New Results on Doubly Coprime Fractional Representations of Generalized Dynamical Systems

Zhi-Wei Gao and Fei-Yue Wang

Abstract—This note first points out that the main results by Wang and Balas regarding the doubly coprime fractional representations for generalized dynamical systems have severe limitation in their applications, that is, the doubly coprime factorization obtained by Wang and Balas cannot characterize the parameterization of all properly stabilizing controllers when a system is singular, therefore, truly generalized. To remedy those results, two new doubly coprime factorizations have been developed here that will parameterize all properly stabilizing controllers for single-input or single-output cases. In addition, the new results can characterize the parameterization of all corresponding causal properly stabilizing controllers. Finally, the extension to the multiple-input–multiple-output case is presented.

Index Terms—Controller parameterization, coprime factorizations, generalized dynamical systems, proportional and derivative feedback.

I. INTRODUCTION

It is well-known that the doubly coprime factorization plays an important role in investigating generalized dynamical systems with the stable fractional approach, e.g., see [1]–[3], [5], and the references therein. In the pioneering work by Wang and Balas [5], by using proportional and derivative (feedback or observation) gains, two doubly coprime factorizations were established by Theorems 2a and 2b. When the derivative coefficient matrix E is nonsingular, all transfer function matrices of the two doubly coprime factorizations in Theorems 2a and 2b are proper stable as pointed out in [5], which can be utilized to characterize the parameterizations of all properly stabilizing controllers (Here, properly stabilizing the plant means that making the corresponding closed-loop system internally proper and stable). However, the case of E being singular is not discussed in [5, Ths. 2a and 2b]. In this note, the focus will be on the situation of E being singular. We prove that the left or right fractional factors of the feedback controller in [5, Th. 2a and Th. 2b] are stable but nonproper when E is singular. This implies that the application of Theorems 2a and 2b has some limitation, e.g., the parameterization of *all properly stabilizing* controllers can not be accomplished as desired. To obtain proper stable factorizations, the results of [5, Ths. 2a and 2b] are modified here for single input or single output generalized dynamical systems. Based on the modified factorizations, the parameterization of all properly stabilizing controllers is established. The corresponding result for multiple-input–multiple-output (MIMO) case is also presented in the end.

II. PROBLEM STATEMENT

In this note, $\mathcal{R}$ denotes the set of real numbers; $\mathcal{C}_+$ denotes the complex, closed right-half plane; $\mathcal{C}_{+e} = \mathcal{C}_+ \bigcup \{\infty\}$; $\mathcal{RH}_\infty$ represents the

class of proper stable rational functions; $\mathcal{RH}_2$ denotes the set of strictly proper and stable rational functions; $\mathcal{S}$ denotes the class of rational functions in the form $m_{sp}(s) + as + b$, where $m_{sp}(s) \in \mathcal{RH}_2$, a, $b \in \mathcal{R}$; A^T denotes the transpose of A; $Im(A)$ and $Ker(A)$ respectively denote the image and kernel of A.

Consider a generalized dynamical plant

$$\begin{cases} E\dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{1}$$

where $x(t) \in \mathcal{R}^n$ is the descriptor vector, $u(t) \in \mathcal{R}^m$ and $y(t) \in \mathcal{R}^p$ are respectively the input and output vectors; $E \in \mathcal{R}^{n \times n}$ may be singular, that is, rank $(E) < n$. The pair (E, A) is assumed to be regular, i.e., $\det(sE - A) \not\equiv 0$. The transfer function matrix of (1) is represented as

$$G(s) = C(sE - A)^{-1}B + D := \left[\begin{array}{c|c} sE - A & B \\ \hline C & D \end{array} \right]. \tag{2}$$

As is known, the plant (1) is called completely stabilizable provided that $\text{rank}(sE - A, B) = n$, $\forall s \in \mathcal{C}_+$ and $\text{rank}(E, B) = n$. The plant is properly stabilizable provided that $\text{rank}(sE - A, B) = n$, $\forall s \in \mathcal{C}_{+e}$. Clearly, a completely stabilizable plant must be properly stabilizable, but not vice versa. The plant (1), simply denoted by (E, A, B, C, D), is completely (or properly) detectable if and only if the dual plant $(E^T, A^T, C^T, B^T, D^T)$ is completely (or properly) stabilizable. For simplicity, we can assume $D = 0$ without loss of generality. Now, we rewrite [5, Ths. 2a and 2b] in the sequel.

