# Robust sliding mode control for a class of underactuated systems with mismatched uncertainties

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**Abstract:** Based on the sliding mode control methodology, this paper presents a robust control strategy for underactuated systems with mismatched uncertainties. The system consists of a nominal system and the mismatched uncertainties. Since the nominal system can be considered to be made up of several subsystems, a hierarchical structure for the sliding surfaces is designed. This is achieved by taking the sliding surface of one of the subsystems as the first-layer sliding surface and using this sliding surface and the sliding surface of another subsystem to construct the second-layer sliding surface. This process continues till the sliding surfaces of all the subsystems are included. A lumped sliding mode compensator is designed at the last-layer sliding surface. The asymptotic stability of all of the layer sliding surfaces and the sliding surface of each subsystem is proven. Simulation results show the validity of this robust control method through stabilization control of a system consisting of two inverted pendulums and mismatched uncertainties.

**Keywords:** robust control, sliding mode, underactuated system, uncertainty

## 1 INTRODUCTION

Mechanical systems that possess a lower number of control inputs than the number of degrees of freedom to be controlled are called underactuated systems. They arise in numerous situations. Some undesired properties of their dynamics, such as non-linearities, non-holonomic constraints, and couplings, make their control design difficult [**1**].

In recent years, there has been increasing interest in the control problem of underactuated systems. Reyhanoglu *et al.* [**2**] and Bloch *et al.* [**3**] established a theoretical framework for the control of underactuated systems and proved that a class of underactuated systems without the gravity term could not be smoothly stabilized. Yabuno *et al.* [**4**] pointed out that there exists a reachable and stabilizable area of an underactuated manipulator without state-feedback control. Tarn *et al.* [**5**] also gave some theoretical results about the output regulation of underactuated systems. Shiriaev *et al.* [**6**] and Kolesnichenko and Shiriaev [**7**] presented a periodic motion planning and a partial stabilization method for underactuated Euler–Lagrange systems, respectively. Control methods for underactuated vessels [**8**, **9**], underactuated fingers [**10**], underactuated vehicles [**11**], underactuated mobile robots [**12**], and underactuated manipulators [**13**] have also been reported.

This paper focuses on a class of underactuated systems. This class that has a control input and multiple outputs is rather large, and includes Acrobot [**1**], inverted pendulum systems [**14–17**], TORA [**18**], ball–beam systems [**15**], Pendubots [**5**, **19**], etc. They are often used as test beds for research on non-linear control and education on various concepts, because they are simple enough to permit complete dynamic analyses and experiments, but there exist strong non-linearities and dynamic couplings. Fang *et al.* [**20**] presented a non-linear coupling control approach for an overhead crane system. Wai *et al.* [**17**] proposed a cascade adaptive

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fuzzy sliding mode control scheme including inner and outer control loops which was used for stabilizing and tracking control of a non-linear two-axis inverted-pendulum servomechanism. Zhang and Tarn [**19**] proposed a hybrid switching control strategy for a Pendubot. Tsai *et al.* [**21**] used a neuro-sliding mode controller to solve the stabilization control problem of a see-saw system. In fact, a canonical state space expression can depict the class. This paper is focused on the development of a general control method for this class of systems.

Uncertainties often exist because of external and internal disturbances, which make the control problems of the class more complicated. Sliding mode control (SMC) is a powerful and robust non-linear feedback control method [**22**]. The sliding mode controller is insensitive to system parameter changes or external disturbances when system states continue to slide on a sliding surface. Under matched conditions, SMC can deal with matched uncertainties effectively (this is the invariable characteristic of SMC [**22**]). It provides a good candidate for a control design of this class. Some papers concerning the control of underactuated systems using the SMC approach have been published in the last few years. There exist two crucial issues associated with the applications of SMC to the class of underactuated systems with mismatched uncertainties. One issue is how to design a suitable sliding-surface structure, because the parameters of the sliding surface of underactuated systems cannot be calculated directly using the Hurwitz condition as for linear systems [**23**]. The other issue is how to handle the mismatched uncertainties, because most physical systems do not satisfy the matched condition of SMC in practice. In this paper, a structure design for the sliding surfaces for the class is proposed and attempts are made to deal with mismatched uncertainties that cannot be destroyed by the invariable characteristic of the sliding mode.

As far as physical structure is concerned, the class of mechanical systems consists of several subsystems. In light of such physical structure, some control methods based on the SMC approach have been presented. Xu and Özgüner [**24**] proposed a SMC approach to stabilize the parts of the class that exist in cascaded form. Martinez *et al.* [**25**] developed a hybrid control synthesis by SMC for some two-degrees-of-freedom (DOF) underactuated systems of the class with Coulomb friction in the joints. However, the methods in references [**24**] and [**25**] were only for second-order underactuated systems with two-DOF (four states) and one input. Lin and Mon [**16**] proposed a hierarchical fuzzy SMC scheme. However, they did not consider the stability of the subsystem sliding surfaces. Lo and Kuo [**15**], designed a decoupled fuzzy SMC law. Unfortunately, it could only be applied to two-level control. Wang *et al.* [**26**] developed a cascade SMC approach. However, some controller parameters needed to be frequently switched to guarantee the system stability. This might make it difficult to select a group of suitable controller parameters. Wang *et al.* [**23**] designed a hierarchical sliding mode controller for all stable sliding surfaces, but the method could not be generalized to underactuated systems with more than two subsystems. In all of these papers, hierarchical sliding surfaces were designed instead of a conventional single-layer sliding surface. An advantage of the hierarchical structure is that the sliding surface of each subsystem can be designed as a second-order system. This provides the opportunity to choose the parameters of the subsystem sliding surface. In order to deal with mismatched uncertainties, distributed compensators were designed in references [**15**] and [**16**] that compensated the uncertainties at every layer sliding surface, but this made their controller structures complex. Both Wang *et al.* [**23**] and Xu and Özgüner [**24**] considered uncertain systems in their theoretical analyses, but they were unable to show the robustness of their controllers in their simulation studies. Moreover, only systems without uncertainties were considered in references [**25**] and [**26**].

