_j+\tilde{l}}^{j+1}\|^2 + \|\mathcal{C}_{\tilde{l}}^{j+1}\|^2}{(\|\mathcal{C}_{k_j+\tilde{l}}^{j+1}\|\widetilde{\mathcal{C}}_{\tilde{l}}^{j+1}(0) + \|\mathcal{C}_{\tilde{l}}^{j+1}\|\widetilde{\mathcal{C}}_{k_j+\tilde{l}}^{j+1}(0))^2}\cdot \mathrm{e}^{-\mathrm{i}\tilde{l}\nu h_j},$$

where we use the fact that

$$\mathcal{D}_l^j(h_{j+1}) = c_l^{j+1}\,\widetilde{\mathcal{C}}_l^{j+1}(0) + c_{k_j+l}^{j+1}\,\widetilde{\mathcal{C}}_{k_j+l}^{j+1}(0).$$

Therefore, we have that

$$Q_j(\omega^l) = \sum_{k=0}^{k_j-1} q_{j,k}\omega^{lk} = \frac{1}{k_j}\frac{\|\mathcal{C}_{k_j+l}^{j+1}\|^2 + \|\mathcal{C}_l^{j+1}\|^2}{\|\mathcal{C}_{k_j+l}^{j+1}\|\,\widetilde{\mathcal{C}}_l^{j+1}(0) + \|\mathcal{C}_l^{j+1}\|\,\widetilde{\mathcal{C}}_{k_j+l}^{j+1}(0))^2}$$

and so by (19) we obtain the formula

$$\|\widetilde{L}_j\|^2 = \sum_{l=0}^{k_j-1}\frac{(\|\mathcal{C}_{k_j+l}^{j+1}\|\,\widetilde{\mathcal{C}}_l^{j+1}(0) + \|\mathcal{C}_l^{j+1}\|\,\widetilde{\mathcal{C}}_{k_j+l}^{j+1}(0))^2}{\|\mathcal{C}_{k_j+l}^{j+1}\|^2 + \|\mathcal{C}_l^{j+1}\|^2}.$$

Since $\|\mathcal{C}_\nu^{j+1}\| = k_{j+1}(q_\nu^{j+1})^{1/2}$ and $\widetilde{\mathcal{C}}_\nu^{j+1}(0) = s_\nu^{j+1}/(q_\nu^{j+1})^{1/2}$, this formula yields the desired conclusion. □

Lemma 3.3. For $j \in \mathbb{Z}_+$, we have that

$$k_j \leqslant \|\widetilde{L}_j\|^2 \leqslant \frac{4}{c^2}k_j, \tag{21}$$

where c is a positive constant such that

$$\inf\left\{\frac{d_l}{s_l^j}\colon\ |l| \leqslant k_{j-1},\ j \in \mathbb{Z}_+\right\} \geqslant c.$$

Proof. From (20) and (6) we have that

$$\|\widetilde{L}_j\|^2 \leqslant 2\sum_{l=-k_{j-1}}^{k_{j-1}-1}\left(\frac{(s_l^{j+1})^2}{d_{-l}^2} + \frac{(s_{k_j+l}^{j+1})^2}{d_{-k_j+l}^2}\right).$$

According to the hypothesis of the lemma we obtain that

$$\|\widetilde{L}_j\|^2 \leqslant \frac{4}{c^2}k_j.$$

On the other hand, from (20) we have that

$$\|\widetilde{L}_j\|^2 \geqslant \sum_{l=0}^{k_j-1} \frac{\left(\sqrt{q_{k_j+l}^{j+1}} + \sqrt{q_l^{j+1}}\, q_{k_j+l}^{j+1}/q_l^{j+1}\right)^2}{q_{k_j+l}^{j+1}\left(1 + q_{k_j+l}^{j+1}/q_l^{j+1}\right)} \geqslant k_j$$

from which we conclude that (21) holds. □

Now, we consider the quantity $\tilde{\lambda}_j$, where

$$\tilde{\lambda}_j := \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}t}\left|\widetilde{L}_j(t)\right|^2 \mathrm{d}t.$$