Theorem 2a [5]: Consider a completely stabilizable and properly detectable plant (1). Choose $F_1, F_2 \in \mathcal{R}^{m \times n}$, $L \in \mathcal{R}^{n \times p}$ such that: 1) $(E + BF_1)$ is nonsingular; 2) $[s(E + BF_1) - (A + BF_2)]^{-1} \in \mathcal{RH}_\infty^{n \times n}$ and $(sE - A - LC)^{-1} \in \mathcal{RH}_\infty^{n \times n}$; and 3) $\lim_{s \to \infty} F_1(sE - A - LC)^{-1}B = 0$ and $\lim_{s \to \infty} F_1(sE - A - LC)^{-1}L = 0$. Define

$$\begin{bmatrix} \tilde{V}(s) & -\tilde{U}(s) \\ -\tilde{N}(s) & \tilde{M}(s) \end{bmatrix} = \left[\begin{array}{c|cc} sE - A - LC & -B & L \\ \hline -(sF_1 - F_2) & I & 0 \\ C & 0 & I \end{array} \right] \tag{3}$$

$$\begin{bmatrix} M(s) & U(s) \\ N(s) & V(s) \end{bmatrix} = \left[\begin{array}{c|cc} s(E + BF_1) - (A + BF_2) & B & -L \\ \hline -(sF_1 - F_2) & I & 0 \\ C & 0 & I \end{array} \right]. \tag{4}$$

Then

a) all the transfer function matrices defined above are *proper* stable;
b) $\tilde{M}(s)$ and $M(s)$ are both nonsingular;
c) $G(s) = \tilde{M}(s)^{-1}\tilde{N}(s) = N(s)M(s)^{-1}$;
d)

$$\begin{bmatrix} \tilde{V}(s) & -\tilde{U}(s) \\ -\tilde{N}(s) & \tilde{M}(s) \end{bmatrix} \cdot \begin{bmatrix} M(s) & U(s) \\ N(s) & V(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{5}$$

Theorem 2b [5]: Consider a properly stabilizable and completely detectable plant (1). Choose $F \in \mathcal{R}^{m \times n}$, $L_1, L_2 \in \mathcal{R}^{n \times p}$ such that: 1) $(E + L_1C)$ is nonsingular; and 2) $(sE - A - BF)^{-1} \in \mathcal{RH}_\infty^{n \times n}$ and $[s(E + L_1C) - (A + L_2C)]^{-1} \in \mathcal{RH}_\infty^{n \times n}$; 3) $\lim_{s \to \infty} C(sE - A - BF)^{-1}L_1 = 0$ and $\lim_{s \to \infty} F(sE - A - BF)^{-1}L_1 = 0$. Define (6)–(7), as shown at the bottom of the next page. Then, the conclusions specified in a)–d) of Theorem 2a still hold.

Remark 1: In both Theorems 2a and 2b as previously shown, Conditions 1) and 2) can be performed easily under the related assumptions.

As pointed out in [5], Condition 3) is valid when E is nonsingular. In this case, all transfer function matrices in the doubly coprime factorizations by Theorems 2a and 2b are guaranteed to be stable and proper. The resultant parameterizations of all properly stabilizing controllers can be characterized readily.

However, (1) is no longer singular in this case. Therefore, our question is: What will happen when E is singular? This motivates us to conduct the study in this note.

III. LIMITATION OF PREVIOUS RESULTS

Now, we discuss the limits of Theorems 2a and 2b when E is singular.

Theorem 3a: Suppose there exist $F_1 \in \mathcal{R}^{m \times n}$, $L \in \mathcal{R}^{n \times p}$ such that $(E + BF_1)$ is nonsingular and $(sE - A - LC)^{-1} \in \mathcal{RH}_{\infty}^{n \times n}$. Then, $\lim_{s \to \infty} F_1(sE - A - LC)^{-1}B = 0$ if and only if E is nonsingular.

Proof: Sufficiency: If E is nonsingular, then $(sE - A - LC)^{-1} \in \mathcal{RH}_2^{n \times n}$, which obviously means that $\lim_{s \to \infty} F_1(sE - A - LC)^{-1}B = 0$.