In this paper, a robust controller based on the SMC methodology is presented for a class of underactuated systems with mismatched uncertainties, which consist of a nominal system and the mismatched uncertainties. The hierarchical structure of the sliding surfaces is designed based on the structural characteristics of the nominal system. For the mismatched uncertainties, a lumped sliding mode compensator is designed at the last-layer sliding surface. The asymptotic stability of all the sliding surfaces is proven. Simulation results show the validity of this robust control method by stabilizing a system consisting of two inverted pendulums with mismatched uncertainties. This paper is organized as follows. Section 2 describes the proposed control method. A stability analysis and simulation results are presented in sections 3 and 4, respectively. Finally, conclusions are drawn in section 5.

## 2 CONTROL STRATEGY DESIGN

The class of underactuated systems with mismatched uncertainties can be expressed in a state space representation in the form of

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f_1(\boldsymbol{X}) + b_1(\boldsymbol{X})u + d_1 \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = f_2(\boldsymbol{X}) + b_2(\boldsymbol{X})u + d_2 \\ \quad \vdots \\ \dot{x}_{2n-1} = x_{2n} \\ \dot{x}_{2n} = f_n(\boldsymbol{X}) + b_n(\boldsymbol{X})u + d_n \end{cases} \tag{1}$$

where $\boldsymbol{X} = [x_1, x_2, \ldots, x_{2n}]^{\mathrm{T}}$ is the state variable vector, $f_i(\boldsymbol{X})$ and $b_i(\boldsymbol{X})$ $(i = 1, 2, \ldots, n)$ are the nonlinear functions of the state variables, they are abbreviated to $f_i$ and $b_i$, $d_i$ consists of the mismatched uncertainties, including system uncertainties and external disturbances and $d_i$ is bounded by $|d_i| \leqslant \bar{d}_i$ where $\bar{d}_i$ is a known and positive constant, and $u$ is the single control input.

Equation (1) is the normal form of single-input multiple-output (SIMO) underactuated systems with mismatched uncertainties that can be described by $n$, $f_i$, $b_i$, and $d_i$. If $n = 2$, it can represent the Acrobot, TORA, and single inverted-pendulum system; if $n = 3$, it can express the two-inverted-pendulum system; if $n = 4$, it can be considered as three-inverted-pendulum system; etc.

Let $d_i = 0$, then equation (1) can be treated as the nominal system of an underactuated system. The physical structure of the nominal system means that it can be treated as being made up of several subsystems. For example, a three-inverted-pendulum system consists of four subsystems: the upper pendulum, the middle pendulum, the lower pendulum, and the cart. Using this viewpoint, it is considered that a hierarchical sliding mode controller can be designed for the nominal system and that a lumped sliding mode compensator can be designed for the mismatched uncertainties. The hierarchical sliding mode controller and the lumped sliding mode compensator work together to create robust control for the class with mismatched uncertainties. In the following two subsections, the design of the robust controller using the SMC methodology will be presented for a system such as equation (1) in a step-by-step manner.

### 2.1 Hierarchical SMC for the nominal system

The nominal system can represent $n$ subsystems with a second-order canonical form in terms of the physical structural characteristic of an underactuated system. The state variables $(x_{2i-1}, x_{2i})$ of the $i$th group can be treated as the states of the $i$th subsystem. And the state space expression of the $i$th subsystem can be described as

$$\begin{cases} \dot{x}_{2i-1} = x_{2i} \\ \dot{x}_{2i} = f_i + b_i u \end{cases} \tag{2}$$

The $i$th subsystem is a second-order system. To be stable its sliding surface must be in the second and fourth quadrants. Thus, define its sliding surface (a linear function) as

$$s_i = c_i x_{2i-1} + x_{2i} \tag{3}$$

where $c_i$ is a positive constant, and is simply the slope of the sliding surface of this second-order system. Differentiating $s_i$ with respect to time $t$ in equation (3) results in

$$\dot{s}_i = c_i \dot{x}_{2i-1} + \dot{x}_{2i} \tag{4}$$

Substituting equation (2) into equation (4) gives

$$\dot{s}_i = c_i x_{2i} + f_i + b_i u \tag{5}$$

Let $\dot{s}_i = 0$, then the equivalent control of the $i$th subsystem can be obtained as

$$u_{\mathrm{eqi}} = -(c_i x_{2i} + f_i)/b_i \tag{6}$$

This equivalent control $u_{\mathrm{eqi}}$ is used to guarantee the $i$th subsystem goes to its subsystem origin when the $i$th subsystem states slides on its subsystem sliding surface.

Depending on the combination of the subsystem sliding surfaces, a variety of hierarchical SMC laws can be designed [**23**, **24**, **26**]. Also, other control methods can be combined with the hierarchical SMC method [**15–17**, **21**]. In this paper, the hierarchical structure of the sliding surfaces is designed in the following manner. The sliding surface of one subsystem is chosen as the first-layer sliding surface $S_1$. Then $S_1$ is used to construct the second-layer sliding surface $S_2$, together with the sliding surface of another subsystem. This process continues until all the subsystem sliding surfaces are included. Without loss of generality, the subsystem sliding surface $s_1$ is selected as $S_1$. The hierarchical structure of the sliding surfaces is shown in Fig. 1.

In the presented hierarchical structure, it is known that the $i$th-layer sliding surface includes information on the $i$th subsystem sliding surface and the other $i - 1$ lower layer sliding surfaces. As a result,