From (17) and (18) we have that

$$\tilde{\lambda}_j = \frac{1}{k_j^2}\sum_{\nu_1,\nu_2=0}^{k_j-1}\sum_{l_1,l_2=0}^{k_j-1}\left(Q_j\left(\omega^{l_1}\right)Q_j\left(\omega^{l_2}\right)\right)^{-1}\omega^{l_2\nu_2-l_1\nu_1}p_{\nu_1,\nu_2},$$

where

$$\begin{aligned}
p_{\nu_1,\nu_2} &:= \left\langle \mathrm{e}^{\mathrm{i}\cdot} L_j(\cdot-\nu_1 h_j), L_j(\cdot-\nu_2 h_j)\right\rangle\\
&= \frac{1}{k_j^2}\sum_{l,k=0}^{k_j-1}\frac{1}{\mathcal{D}_l^j(h_{j+1})\overline{\mathcal{D}_k^j(h_{j+1})}}(J_1+J_2+J_3+J_4)\mathrm{e}^{\mathrm{i}h_j(l\nu_1-k\nu_2)}\mathrm{e}^{\mathrm{i}h_{j+1}(l-k)},\\
J_1 &:= \frac{c_l^{j+1}c_k^{j+1}}{\|\mathcal{C}_l^{j+1}\|\|\mathcal{C}_k^{j+1}\|}\widetilde{J}_1, \qquad J_2 := -\frac{c_l^{j+1}c_{k_j+k}^{j+1}}{\|\mathcal{C}_l^{j+1}\|\|\mathcal{C}_{k_j+k}^{j+1}\|}\widetilde{J}_2,\\
J_3 &:= -\frac{c_{k_j+l}^{j+1}c_k^{j+1}}{\|\mathcal{C}_{k_j+l}^{j+1}\|\|\mathcal{C}_k^{j+1}\|}\widetilde{J}_3, \qquad J_4 := \frac{c_{k_j+l}^{j+1}c_{k_j+k}^{j+1}}{\|\mathcal{C}_{k_j+l}^{j+1}\|\|\mathcal{C}_{k_j+k}^{j+1}\|}\widetilde{J}_4.\\
\widetilde{J}_1 &:= k_{j+1}^2\sum_{n\in\mathbb{Z}} d_{nk_{j+1}-l}d_{nk_{j+1}-k}\delta_{l,k+1},\\
\widetilde{J}_2 &:= k_{j+1}^2\sum_{n\in\mathbb{Z}} d_{nk_{j+1}-l}d_{(n+1)k_{j+1}-k-k_j}\delta_{l,0}\delta_{k,k_j-1},\\
\widetilde{J}_3 &:= k_{j+1}^2\sum_{n\in\mathbb{Z}} d_{nk_{j+1}-k_j-l}d_{nk_{j+1}-k}\delta_{l,0}\delta_{k,k_j-1},
\end{aligned}$$

and

$$\widetilde{J}_4 := k_{j+1}^2\sum_{n\in\mathbb{Z}} d_{nk_{j+1}-k_j-l}d_{nk_{j+1}-k_j-k}\delta_{l,k+1}.$$

Therefore, we obtain that

$$\tilde{\lambda}_j = E_1 + E_2 + E_3 + E_4,$$

where

$$E_1 := \sum_{k=0}^{k_j-1} \frac{\mathrm{e}^{\mathrm{i}h_{j+1}} \Delta_{k+1} \Delta_k v_{k+1,k}^{j+1}}{q_{k_j+k}^{j+1} q_{k_j+k+1}^{j+1}} m_{k+1}^{j+1} m_k^{j+1}, \tag{22}$$

$$E_2 := \mathrm{e}^{\mathrm{i}h_{j+1}} \frac{\Delta_0 \Delta_{k_j-1}}{q_{k_j}^{j+1} q_{k_{j+1}-1}^{j+1}} v_{0,k_{j+1}-1} m_0^{j+1} m_{k_{j+1}-1}^{j+1}, \tag{23}$$

$$E_3 := \mathrm{e}^{\mathrm{i}h_{j+1}} \frac{\Delta_0 \Delta_{k_j-1}}{q_{k_j}^{j+1} q_{k_{j+1}-1}^{j+1}} v_{k_j,k_j-1} m_{k_j}^{j+1} m_{k_j-1}^{j+1} \tag{24}$$

and

$$E_4 := \sum_{k=0}^{k_j-1} \frac{\mathrm{e}^{\mathrm{i}h_{j+1}} \Delta_{k+1} \Delta_k v_{k_j+k+1,k_j+k}^{j+1}}{q_{k_j+k}^{j+1} q_{k_j+k+1}^{j+1}} m_{k_j+k+1}^{j+1} m_{k_j+k}^{j+1}, \tag{25}$$

where $\Delta_l := m_l^{j+1} s_l^{j+1} + m_{k_j+l}^{j+1} s_{k_j+l}^{j+1}$.