Necessity: For the purpose of the contradiction, we assume that $\lim_{s \to \infty} F_1(sE - A - LC)^{-1}B = 0$ implies E is singular. Choose nonsingular constant matrices $P, Q \in \mathcal{R}^{n \times n}$ to make the following restricted equivalent transformation [4]:

$$
\begin{aligned}
F_1(sE - A - LC)^{-1}B &= F_1 Q \left[P(sE - A - LC)Q \right]^{-1} PB \\
&= F_s^1 (sI - A_s)^{-1} B_s - F_f^1 B_f
\end{aligned} \tag{8}
$$

where $PEQ = \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix}$, $P(A + LC)Q = \begin{pmatrix} A_s & 0 \\ 0 & I_f \end{pmatrix}$, $PB = \begin{pmatrix} B_s \\ B_f \end{pmatrix}$, $F_1 Q = (F_s^1 \quad F_f^1)$, $A_s \in \mathcal{R}^{n_s \times n_s}$, $I_f \in \mathcal{R}^{n_f \times n_f}$, and $n_s + n_f = n$. Since A_s is a stable matrix and $\lim_{s \to \infty} F_1(sE - A - LC)^{-1}B = 0$, then (8) yields

$$
F_f^1 B_f = 0. \tag{9}
$$

Note that $(E + BF_1)$ is assumed to be nonsingular and $B_f \in \mathcal{R}^{n_f \times m}$. It follows that

$$
\begin{aligned}
n &= \operatorname{rank}(E + BF_1) \\
&= \operatorname{rank} \begin{bmatrix} I_s + B_s F_s^1 & B_s F_f^1 \\ B_f F_s^1 & B_f F_f^1 \end{bmatrix} \\
&= \operatorname{rank} \left[\begin{pmatrix} I_s & 0 & B_s \\ 0 & 0 & B_f \end{pmatrix} \begin{pmatrix} I_s & 0 \\ 0 & I_f \\ F_s^1 & F_f^1 \end{pmatrix} \right] \\
&\leq n_s + \operatorname{rank}(B_f) \leq n.
\end{aligned} \tag{10}
$$

Equation (10) indicates that

$$
\operatorname{rank}(B_f) = n - n_s = n_f. \tag{11}
$$

From (11), there exists a right inverse $B_f^r \in \mathcal{R}^{m \times n_f}$ to $B_f \in \mathcal{R}^{n_f \times m}$ [6], i.e.,

$$
B_f B_f^r = I_f. \tag{12}
$$

By postmultiplying B_f^r to the both sides of (9), and using (12), one concludes immediately

$$
F_f^1 = 0. \tag{13}
$$

Therefore, according to (13)

$$
\begin{aligned}
\operatorname{rank}(E + BF_1) &= \operatorname{rank} \left[\begin{pmatrix} I_s & B_s \\ 0 & B_f \end{pmatrix} \begin{pmatrix} I_s \\ F_s^1 \end{pmatrix} \right] \\
&\leq n_s < n.
\end{aligned} \tag{14}
$$

Equation (14) contradicts the assumption that $(E + BF_1)$ is nonsingular. This completes the proof.

Similarly, the dual form of Theorem 3a can be proved easily, as follows.

Theorem 3b: Suppose there exist $F \in \mathcal{R}^{m \times n}$, $L_1 \in \mathcal{R}^{n \times p}$ such that $(E + L_1 C)$ is nonsingular and $(sE - A - BF)^{-1} \in \mathcal{RH}_{\infty}^{n \times n}$. Then, $\lim_{s \to \infty} C(sE - A - BF)^{-1}L_1 = 0$ if and only if E is nonsingular.

Remark 2: From Theorem 3a, the left fractional factor $\tilde{V}(s)$ of the feedback controller $K(s) = \tilde{V}^{-1}(s)\tilde{U}(s)$ in Theorem 2a must be stable but nonproper when E is singular. Similarly, Theorem 3b implies that the right fractional factor $V(s)$ of the feedback controller $K(s) = U(s)V^{-1}(s)$ in Theorem 2b must be stable but nonproper when E is singular. Thus, when E is singular, Theorems 2a and 2b both give stable doubly coprime factorizations, but not *proper* stable doubly coprime factorizations. This imposes significant limitations on the application of Theorems 2a and 2b, e.g., one can not characterize the class of *all properly stabilizing* controllers by these factorizations. Thus, it is imperative to provide remedial modifications if possible.