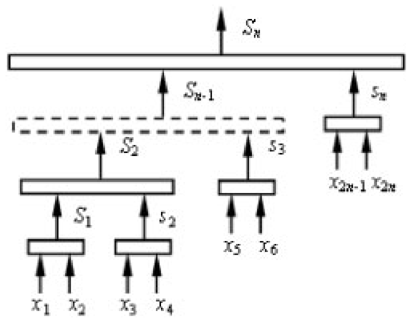


**Fig. 1** Structure of the hierarchical sliding mode surfaces

the $i$th-layer sliding surface $S_i$ and its control law $u_i$ can be defined as follows

$$S_i = \lambda_{i-1} S_{i-1} + s_i \tag{7}$$

$$u_i = u_{i-1} + u_{\mathrm{eq}i} + u_{\mathrm{sw}i} \tag{8}$$

where $\lambda_{i-1}(i = 1, 2, \ldots, n)$ is a constant, $\lambda_0$, $S_0$, $u_0$ are defined to have a value of zero, $u_{\mathrm{sw}i}$ $(i = 1, 2, \ldots, n)$ is the switching control of the $i$th-layer sliding surface. From the recursive formulas (7) and (8), it is possible to write that

$$S_i = \sum_{r=1}^{i} \left( \prod_{j=r}^{i} a_j \right) s_r \tag{9}$$

$$u_i = \sum_{r=1}^{i} \left( u_{\mathrm{sw}r} + u_{\mathrm{eq}r} \right) \tag{10}$$

here $a_j = \lambda_j$ $(j \neq i)$ is a constant for a given $i$, and $a_j = 1$ $(j = i)$. The control law can be derived using the Lyapunov theorem. The Lyapunov function of the $i$th layer is selected as

$$V_i(t) = S_i^2/2 \tag{11}$$

Differentiating $V_i$ with respect to time $t$ results in

$$\dot{V}_i = S_i \dot{S}_i \tag{12}$$

Differentiating $S_i$ with respect to time $t$ in equation (9) and substituting it into equation (12) yields

$$\dot{V}_i = S_i \dot{S}_i = S_i \left[ \sum_{r=1}^{i} \left( \prod_{j=r}^{i} a_j \right) \dot{s}_r \right] \tag{13}$$

Substituting equations (2), (4), and (10) into equation (13) results in

$$\begin{aligned} \dot{V}_i &= S_i \left\{ \sum_{r=1}^{i} \left[ \left( \prod\nolimits_{j=r}^{i} a_j \right) \times \left( c_r x_{2r} + f_r + b_r u_i \right) \right] \right\} \\ &= S_i \left\{ \sum_{r=1}^{i} \left[ \left( \prod_{j=r}^{i} a_j \right) \times b_r \right. \right. \\ &\quad \left. \left. \times \left( \sum_{\substack{l=1 \\ l \neq r}}^{i} u_{\mathrm{eq}l} + \sum_{l=1}^{i} u_{\mathrm{sw}l} \right) \right] \right\} \\ &= S_i \left\{ \sum_{l=1}^{i} \left[ \sum_{\substack{r=1 \\ r \neq l}}^{i} \left( \prod_{j=r}^{i} a_j \right) b_r \right] \times u_{\mathrm{eq}l} \right. \\ &\quad \left. + \sum_{l=1}^{i} \left[ \sum_{r=1}^{i} \left( \prod_{j=r}^{i} a_j \right) b_r \right] \times u_{\mathrm{sw}l} \right\} \end{aligned} \tag{14}$$

In order to have the stability of the $i$th-layer sliding surface, let

$$\begin{aligned} &\sum_{l=1}^{i} \left[ \sum_{\substack{r=1 \\ r \neq l}}^{i} \left( \prod_{j=r}^{i} a_j \right) b_r \right] u_{\mathrm{eq}l} \\ &\quad + \sum_{l=1}^{i} \left[ \sum_{r=1}^{i} \left( \prod_{j=r}^{i} a_j \right) b_r \right] u_{\mathrm{sw}l} = -k_i S_i - \eta_i \operatorname{sgn} S_i \end{aligned} \tag{15}$$

where $k_i$ and $\eta_i$ are positive constants. The switching control law $u_{\mathrm{sw}i}$ is obtained from equation (15)

$$\begin{aligned} u_{\mathrm{sw}i} = &- \sum\nolimits_{l=1}^{i-1} u_{\mathrm{sw}l} - \frac{\sum_{l=1}^{i} \left[ \sum_{\substack{r=1 \\ r \neq l}}^{i} \left( \prod_{j=r}^{i} a_j \right) b_r \right] u_{\mathrm{eq}l}}{\sum_{r=1}^{i} \left( \prod_{j=r}^{i} a_j \right) b_r} \\ &- \frac{k_i S_i + \eta_i \operatorname{sgn} S_i}{\sum_{r=1}^{i} \left( \prod_{j=r}^{i} a_j \right) b_r} \end{aligned} \tag{16}$$

Substituting equation (16) into equation (10) and letting $i = n$, the hierarchical SMC law can be obtained as

$$\begin{aligned} u_n &= \sum_{l=1}^{n-1} u_{\mathrm{sw}l} + u_{\mathrm{sw}n} + \sum_{l=1}^{n} u_{\mathrm{eq}l} \\ &= \frac{\sum_{r=1}^{n} \left( \prod_{j=r}^{n} a_j \right) b_r u_{\mathrm{eq}r}}{\sum_{r=1}^{n} \left( \prod_{j=r}^{n} a_j \right) b_r} - \frac{k_n S_n + \eta_n \operatorname{sgn} S_n}{\sum_{r=1}^{n} \left( \prod_{j=r}^{n} a_j \right) b_r} \end{aligned} \tag{17}$$

*Remark 1*

Although the methods in [**23**], [**24**], and [**26**] use a hierarchical SMC, the hierarchical structures have some differences between themselves. In [**23**] the hierarchical structure consists of only two levels, and is designed for second-order systems. The structure in [**26**] generalizes the one in [**23**] for underactuated systems with $n$ subsystems. Two-layer sliding surfaces still exist in [**23**], with the second layer being the aggregate of the sliding surfaces of the $n$ subsystems defined as the first layer. The sliding surface of [**24**] has a single-layer structure and is defined on the basis of a cascade model. The structure with $n$ layers shown in Fig. 1 and used in this paper is different from all of them.

*Remark 2*

Equation (17) is a recursive formula. As equation (17) has shown, only the switching control of the last-layer sliding mode controller works and the switching controls of the other $n-1$ layers are merged during the derivation. If any state deviates from its sliding surface in a dynamic process, then the switching control of the last layer will drive it back to its own sliding surface. This makes the system states slide on the last-layer sliding surface. Moreover, the states of every subsystem continue to slide on their own sliding surface due to the control action of its own subsystem equivalent control in equation (6).

### 2.2 Compensator for mismatched uncertainties

The proposed hierarchical sliding mode controller is insensitive to matched uncertainties because of the invariant characteristic of SMC [**22**]. However, mismatched uncertainties can not be handled using this invariant characteristic. In this section a sliding mode compensator that is able to resist mismatched uncertainties is designed. To the best of the authors' knowledge, this is the first time that such a lumped sliding compensator for mismatched uncertainties based on the hierarchical structure of Fig. 1 has been designed.