To simplify our notation, we shall now delete the superscript $j+1$ as the context makes its presence clear. By using (20), (22)–(25), we have that

$$\|\widetilde{L}_j\|^2 - |\tilde{\lambda}_j| = \sum_{l=0}^{k_j-1} \left\{ \frac{(q_{k_j+l} s_l + q_l s_{k_j+l})^2}{q_l q_{k_j+l}(q_l + q_{k_j+l})} - \frac{\Delta_{l+1}\Delta_l}{q_{k_j+l} q_{k_j+l+1}} \nabla_1 \right\} - \frac{\Delta_0 \Delta_{k_j-1}}{q_{k_j} q_{k_{j+1}-1}} \nabla_2, \tag{26}$$

where

$$\nabla_1 := v_{l+1,l} m_l m_{l+1} + v_{k_j+l+1,k_j+l} m_{k_j+l+1} m_{k_j+l}$$

and

$$\nabla_2 := v_{0,k_{j+1}-1} m_0 m_{k_{j+1}-1} + v_{k_j,k_j-1} m_{k_j} m_{k_j-1}.$$

To estimate the right-hand side of equation (26) we set

$$x_l := m_l s_l, \qquad y_l := m_l m_{l+1} v_{l,l+1},$$

and observe that

$$\|\widetilde{L}_j\|^2 - |\tilde{\lambda}_j| \leqslant \sum_{l=0}^{k_j-1} \left[\frac{(x_l + x_{l+k_j})^2}{q_l + q_{k_j+l}} - \frac{(x_l + x_{l+k_j})(x_{l+1} + x_{l+1+k_j})(y_l + y_{l+k_j})}{q_{l+k_j} q_{l+1+k_j}} \right]$$

which gives the estimate

$$\|\widetilde{L}_j\|^2 - \tilde{\lambda}_j \leqslant I_1 + I_2 + I_3 + I_4, \tag{27}$$

where

$$I_1 := \sum_{l=0}^{k_j-1} \left(\frac{x_l^2}{q_l + q_{l+k_j}} - \frac{x_l x_{l+1} y_l}{q_{l+k_j} q_{l+1+k_j}} \right), \tag{28}$$

$$I_2 := \sum_{l=0}^{k_j-1}\left(\frac{x_{l+k_j}^2}{q_l + q_{l+k_j}} - \frac{x_{l+k_j}x_{l+1+k_j}y_l}{q_{l+k_j}q_{l+1+k_j}}\right), \tag{29}$$

$$I_3 := \sum_{l=0}^{k_j-1}\left(\frac{x_l x_{l+k_j}}{q_l + q_{l+k_j}} - \frac{x_l x_{l+1+k_j}y_l}{q_{l+k_j}q_{l+1+k_j}}\right) \tag{30}$$

and

$$I_4 := \sum_{l=0}^{k_j-1}\left(\frac{x_l x_{l+k_j}}{q_l + q_{l+k_j}} - \frac{x_{l+k_j}x_{l+1}y_l}{q_{l+k_j}q_{l+1+k_j}}\right). \tag{31}$$

We establish the following results.