IV. MODIFIED DOUBLY COPRIME FACTORIZATIONS

A. Single-Input Plant Case

Let $\Delta(s) = s(E + BF_1) - (A + BF_2)$, then

$$
\begin{aligned}
sF_1 \Delta^{-1}(s) &= F_1(E + BF_1)^{-1}[\Delta(s) + (A + BF_2)]\Delta^{-1}(s) \\
&= F_1(E + BF_1)^{-1}(A + BF_2)\Delta^{-1}(s) \\
&\quad + F_1(E + BF_1)^{-1}.
\end{aligned} \tag{15}
$$

Using (15), (4) in Theorem 2a can be reformulated equivalently, as shown in (16) at the bottom of the next page, where $F_c = F_1(E + BF_1)^{-1}(A + BF_2) - F_2$.

Theorem 4a: For a single-input plant (E, A, B, C), and suppose $(E + BF_1)^{-1}$ exists. Then, $F_1(E + BF_1)^{-1}B = 1$ if and only if $\det(E) = 0$.

$$
\begin{bmatrix} \tilde{V}(s) & -\tilde{U}(s) \\ -\tilde{N}(s) & \tilde{M}(s) \end{bmatrix} = \left[\begin{array}{c|cc} s(E + L_1 C) - (A + L_2 C) & -B & -(sL_1 - L_2) \\ \hline F & I & 0 \\ C & 0 & I \end{array} \right] \tag{6}
$$

$$
\begin{bmatrix} M(s) & U(s) \\ N(s) & V(s) \end{bmatrix} = \left[\begin{array}{c|cc} sE - A - BF & B & sL_1 - L_2 \\ \hline F & I & 0 \\ C & 0 & I \end{array} \right]. \tag{7}
$$

Proof: We will prove the result for two-dimensional plant, which can be generalized to n-dimensional plant readily. Let $E = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}$, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, $F_1 = (\, f_1^1, \quad f_2^1 \,)$, then

$$E + BF_1 = \begin{pmatrix} e_{11} + b_1 f_1^1 & e_{12} + b_1 f_2^1 \\ e_{21} + b_2 f_1^1 & e_{22} + b_2 f_2^1 \end{pmatrix} \tag{17}$$

$$\det(E + BF_1) = \det \begin{pmatrix} e_{11} & b_1 f_2^1 \\ e_{21} & b_2 f_2^1 \end{pmatrix} + \det \begin{pmatrix} b_1 f_1^1 & e_{12} \\ b_2 f_1^1 & e_{22} \end{pmatrix}$$
$$+ \det \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}$$
$$= \det \begin{pmatrix} 1 & -f_1^1 & -f_2^1 \\ b_1 & e_{11} & e_{12} \\ b_2 & e_{21} & e_{22} \end{pmatrix} \tag{18}$$

$$(E + BF_1)^{-1} = \frac{1}{\det(E + BF_1)} \begin{pmatrix} e_{11}^{\star} & e_{21}^{\star} \\ e_{12}^{\star} & e_{22}^{\star} \end{pmatrix} \tag{19}$$

where e_{ij}^* $(i, j \in \{1, 2\})$, is the cofactor of $e_{ij} + b_i f_j^1$, that is, $(-1)^{i+j} \det[(E + BF_1)(i|j)]$, where $(E + BF_1)(i|j)$ stands for the submatrix of $(E + BF_1)$ obtained by deleting row i and column j[6]. Now, it follows that