Generally speaking, there are two methods to design a compensator for the hierarchical sliding surfaces. One is to design a distributed compensator and compensate the mismatched uncertainties at every layer of the sliding mode surface [**15**, **16**]. Two disadvantages of this idea are that this makes the controller structure complex and that if the compensator at a lower layer does not eliminate the uncertainties, it will affect the stabilities of higher layers. The other method is to design a lumped compensator and compensate the mismatched uncertainties at the last layer. Its advantage is that this method simplifies the control design. Thus, a lumped sliding mode compensator at the last layer is designed in this work. Based on the above viewpoints, the lumped sliding mode compensator can be defined as

$$u_{cn} = \frac{\sum_{r=1}^{n}\left(\prod_{j=r}^{n} a_j\right)\bar{d}_r}{\sum_{r=1}^{n}\left(\prod_{j=r}^{n} a_j\right)b_r} \tag{18}$$

*Remark 3*

According to the invariable characteristics of SMC, the matched part in equation (1) could be handled by the hierarchical sliding mode controller using the nominal system designed in Section 2.1. However, there exists a mismatched part in equation (1), which means that the output of each subsystem contains uncertainties that cannot be removed using a controller designed for the nominal system. A special compensator is needed for this condition. Using this lumped compensator all the mismatched uncertainties are removed at the last layer so that the mismatched uncertainties in every subsystem can be treated as if they are added to the last-layer sliding surface where they are compensated. Therefore, the qualitative stability analysis of the $n-1$ lower layers of the nominal system can represent the system with uncertainties as for equation (1), but the uncertainties actually exist so that the system output in equation (1) contains them, which is different from the nominal system.

### 2.3 Total control law

For the control design, the hierarchical sliding mode controller for the nominal system and the lumped sliding compensator of the mismatched uncertainties should be able to work together to realize the robust control of the SIMO underactuated systems with mismatched uncertainties. Therefore, the total control law can be written as

$$u = u_n + u_{cn} \tag{19}$$

where $u_n$ is the hierarchical sliding mode control law of the $n$th layer; and $u_{cn}$ is the sliding mode compensator at the $n$th layer sliding surface. Here $u_n$ and $u_{cn}$ are given by equations (17) and (18).

## 3 STABILITY ANALYSIS

In this section, the stability of the entire sliding surfaces is analysed. The stability of the lumped sliding mode compensator at the last-layer sliding surface is first analysed.

*Theorem 1*

Consider an underactuated system with mismatched uncertainties as described by equation (1), if the robust control law is adopted as in equation (19) and the last-layer sliding surface is defined as in equation (6) $(i=n)$, then the last-layer sliding surface is asymptotically stable.

*Proof*

Because of the existence of mismatched uncertainties, the Lyapunov function of the actual system with uncertainties at the last-layer sliding surface becomes

$$\bar{V}_n=\bar{S}_n^2/2 \tag{20}$$

The mismatched uncertainties make the dynamic process of the actual system different from the nominal system. Furthermore, the actual system become a nominal one when the uncertainties are compensated. Thus

$$\bar{S}_n=S_n \tag{21}$$

and

$$\dot{\bar{S}}_n=\left[\sum_{r=1}^{n}\left(\prod_{j=r}^{n}a_j\right)\bar{d}_r+\sum_{r=1}^{n}\left(\prod_{j=r}^{n}a_j\right)b_r u_{\text{eqr}}\right] -k_nS_n-\eta_n\text{sgn}S_n \tag{22}$$

Differentiating equation (20) with respect to time $t$, results in

$$\dot{\bar{V}}_n=\bar{S}_n\dot{\bar{S}}_n=S_n\dot{\bar{S}}_n \tag{23}$$

Substituting equations (21) and (22) into equations (23) yields

$$\dot{\bar{V}}_n=S_n\left[\sum_{r=1}^{n}\left(\prod_{j=r}^{n}a_j\right)d_r\right]-|S_n|\left[\sum_{r=1}^{n}\left(\prod_{j=r}^{n}a_j\right)\bar{d}_r\right] -\eta_n|S_n|-k_nS_n^2 \tag{24}$$

Integrating both sides of equation (24) leads to

$$\int_0^t\dot{\bar{V}}_n\mathrm{d}\tau=\int_0^t\left[S_n\left[\sum_{r=1}^{n}\left(\prod\nolimits_{j=r}^{n}a_j\right)d_r\right] -|S_n|\left[\sum\nolimits_{r=1}^{n}\left(\prod\nolimits_{j=r}^{n}a_j\right)\bar{d}_r\right] -\eta_n|S_n|-k_nS_n^2\right]\mathrm{d}\tau \tag{25}$$

Then it is possible to write that

$$\begin{aligned}&\bar{V}_n(0)-\bar{V}_n(t)\\&=\int_0^t\left[\eta_n|S_n|+k_nS_n^2+|S_n|\left[\sum_{r=1}^{n}\left(\prod\nolimits_{j=r}^{n}a_j\right)d_r\right] -S_n\left[\sum_{r=1}^{n}\left(\prod\nolimits_{j=r}^{n}a_j\right)d_r\right]\right]\mathrm{d}\tau\\&\geqslant\int_0^t\left(\eta_n|S_n|+k_nS_n^2\right)\mathrm{d}\tau\end{aligned} \tag{26}$$

Furthermore

$$\bar{V}_n(0)\geqslant\int_0^t\left(\eta_n|S_n|+k_nS_n^2\right)\mathrm{d}\tau \tag{27}$$

Finally, the following equation is obtained

$$\lim_{t\to\infty}\int_0^t\left(\eta_n|S_n|+k_nS_n^2\right)\mathrm{d}\tau\leqslant\bar{V}_n(0)<\infty \tag{28}$$

In the light of Barbalat's lemma, $\eta_n|S_n|+k_nS_n^2\to0$ as $t\to\infty$, which means $\lim_{t\to\infty}S_n=0$, namely, the last-layer sliding surface $S_n$ is asymptotically stable.