Lemma 3.4. Under the assumption (4) and (5) there exists a constant c such that for all $j \in \mathbb{Z}_+$, we have

$$|I_1| \leqslant ck_j^{1/2} \quad \text{and} \quad |I_2| \leqslant ck_j^{1/2}. \tag{32}$$

Proof. From (28) we obtain that

$$|I_1| \leqslant \sum_{l=0}^{k_j-1}\frac{s_l}{(q_l q_{l+k_j})^{1/2}}\left|m_l s_l - \frac{m_{l+1}^2 m_l s_{l+1} v_{l,l+1}}{q_{l+1+k_j}}\right|.$$

Consequently, using the condition of (5), we have that

$$|I_1| \leqslant \frac{1}{c}\sum_{l=0}^{k_j-1}\frac{s_l}{q_l^{1/2}}\left|1 - \frac{s_{l+1}}{s_l}\left(\frac{q_l}{q_{l+1}}\right)^{1/2}\frac{v_{l,l+1}}{(q_l q_{l+1})^{1/2}}\right|.$$

By the Cauchy–Schwarz inequality and the assumption (5), we obtain the inequality

$$|I_1| \leqslant \frac{k_j^{1/2}}{c^2}\left[\sum_{l=0}^{k_j-1}\left|1 - \frac{s_{l+1}}{s_l}\frac{v_{l,l+1}}{q_{l+1}}\right|^2\right]^{1/2}. \tag{33}$$

We shall establish the existence of the constants c_1 and c_2 such that for all $j \in \mathbb{Z}_+$,

$$\sum_{l=0}^{k_j-1}\left(1 - \frac{s_{l+1}}{s_l}\right)^2 < c_1 \tag{34}$$

and

$$\sum_{l=0}^{k_j-1}\left(1 - \frac{v_{l,l+1}}{q_l}\right)^2 < c_2. \tag{35}$$

In fact, from equations (5) and (6), we obtain that

$$\sum_{l=0}^{k_j-1}\left(1-\frac{s_{l+1}}{s_l}\right)^2 \leqslant \sum_{l=0}^{k_j-1}\sum_{n\in\mathbb{Z}}\left(1-\frac{d_{nk_{j+1}-l-1}}{d_{nk_{j+1}-l}}\right)^2$$

which proves (34). As for the estimate (35), we first observe that

$$\sum_{l=0}^{k_j-1}\left(1-\frac{v_{l,l+1}}{q_{l+1}}\right)^2 = \sum_{l=0}^{k_j-1}\frac{\left[\sum_n d^2_{nk_{j+1}-l-1}(1-d_{nk_{j+1}-l}/d_{nk_{j+1}-l-1})\right]^2}{q^2_{l+1}},$$

that is,

$$\sum_{l=0}^{k_j-1}\left(1-\frac{v_{l,l+1}}{q_{l+1}}\right)^2 \leqslant \sum_{l=0}^{k_j-1}\sum_{n}\left(1-\frac{d_{nk_{j+1}-l}}{d_{nk_{j+1}-l-1}}\right)^2 \leqslant c_2,$$

from which we obtain (35). Also, using the Cauchy–Schwarz inequality, we obtain that

$$\left[\sum_{l=0}^{k_j-1}\left|1-\frac{s_{l+1}}{s_l}\frac{v_{l,l+1}}{q_{l+1}}\right|^2\right]^{1/2} \leqslant \left[2\sum_{l=0}^{k_j-1}\left(1-\frac{s_{l+1}}{s_l}\right)^2+2\sum_{l=0}^{k_j-1}\left(\frac{s_{l+1}}{s_l}\right)^2\left(1-\frac{v_{l,l+1}}{q_{l+1}}\right)^2\right]^{1/2}.$$

Therefore from (34), (35) and the boundedness of the sequence $\{s_{l+1}/s_l\colon\ l\in\mathbb{Z}_+\}$, we have that

$$\left[\sum_{l=0}^{k_j-1}\left|1-\frac{s_{l+1}}{s_l}\frac{v_{l,l+1}}{q_{l+1}}\right|^2\right]^{1/2} \leqslant c. \tag{36}$$

We now employ (33) and (36) to complete the proof of the first inequality of (32).