$$F_1(E + BF_1)^{-1} B$$
$$= \frac{1}{\det(E + BF_1)} (\, f_1^1, \quad f_2^1 \,) \begin{pmatrix} e_{11}^{\star} & e_{21}^{\star} \\ e_{12}^{\star} & e_{22}^{\star} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$
$$= \frac{1}{\det(E + BF_1)}$$
$$\times \left(b_1 f_1^1 e_{11}^{\star} + b_1 f_2^1 e_{12}^{\star} + b_2 f_1^1 e_{21}^{\star} + b_2 f_2^1 e_{22}^{\star} \right)$$
$$= \frac{1}{\det(E + BF_1)}$$
$$\times \left[f_1^1 \det \begin{pmatrix} b_1 & e_{12} + b_1 f_2^1 \\ b_2 & e_{22} + b_2 f_2^1 \end{pmatrix} \right.$$
$$\left. + f_2^1 \det \begin{pmatrix} e_{11} + b_1 f_1^1 & b_1 \\ e_{21} + b_2 f_1^1 & b_2 \end{pmatrix} \right]$$
$$= \frac{1}{\det(E + BF_1)} \det \begin{pmatrix} 0 & -f_1^1 & -f_2^1 \\ b_1 & e_{11} + b_1 f_1^1 & e_{12} + b_1 f_2^1 \\ b_2 & e_{21} + b_2 f_1^1 & e_{22} + b_2 f_2^1 \end{pmatrix}$$
$$= \frac{1}{\det(E + BF_1)} \det \begin{pmatrix} 0 & -f_1^1 & -f_2^1 \\ b_1 & e_{11} & e_{12} \\ b_2 & e_{21} & e_{22} \end{pmatrix}. \tag{20}$$

In comparison with (18) and (20), one has

$$F_1(E + BF_1)^{-1} B$$
$$= \frac{\det \begin{pmatrix} 0 & -f_1^1 & -f_2^1 \\ b_1 & e_{11} & e_{12} \\ b_2 & e_{21} & e_{22} \end{pmatrix}}{\det \begin{pmatrix} 0 & -f_1^1 & -f_2^1 \\ b_1 & e_{11} & e_{12} \\ b_2 & e_{21} & e_{22} \end{pmatrix} + \det \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}} \tag{21}$$

which implies that $F_1(E + BF_1)^{-1} B = 1$ if and only if $\det(E) = \det \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} = 0$. This completes the proof.

Remark 3: Theorem 4a shows that $M(s)$ and $N(s)$ in (16) are both strictly proper and stable for single-input generalized plant with singular E. This result paves a way for obtaining proper stable doubly coprime factorizations from Theorem 2a.

Theorem 5a: Consider a completely stabilizable and properly detectable single-input plant (1) with E being singular. Choose $F_1, F_2 \in \mathcal{R}^{1 \times n}$, $L \in \mathcal{R}^{n \times p}$ such that Conditions 1) and 2) hold in Theorem 2a. Let α be a negative-real number. Define (22)–(23), as shown at the bottom of the next page. Then, all the corresponding conclusions specified in a)–d) of Theorem 2a still hold.

Proof: Note that in Theorem 2a, $\tilde{N}(s), \tilde{M}(s), U(s), V(s)$ are all proper stable, while $\tilde{V}(s) \in \mathcal{S}$, $\tilde{U}(s) \in \mathcal{S}^{1 \times p}$, $M(s) \in \mathcal{RH}_2$, and $N(s) \in \mathcal{RH}_2^{p \times 1}$ from the aforementioned analysis. Let

$$\tilde{V}_{ma}(s) = (s - \alpha)^{-1} \tilde{V}(s)$$
$$\tilde{U}_{ma}(s) = (s - \alpha)^{-1} \tilde{U}(s)$$
$$M_{ma}(s) = M(s)(s - \alpha)$$
$$N_{ma}(s) = N(s)(s - \alpha). \tag{24}$$

Then, $\tilde{V}_{ma}(s), \tilde{U}_{ma}(s), M_{ma}(s), N_{ma}(s)$ are all proper stable. From (24) and the conclusions b)–d) of Theorem 2a, it can be shown easily that $M_{ma}(s)$ is nonsingular, $N_{ma}(s) M_{ma}^{-1}(s) = G(s)$ and

$$\begin{bmatrix} \tilde{V}_{ma}(s) & -\tilde{U}_{ma}(s) \\ -\tilde{N}(s) & \tilde{M}(s) \end{bmatrix} \cdot \begin{bmatrix} M_{ma}(s) & U(s) \\ N_{ma}(s) & V(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{25}$$

From (3), one has

$$(\, \tilde{V}_{ma}(s), \quad -\tilde{U}_{ma}(s) \,) = (s - \alpha)^{-1} (\, \tilde{V}(s), \quad -\tilde{U}(s) \,)$$
$$= (s - \alpha)^{-1} \left[\begin{array}{c|cc} sE - A - LC & -B & L \\ \hline -(sF_1 - F_2) & 1 & 0 \end{array} \right]$$
$$= \left[\begin{array}{cc|cc} s - \alpha & sF_1 - F_2 & 1 & 0 \\ \hline 0 & sE - A - LC & -B & L \\ \hline 1 & 0 & 0 & 0 \end{array} \right]. \tag{26}$$

Using (16) and noting that $1 - F_1(E + BF_1)^{-1} B = 0$, then (27), shown at the bottom of the next page, holds.