*Theorem 2*

Consider an uncertain underactuated system such as equation (1), and define every layer sliding surface as in equation (6) $(i=1,\ldots,n-1)$. If the robust control law is adopted as in equation (19), then the sliding surfaces of the lower $n-1$ layers are still asymptotically stable.

*Proof*

As previously stated the lumped sliding mode compensator compensates the mismatched uncertainties at the last layer. Thus, the lower $n-1$ layers can be treated as being sliding surfaces of a nominal system.

From equation (2), the Lyapunov function of the $i$th-layer sliding surface is $V_i(t)=S_i^2/2$ for

$1 \leqslant i \leqslant n-1$. Differentiating $V_i(t)$ with respect to time $t$, leads to

$$\dot{V}_i = S_i \dot{S}_i \tag{29}$$

Using the Lyapunov stabilization theorem, let $\dot{S}_i = -\eta_i \mathrm{sgn} S_i - k_i S_i$ ($k_i$ and $\eta_i$ are positive constants) as in equation (15). Thus, it is possible to write that

$$\dot{V}_i = -\eta_i |S_i| - k_i S_i^2 \tag{30}$$

Integrating both sides of equation (30) yields

$$V_i(0) - V_i(t) = \int_0^t \left(\eta_i |S_i| + k_i S_i^2\right) \mathrm{d}\tau \tag{31}$$

Furthermore, it is possible to write that

$$\begin{aligned} V_i(0) &= V_i(t) + \int_0^t \left(\eta_i |S_i| + k_i S_i^2\right) \mathrm{d}\tau \\ &\geqslant \int_0^t \left(\eta_i |S_i| + k_i S_i^2\right) \mathrm{d}\tau \end{aligned} \tag{32}$$

Finally the following equation is obtained

$$\lim_{t\to\infty} \int_0^t \left(\eta |S_i| + k S_i^2\right) \mathrm{d}\tau \leqslant V_i(0) < \infty \tag{33}$$

In light of Barbalat's lemma, $\eta_i |S_i| + k_i S_i^2 \to 0$ as $t \to \infty$, which means that $\lim_{t\to\infty} S_i = 0$, namely, the $i$th-layer sliding surface ($1 \leqslant i \leqslant n-1$) is asymptotically stable.

*Theorem 3*

Consider an uncertain underactuated system as in equation (1) and define the sliding surfaces of all the subsystems as in equation (3). If the control law as in equation (19) is adopted, then the sliding surfaces of all the subsystems are asymptotically stable.

*Proof*

From Theorem 1 and Theorem 2, there exist

$$\lim_{t\to\infty} S_i = 0, \qquad \text{for} \quad 1 \leqslant i \leqslant n \tag{34}$$

If as previously defined $S_1 = s_1$, then the sliding surface of the first subsystem is asymptotically stable.

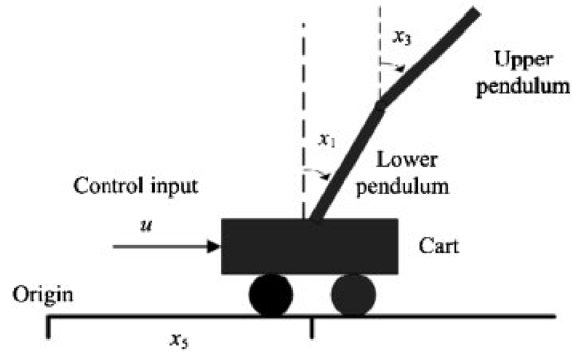


**Fig. 2** Structure of system consisting of two inverted pendulums

In the following, it will be proved that the sliding surfaces of the other $n-1$ subsystems are asymptotically stable by contradiction.

If it is assumed that $s_i$ ($2 \leqslant i \leqslant n$) is not asymptotically stable, that is

$$\lim_{t\to\infty} s_i \neq 0 \tag{35}$$

From equation (7), it can be obtained that

$$S_i = \lambda_{i-1} S_{i-1} + s_i, \qquad \text{for } 2 \leqslant i \leqslant n \tag{36}$$

Calculating the limit of both sides of equation (7) yields

$$\begin{aligned} \lim_{t\to\infty} S_i &= \lim_{t\to\infty} (\lambda_{i-1} S_{i-1} + s_i) = \lim_{t\to\infty} \lambda_{i-1} S_{i-1} + \lim_{t\to\infty} s_i \\ &= \lim_{t\to\infty} s_i \neq 0 \end{aligned} \tag{37}$$

This case contradicts the case $\lim_{t\to\infty} S_i = 0$ ($2 \leqslant i \leqslant n$) that was obtained using Theorem 1 and Theorem 2. Therefore, the initial assumption of equation (17) is false and the opposite case of equation (17) that $\lim_{t\to\infty} s_i = 0$ $(2 \leqslant i \leqslant n)$ must be correct.

In summary, the sliding surfaces of all the subsystems are asymptotically stable.

## 4 SIMULATION RESULTS

In this section, it will be demonstrated that the proposed robust control strategy is able to stabilize a system consisting of two inverted pendulums. The structure of such a system is shown in Fig. 2. It consists of three subsystems: the lower pendulum, the upper pendulum, and the cart. The control objective of stabilizing the system is to balance both of the pendulums upright and to move the cart to the start of the rail.

The symbols in Fig. 2 are defined as follows: $x_1$ is the lower pendulum angle with respect to the vertical line, $x_3$ is the upper pendulum angle with respect to the vertical line, $x_5$ is the cart position with respect to the origin, and $u$ is the control force. Let $n = 3$ in equation (1), then we have its state equation

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f_1 + b_1 u + d_1 \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = f_2 + b_2 u + d_2 \\ \dot{x}_5 = x_6 \\ \dot{x}_6 = f_3 + b_3 u + d_3 \end{cases} \tag{38}$$

where $x_2$ is the angular velocity of the lower pendulum, $x_4$ is the angular velocity of the upper pendulum, and $x_6$ is the velocity of the cart, the expressions of $f_i$ and $b_i$ $(i = 1, 2, 3)$ are given in [**15**], and $d_i$ is the mismatched uncertain term whose bound is known as $\bar{d}_i$. The mismatched uncertain terms of the system are assumed to be

$$\begin{cases} d_1 = 0.0872 + 0.5\rho \\ d_2 = 0.0872 + 0.5\rho \\ d_3 = 0.5\rho \end{cases} \tag{39}$$

where $\rho$ is a random number whose range is from $-1$ to 1.