Now, we start the proof of the second assertion. First, we observe that

$$|I_2| \leqslant \sum_{l=0}^{k_j-1}\frac{(m_{l+k_j}s_{l+k_j})^2}{q_l+q_{l+k_j}}\left[1-\frac{q_l+q_{l+k_j}}{q_{l+k_j}q_{l+1+k_j}}\cdot\frac{x_{l+1+k_j}y_l}{x_{l+k_j}}\right] \leqslant \Omega_1\Omega_2,$$

where

$$\Omega_1 := \left[\sum_{l=0}^{k_j-1}\left(\frac{(m_{l+k_j}s_{l+k_j})^2}{q_l+q_{l+k_j}}\right)^2\right]^{1/2}$$

and

$$\Omega_2 := \left[\sum_{l=0}^{k_j-1}\left(1-\frac{q_l+q_{l+k_j}}{q_{l+k_j}q_{l+1+k_j}}\cdot\frac{x_{l+1+k_j}y_l}{x_{l+k_j}}\right)^2\right]^{1/2}.$$

By using (5) we have that

$$\Omega_1 \leqslant \left[\sum_{l=0}^{k_j-1}\left(\frac{s^2_{l+k_j}}{q_{l+k_j}}\right)^2\right]^{1/2} \leqslant ck_j^{1/2},\quad j\in\mathbb{Z}_+. \tag{37}$$

Similarly, we obtain that

$$\Omega_2 \leqslant \sqrt{2}(\Omega_{2,1} + \Omega_{2,2})^{1/2},$$

where

$$\Omega_{2,1} := \sum_{l=0}^{k_j-1}\left(1 - \frac{m_{l+1+k_j}}{m_{l+k_j}}\right)^2$$

and

$$\Omega_{2,2} := \sum_{l=0}^{k_j-1}\left(\frac{m_{l+1+k_j}}{m_{l+k_j}}\right)^2\left(1 - \frac{s_{l+1+k_j}}{s_{l+k_j}}\frac{(q_l + q_{l+k_j})y_l}{q_{l+k_j}q_{l+1+k_j}}\right)^2.$$

By using the formula given in [5, (4.8.12), p. 195], we have

$$\Omega_{2,1} \leqslant \sum_{l=0}^{k_j-1}\left|1 - \left(\frac{m_{l+1+k_j}}{m_{l+k_j}}\right)^2\right|^2 \leqslant c.$$

We estimate $\Omega_{2,2}$ next. Specifically, we have that

$$\Omega_{2,2} \leqslant c\sum_{l=0}^{k_j-1}\left(1 - \frac{s_{l+1+k_j}}{s_{l+k_j}}\frac{(q_l + q_{l+k_j})m_l m_{l+1}v_{l,l+1}}{q_{l+k_j}q_{l+1+k_j}}\right)^2 \leqslant 2c\Omega^0_{2,1} + 2c\Omega^0_{2,2},$$

where

$$\Omega^0_{2,1} := \sum_{l=0}^{k_j-1}\left(1 - \frac{s_{l+1+k_j}}{s_{l+k_j}}\right)^2$$

and

$$\Omega^0_{2,2} := \sum_{l=0}^{k_j-1}\left[1 - \frac{q_l + q_{l+k_j}}{q_{l+k_j}q_{l+1+k_j}}\frac{(q_{l+k_j}q_{l+1+k_j})^{1/2}}{(q_l q_{l+1})^{1/2}}v_{l,l+1}\right]^2.$$

By using the same method used to confirm (35), we have that $\Omega^0_{2,1} < c$. For the second inequality, one can also refer to [5, (4.8.16), p. 199], specifically, we have that

$$\begin{aligned}\Omega^0_{2,2} &\leqslant \sum_{l=0}^{k_j-1}\left[1 - \frac{(q_{l+k_j}q_{l+1+k_j})^{1/2}}{(q_l q_{l+1})^{1/2}}\frac{v_{l,l+1}}{q_{l+1}}\right]^2\\ &\leqslant 2\sum_{l=0}^{k_j-1}\left(1 - \frac{m_l}{m_{l+1}}\right)^2 + 2\sum_{l=0}^{k_j-1}\left(\frac{m_l}{m_{l+1}}\right)^2\left(1 - \frac{v_{l,l+1}}{q_{l+1}}\right)^2,\end{aligned}$$

from which the technique used to derive the inequality (37) and (35), yields the bound $\Omega^0_{2,2} \leqslant c$ thereby providing the inequality $\Omega_2 < c$. This establishes the second inequality in (32). $\square$