By substituting $(\, \tilde{V}_{ma}(s), \quad -\tilde{U}_{ma}(s) \,)$ in (26), $(-\tilde{N}(s), \quad \tilde{M}(s))$ in (3), $\begin{pmatrix} M_{ma}(s) \\ N_{ma}(s) \end{pmatrix}$ in (27), $\begin{pmatrix} U(s) \\ V(s) \end{pmatrix}$ in (16), respectively, into $\begin{bmatrix} \tilde{V}_{ma}(s) & -\tilde{U}_{ma}(s) \\ -\tilde{N}(s) & \tilde{M}(s) \end{bmatrix}$ and $\begin{bmatrix} M_{ma}(s) & U(s) \\ N_{ma}(s) & V(s) \end{bmatrix}$, one completes the proof.

Remark 4: Theorem 5a is an improved modification of Theorem 2a for single-input plants with E singular, because all transfer functions in (22)–(23) are proper and stable. Hence, two parameterizations of all causal properly stabilizing controllers can be obtained, which

$$\begin{bmatrix} M(s) & U(s) \\ N(s) & V(s) \end{bmatrix} = \left[\begin{array}{c|cc} s(E + BF_1) - (A + BF_2) & B & -L \\ \hline -F_c & I - F_1(E + BF_1)^{-1} B & F_1(E + BF_1)^{-1} L \\ C & 0 & I \end{array} \right] \tag{16}$$

read as shown in (28)–(29) at the bottom of the page. Note that if a) the condition $\det[(\tilde{V}_{ma}(s) + Q(s)\tilde{N}(s))(\infty)] \neq 0$ is replaced by $\det[\tilde{V}_{ma}(s) + Q(s)\tilde{N}(s)] \not\equiv 0$ in (28), and b) a similar manipulation is made in (29), then one gets two parameterizations of all properly stabilizing controllers.

B. Single-Output Plant Case

Similar to the discussion above for single-input plant, one can show the following results for single-output plant without proofs.

Theorem 4b: For a single-output plant (E, A, B, C), and assuming $(E + L_1 C)^{-1}$ exists, then $C(E + L_1 C)^{-1} L_1 = 1$ if and only if det $(E) = 0$.

Theorem 5b: Consider a properly stabilizable and completely detectable single-output plant (1) with E being singular. Choose $F \in \mathcal{R}^{m \times n}$, $L_1, L_2 \in \mathcal{R}^{n \times 1}$ such that Conditions 1) and 2) hold in The-

orem 2b. Let α be a negative real number. Define (30)–(31), shown at the top of the next page, where $L_c = (A+L_2 C)(E+L_1 C)^{-1} L_1 - L_2$. Then, all the corresponding conclusions specified in a)–d) of Theorem 2a still hold.

Remark 5: Theorem 5b is an appropriate modification of Theorem 2b for single-output plants with E singular, because all transfer functions in (30)–(31) are proper stable. Hence, two parameterizations of all causal properly stabilizing controllers can be obtained, which read as

$$K(s) = \left\{ \left[\tilde{V}(s) + Q(s)\tilde{N}_{mb}(s) \right]^{-1} \left[\tilde{U}(s) + Q(s)\tilde{M}_{mb}(s) \right] \; \middle| \right.$$
$$\left. Q(s) \in \mathcal{RH}_{\infty}^{m \times 1}, \det\left[(\tilde{V}(s) + Q(s)\tilde{N}_{mb}(s))(\infty) \right] \neq 0 \right\} \quad (32)$$

$$\begin{bmatrix} \tilde{V}_{ma}(s) & -\tilde{U}_{ma}(s) \\ -\tilde{N}(s) & \tilde{M}(s) \end{bmatrix} = \left[\begin{array}{cc|cc} s - \alpha & sF_1 - F_2 & 1 & 0 \\ 0 & sE - A - LC & -B & L \\ \hline 1 & 0 & 0 & 0 \\ 0 & C & 0 & I \end{array} \right] \quad (22)$$