The structural parameters are a cart mass $M = 1\,\text{kg}$, the lower-pendulum mass $m_1 = 1\,\text{kg}$, the upper-pendulum mass $m_2 = 1\,\text{kg}$, the lower-pendulum length $l_1 = 0.1\,\text{m}$, the upper-pendulum length $l_2 = 0.1\,\text{m}$, the gravitational acceleration $g = 9.81\,\text{m/s}^2$. The control objective is to go from the initial states $[\pi/6,\ 0,\ \pi/18,\ 0,\ 0,\ 0]^{\text{T}}$ to the desired states $[0, 0, 0, 0, 0, 0]^{\text{T}}$. The initial states, the desired states, the structure parameters, and the control objective are the same as those in Lin and Mon [**16**].

From equation (39), the bounds of the mismatched uncertain terms $d_1$, $d_2$, and $d_3$ can be determined to be 0.5872, 0.5872, and 0.5. The lumped sliding mode compensator can now be obtained. The parameters of the hierarchical sliding mode controller are selected by a trial-and-error approach to be $c_1 = 184.26$, $c_2 = 15.96$, $c_3 = 0.72$, $a_1 = -0.06$, $a_2 = 0.45$, $k = 1.50$, and $\eta = 0.02$. The simulation results are now discussed.

Figure 3 shows the angular curves and the positional curves. The solid curves are with a compensator and the dashed curves are without a compensator. The parameters of the hierarchical sliding mode controller for the nominal systems are the same. The only difference is that one has a compensator and the other does not have a compensator. This allows a fair comparison of the systems to highlight the effect of the lumped compensator. As can be seen from the plots the positional curves show that the proposed robust