Lemma 3.5. Under the conditions (4) and (5), there is a positive constant c such that for all $j \in \mathbb{Z}_+$,

$$|I_3| \leqslant c k_j^{1/2}, \qquad |I_4| \leqslant c k_j^{1/2}. \tag{38}$$

Proof. From the definition of x_l and y_l, we have that

$$|I_3| \leqslant \sum_{l=0}^{k_j-1} \frac{s_l s_{l+k_j}}{2(q_l q_{l+k_j})^{1/2}} \left\{ 1 - \frac{m_{l+1+k_j} s_{l+1+k_j} m_l m_{l+1} v_{l,l+1}}{q_{l+1+k_j} m_{l+k_j} s_{l+k_j}} \right\}.$$

Consequently, the Cauchy–Schwarz inequality yields the bound

$$\begin{aligned} |I_3| &\leqslant \left\{ \sum_{l=0}^{k_j-1} \left(\frac{s_l s_{l+k_j}}{2(q_l q_{l+k_j})^{1/2}} \right)^2 \right\}^{1/2} \left\{ \sum_{l=0}^{k_j-1} \left[1 - \frac{q_{l+k_j} s_{l+1+k_j} v_{l,l+1}}{q_{l+1+k_j} q_l s_{l+k_j}} \right]^2 \right\}^{1/2} \\ &\leqslant c k_j^{1/2} \left\{ \sum_{l=0}^{k_j-1} \left[1 - \frac{s_{l+1+k_j}}{s_{l+k_j}} + \frac{s_{l+1+k_j}}{s_{l+k_j}} \left(1 - \frac{q_{l+k_j}}{q_{l+1+k_j}} \frac{v_{l,l+1}}{q_l} \right) \right]^2 \right\}^{1/2} \\ &\leqslant c k_j^{1/2} \left\{ c + c \sum_{l=0}^{k_j-1} \left(1 - \frac{q_{l+k_j}}{q_{l+1+k_j}} \frac{v_{l,l+1}}{q_l} \right)^2 \right\}^{1/2}, \quad j \in \mathbb{Z}_+. \end{aligned}$$

We use (5) and (37) to obtain that

$$|I_3| \leqslant c k_j^{1/2}, \quad j \in \mathbb{Z}_+,$$

since the sequence $(q_{l+k_j}/q_{l+1+k_j})^2$ is bounded for $j \in \mathbb{Z}_+$, and

$$\begin{aligned} &\sum_{l=0}^{k_j-1} \left(1 - \frac{q_{l+k_j}}{q_{l+1+k_j}} \frac{v_{l,l+1}}{q_l} \right)^2 \\ &\quad = \sum_{l=0}^{k_j-1} \left[1 - \frac{q_{l+k_j}}{q_{l+1+k_j}} + \frac{q_{l+k_j}}{q_{l+1+k_j}} \left(1 - \frac{v_{l,l+1}}{q_l} \right) \right]^2 \\ &\quad \leqslant 2 \sum_{l=0}^{k_j-1} \left(1 - \frac{q_{l+k_j}}{q_{l+1+k_j}} \right)^2 + 2 \sum_{l=0}^{k_j-1} \left(\frac{q_{l+k_j}}{q_{l+1+k_j}} \right)^2 \left(1 - \frac{v_{l,l+1}}{q_l} \right)^2. \end{aligned}$$

Now, we prove the second inequality in (38). To this end, we note that

$$\begin{aligned} |I_4| &= \sum_{l=0}^{k_j-1} \frac{x_l x_{l+k_j}}{q_l + q_{l+k_j}} \left| 1 - \frac{(q_l + q_{l+k_j}) x_{l+1} y_l}{x_l q_{l+k_j} q_{l+1+k_j}} \right| \\ &\leqslant \sum_{l=0}^{k_j-1} \frac{s_l s_{l+k_j}}{q_l + q_{l+k_j}} \left| 1 - \frac{s_{l+1}}{s_l} \frac{q_l}{q_{l+1}} \frac{v_{l,l+1}}{q_l} \right|. \end{aligned}$$

We estimate this sum just as above and conclude that

$$|I_4| \leqslant ck_j^{1/2}, \quad j \in \mathbb{Z}_+. \qquad \square$$

Proof of theorem 3.1. The proof follows directly from (27)–(31) and lemmas 3.3–3.5.

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