$$\begin{bmatrix} M_{ma}(s) & U(s) \\ N_{ma}(s) & V(s) \end{bmatrix} = \left[\begin{array}{c|cc} s(E + BF_1) - (A + BF_2) & -\alpha B + (A + BF_2)(E + BF_1)^{-1}B & -L \\ \hline -F_c & -F_c(E + BF_1)^{-1}B & F_1(E + BF_1)^{-1}L \\ C & C(E + BF_1)^{-1}B & I \end{array} \right]. \quad (23)$$

$$\begin{pmatrix} M_{ma}(s) \\ N_{ma}(s) \end{pmatrix} = \begin{pmatrix} M(s) \\ N(s) \end{pmatrix}(s - \alpha)$$
$$= \left[\begin{array}{c|c} s(E + BF_1) - (A + BF_2) & B \\ \hline -F_c & 0 \\ C & 0 \end{array} \right](s - \alpha)$$
$$= \left[\begin{array}{c|c} s(E + BF_1) - (A + BF_2) & -\alpha B + (A + BF_2)(E + BF_1)^{-1}B \\ \hline -F_c & -F_c(E + BF_1)^{-1}B \\ C & C(E + BF_1)^{-1}B \end{array} \right]. \quad (27)$$

$$K(s) = \left\{ \left[\tilde{V}_{ma}(s) + Q(s)\tilde{N}(s) \right]^{-1} \left[\tilde{U}_{ma}(s) + Q(s)\tilde{M}(s) \right] \; \middle| \right.$$
$$\left. Q(s) \in \mathcal{RH}_{\infty}^{1 \times p}, \det\left[(\tilde{V}_{ma}(s) + Q(s)\tilde{N}(s))(\infty) \right] \neq 0 \right\} \quad (28)$$
$$= \left\{ [U(s) + M_{ma}(s)Q(s)] [V(s) + N_{ma}(s)Q(s)]^{-1} \; \middle| \right.$$
$$\left. Q(s) \in \mathcal{RH}_{\infty}^{1 \times p}, \det[(V(s) + N_{ma}(s)Q(s))(\infty)] \neq 0 \right\}. \quad (29)$$

$$\begin{bmatrix} \tilde{V}(s) & -\tilde{U}(s) \\ -\tilde{N}_{mb}(s) & \tilde{M}_{mb}(s) \end{bmatrix} = \left[\begin{array}{c|cc} s(E+L_1C)-(A+L_2C) & -B & -L_c \\ \hline F & I & -F(E+L_1C)^{-1}L_1 \\ -\alpha C + C(E+L_1C)^{-1}(A+L_2C) & -C(E+L_1C)^{-1}B & -C(E+L_1C)^{-1}L_c \end{array} \right] \tag{30}$$

$$\begin{bmatrix} M(s) & U_{mb}(s) \\ N(s) & V_{mb}(s) \end{bmatrix} = \left[\begin{array}{cc|cc} sE-A-BF & -(sL_1-L_2) & B & 0 \\ 0 & s-\alpha & 0 & 1 \\ \hline F & 0 & I & 0 \\ C & 1 & 0 & 0 \end{array} \right] \tag{31}$$

$$= \left\{ [U_{mb}(s)+M(s)Q(s)]\,[V_{mb}(s)+N(s)Q(s)]^{-1} \mid \right.$$
$$\left. Q(s) \in \mathcal{RH}_\infty^{m\times 1},\ \det\left[(V_{mb}(s)+N(s)Q(s))(\infty)\right] \neq 0 \right\}. \tag{33}$$

C. MIMO Case

In this section, we shall extend our work for single-input and single-output systems to the MIMO case.

Theorem 6a: Consider a triple (E,A,B) with $E,A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$, $q := \mathrm{rank}(E)$, and $\mathrm{rank}(B) = m \leq n$. Assume there exists $F_1 \in \mathcal{R}^{m\times n}$ such that $E+BF_1$ is nonsingular. Then $F_1(E+BF_1)^{-1}B = I_m$ if and only if $q = n-m$.

Proof: Let $U := (E+BF_1)^{-1}$ and $V := F_1U$, it is clear that $EU + BV = I_n$.