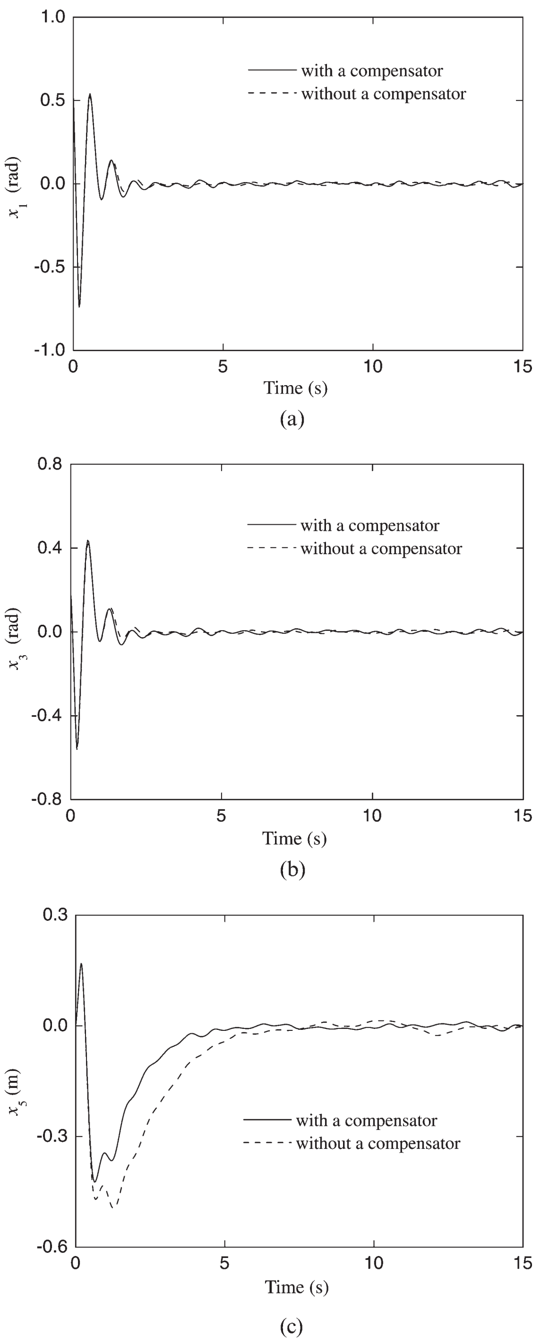


**Fig. 3** (a) The angle of the lower pendulum, (b) the angle of the upper pendulum, and (c) the position of the cart

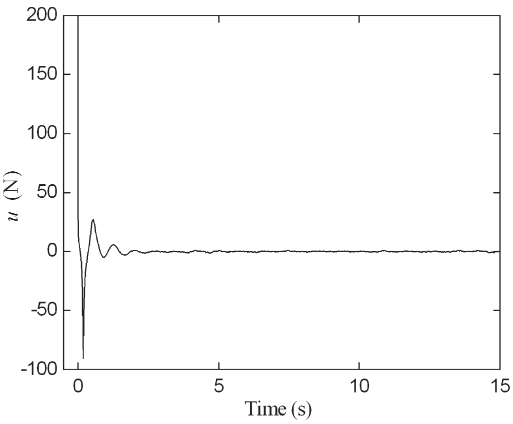


**Fig. 4** Control force applied to the cart

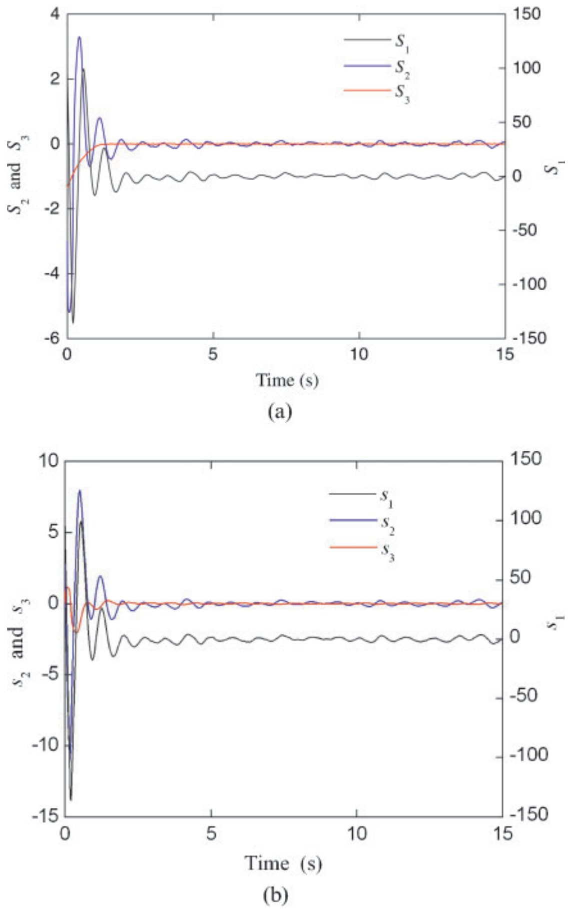


**Fig. 5** (a) The single-layer sliding surfaces and (b) the subsystem sliding surfaces

controller can compensate for the mismatched uncertainties and realize the control objective although both controllers are able to make the double pendulums reach an upright position.

Figure 4 is the control force applied to the cart. Figure 5 shows the sliding surfaces. Using the robust control method, not only is every layer sliding surface asymptotically stable, but also the sliding surfaces of all the subsystems possess asymptotic stability, as proven in Theorems 1, 2, 3. In Fig. 5(a), $S_3$ is smoother than $S_2$ and $S_1$. This could be explained as follows. Using the proposed method, the lumped sliding mode compensator is designed at the third-layer sliding surface. It works and eliminates all the uncertainties at the sliding surface of the third layer. However, uncertainties actually exist in the first and second subsystems. Thus, $S_1$ and $S_2$ are not as even as $S_3$.

Figure 6, is a performance comparison between a traditional single-layer sliding mode controller without a compensator and the presented hierarchical method with a compensator from the initial condition vector $\boldsymbol{X}_0 = [\pi/18, 0, \pi/36, 0, 0, 0]^{\mathrm{T}}$ to the origin. To design the traditional controller, the nominal system model was linearized at the origin and then the SMC law is equivalent to control plus switching control. When the linearized nominal system arrives at the sliding mode stage, the sliding mode function $S_{\mathrm{s}}$ can be obtained using equation (40) by the pole placement approach of linear systems

$$S_{\mathrm{s}} = -2.2296x_1 - 0.0121x_2 + 3.1639x_3 + 0.1982x_4 + 0.2026x_5 + 0.2813x_6 \tag{40}$$

The parameters used for the switching control are those used in the hierarchical system.

From the simulations it can be seen that the single-layer sliding mode controller designed by the linearization method could not stabilize the system with mismatched uncertainties (equation (1)) from the initial condition $\boldsymbol{X}_0 = [\pi/6, 0, \pi/18, 0, 0, 0]^{\mathrm{T}}$ to the desired $[0, 0, 0, 0, 0, 0]^{\mathrm{T}}$ whereas the hierarchical sliding mode controller can achieve this goal. This fact means that this initial condition is out of the stability domain of the single-layer controller, which confirms that the presented hierarchical controller with a compensator is more robust than the single-layer controller. As the initial condition approaches the operating point $[0,0,0,0,0,0]^{\mathrm{T}}$ for the linearization, not only can the single-layer controller stabilize the uncertain system, but also the performance of the two pendulums may be better, which is shown in Figs 6(a) and (b). In Fig. 6(c), the cart is moved a shorter distance by the hierarchical controller with a compensator, which is in agreement with Fig. 3(c).

*Remark 4*

In [**15**], the control objective was simply to make the two pendulums become upright and the cart posi-

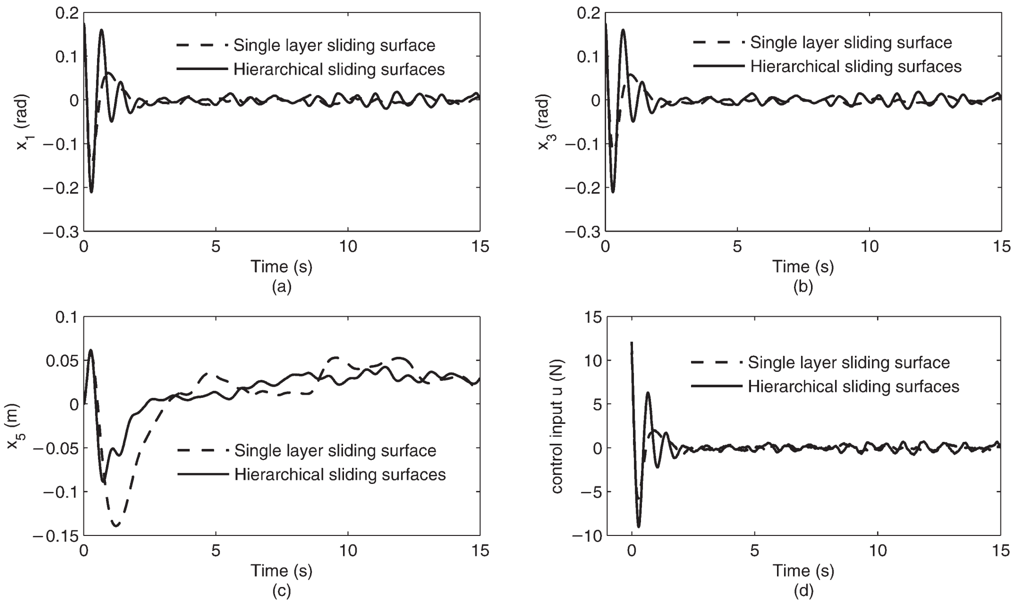


**Fig. 6** Performance comparison between single-layer SMC without a compensator and the hierarchical SMC with a compensator from $\boldsymbol{X}_0 = [\pi/18, 0, \pi/36, 0, 0, 0]^{\mathrm{T}}$ to the origin

tion was not considered. Compared with [**15**] the objective in this work is more difficult. Compared with [**16**], the curves obtained in this study are smoother and the response time is shorter. Furthermore, the presented controller is robust although it needs a large control force (see Fig. 4) that exceed 200 N at the start of the simulation. For most physical systems, it is generally very difficult to produce such a force. Also, this force may make the input saturated, which may influence the operation and performance characteristics of the control system. This drawback will restrict the control method in practice. However, this phenomenon suggests two interesting areas for future study. One is how to design a controller with a smaller control force. The other is how to analyse the system stability of this robust control method when the control input becomes saturated.