Necessity. Suppose $F_1(E+BF_1)^{-1}B = I_m$, i.e., $VB = I_m$, which implies $EUB = 0$. Then $Im(B) \subset Ker(EU)$. Thus $m = \mathrm{rank}(B) \leq n-q$. Notice that, $(E+BF_1)^{-1}$ exists if and only if $\mathrm{rank}(E,B) = n$, i.e., $Im(E)+Im(B) = \mathcal{R}^n$, giving $m \geq n-q$. Therefore, one has shown $m = n-q$, equivalently $q = n-m$.

Sufficiency. Suppose $q = n-m$. Since $\mathrm{rank}(E,B) = n$, there exists a nonsingular matrix $P \in \mathcal{R}^{n\times n}$ such that

$$(\tilde{E},\tilde{B}) := P(E,B) = \begin{pmatrix} E_1 & 0 \\ 0 & I_m \end{pmatrix}$$

where $E_1 \in \mathcal{R}^{(n-m)\times n}$ has full-row rank. Next, put

$$\tilde{U} := UP^{-1} = (\,U_1 \quad U_2\,), \text{ and } \tilde{V} := VP^{-1} = (\,V_1 \quad V_2\,)$$

where the first and second submatrices have respectively $n-m$ and m columns. Then, $EU+BV = I_n$ implies $\tilde{E}\tilde{U} + \tilde{B}\tilde{V} = I_n$ giving $E_1U_1 = I_{n-m}$, $E_1U_2 = 0$, $V_1 = 0$ and $V_2 = I_m$. Thus

$$F_1(E+BF_1)^{-1}B = VB = \tilde{V}\tilde{B} = (\,0 \quad I_m\,)\begin{pmatrix} 0 \\ I_m \end{pmatrix} = I_m.$$

Remark 6: $\mathrm{rank}(B) = m$ is a necessary condition for the validity of $F_1(E+BF_1)^{-1}B = I_m$. Since this condition is standard in control, it has been added to the assumptions. For $m = 1$, the theorem above coincides with the result presented by Theorem 4a for single-input case. Based on Theorem 6a, the *proper stable* doubly coprime factorizations of Theorem 5a can be extended to multi-input plants with $\mathrm{rank}(B) = m \leq n$ and $\mathrm{rank}(E) = n-m$. Specifically, we can replace 1, $s-\alpha$,

αB in Theorem 5a, respectively, with I, $sI - \Lambda_a$, $B\Lambda_a$ for multi-input systems, where $\Lambda_a \in \mathcal{R}^{m\times m}$ is a stable matrix. The resulting parameterizations of all causal properly stabilizing controllers can be obtained in the forms (28)–(29), where $Q(s) \in \mathcal{RH}_\infty^{1\times p}$ is replaced by $Q(s) \in \mathcal{RH}_\infty^{m\times p}$.

Theorem 6b: Consider a triple (E,A,C) with $E,A \in \mathcal{R}^{n\times n}$, $C \in \mathcal{R}^{p\times n}$, $q := \mathrm{rank}(E)$, and $\mathrm{rank}(C) = p \leq n$. Assume there exists $L_1 \in \mathcal{R}^{n\times p}$ such that $E+L_1C$ is nonsingular. Then, $C(E+L_1C)^{-1}L_1 = I_p$ if and only if $q = n-p$.

Proof: This proof is similar to that for Theorem 6a, so it is omitted.

Remark 7: Based on Theorem 6b, the result of Theorem 5b can be extended to multi-output plants with $\mathrm{rank}(C) = p \leq n$ and $\mathrm{rank}(E) = n-p$. Specifically, we can replace 1, $s-\alpha$, αC in Theorem 5b respectively with I, $sI - \Lambda_b$, $\Lambda_b C$ for multi-output systems, where $\Lambda_b \in \mathcal{R}^{p\times p}$ is a stable matrix. The resulting parameterizations of all causal properly stabilizing controllers can also be presented in the forms (32)–(33), where $Q(s) \in \mathcal{RH}_\infty^{m\times 1}$ is replaced by $Q(s) \in \mathcal{RH}_\infty^{m\times p}$.

V. CONCLUSION

For generalized dynamical systems (E,A,B,C) with E singular, which are either single-input or single-output, proper stable doubly coprime factorizations are obtained by using proportional and derivative state feedback (or output injection), enabling the parameterization of all causal properly stabilizing unity feedback controllers. The extensions for the multi-input case and the multi-output case are also presented. The results of [5] are hereby significantly improved.

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