## 5 CONCLUSIONS

In this paper, a robust controller for a class of underactuated systems with mismatched uncertainties, consisting of a nominal system and mismatched uncertainties has been proposed on the basis of SMC. The nominal system is made up of several subsystems and this allows a hierarchical sliding mode controller to be designed. A lumped sliding mode compensator has been designed to deal with the mismatched uncertainties. The hierarchical sliding mode controller and the lumped sliding mode compensator work together to create robust control for the system. The asymptotic stability of all the sliding surfaces has been proven. In the simulations, the proposed control method was applied to the stabilization control of a system consisting of two inverted pendulums. The simulation results show the validity of the control strategy. Although there exist some weak points in the presented robust control method, such as how to select suitable controller parameters, how to deal with the problem of the input saturation, and so on, this method provides a robust control strategy for the class of underactuated systems with mismatched uncertainties.

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## REFERENCES

1 **Spong, M.** Underactuated mechanical systems, control problems in robotics and automation. *Lect. Notes Control Inf. Sci.*, 1998, **230**, 135–150.

2 **Reyhanoglu, M., van der Schaft, A., McClamroch, N. H.,** and **Kolmanovsky, I.** Dynamics and control of a class of underactuated mechanical systems. *IEEE Trans. Autom. Control*, 1999, **44**(9), 1663–1671.

3 **Bloch, A. M., Reyhanoglu, M.,** and **McClamroch, N. H.** Control and stabilization of nonholonomic dynamic systems. *IEEE Trans. Autom. Control*, 1992, **37**(11), 1746–1757.

4 **Yabuno, H., Matsuda, T.,** and **Aoshima, N.** Reachable and stabilizable area of an underactuated manipulator without state feedback control. *IEEE/ASME Trans. Mechatron.*, 2005, **10**(4), 397–403.

5 **Tarn, T.-J., Sanposh, P., Cheng, D.,** and **Zhang, M.** Output regulation for nonlinear systems: some recent theoretical and experimental results. *IEEE Trans. Control Syst. Technol.*, 2005, **13**(4), 605–610.

6 **Shiriaev, A., Robertsson, A., Perram, J.,** and **Sandberg, A.** Periodic motion planning for virtually constrained Euler – Lagrange systems. *Syst. Control Lett.*, 2006, **55**, 900–907.

7 **Kolesnichenko, O.** and **Shiriaev, A. S.** Partial stabilization of underactuated Euler–Lagrange systems via a class of feedback transformations. *Syst. Control Lett.*, 2002, **45**, 121–132.

8 **Do, K. D.** and **Pan, J.** Global robust adaptive path following of underactuated ships. *Automatica*, 2006, **42**, 1713–1722.

9 **Fahimi, F.** Sliding-mode formation control for underactuated surface vessels. *IEEE Trans. Robot.*, 2007, **23**, 617–622.

10 **Birglen, L.** and **Gosselin, C. M.** Kinetostatic analysis of underactuated fingers. *IEEE Trans. Robot. Autom.*, 2004, **20**(2), 211–221.

11 **Aguiar, A. P.** and **Hespanha, J. P.** Trajectory-tracking and path-following of underactuated autonomous vehicles with parametric modeling uncertainty. *IEEE Trans. Autom. Control*, 2007, **52**, 1362–1379.

12 **Chen, J., Dixon, W. E., Dawson, M.,** and **McIntyre, M.** Homography-based visual servo tracking control of a wheeled mobile robot. *IEEE Trans. Robot.*, 2006, **22**(2), 406–415.

13 **Tinos, R., Terra, M. H.,** and **Ishihara, J. Y.** Motion and force control of cooperative robotic manipulators with passive joints. *IEEE Trans. Control Syst. Technol.*, 2006, **14**(4), 725–734.

14 **Yi, J., Yubazaki, N.,** and **Hirota, K.** Upswing and stabilization control of inverted pendulum system based on the SIRMs dynamically connected fuzzy inference model. *Fuzzy Sets Syst.*, 2001, **122**, 139–152.

15 **Lo, J.** and **Kuo, Y.** Decoupled fuzzy sliding-mode control. *IEEE Trans. Fuzzy Syst.*, 1998, **6**(3), 426–435.

16 **Lin, C. M.** and **Mon, Y. J.** Decoupling control by hierarchical fuzzy sliding-mode controller. *IEEE Trans. Control Syst. Technol.*, 2005, **13**(4), 593–598.

17 **Wai, R.-J., Kuo, M.-A.,** and **Lee, J.-D.** Design of cascade adaptive fuzzy sliding-mode control for nonlinear two-axis inverted-pendulum servomechanism. *IEEE Trans. Fuzzy Syst.*, 2008, **16**(5), 1232–1244.

18 **Alleyne, A.** Physical insights on passivity-based TORA control designs. *IEEE Trans. Control Syst. Technol.*, 1998, **6**(3), 436–439.

19 **Zhang, M.** and **Tarn, T. J.** A hybrid switching control strategy for nonlinear and underactuated mechanical systems. *IEEE Trans. Autom. Control*, 2003, **48**(10), 1777–1782.

20 **Fang, Y., Dixon, W. E., Dawson, D. M.,** and **Zergeroglu, E.** Nonlinear coupling control laws for an underactuated overhead crane system. *IEEE/ASME Trans. Mechatron.*, 2003, **3**(8), 418–423.

21 **Tsai, C. H., Chung, H. Y.,** and **Yu, F. M.** Neuro-sliding mode control with its applications to seesaw systems. *IEEE Trans. Neural Netw.*, 2004, **15**(1), 124–134.

22 **Utkin, V. I.** *Sliding modes in control and optimization*, 1992 (Springer-Verlag, New York).

23 **Wang, W., Yi, J., Zhao, D.,** and **Liu, D.** Design of a stable sliding-mode controller for a class of second-order underactuated systems. *IEE Proc. – Control Theory Appl.*, 2004, **151**(6), 683–690.

24 **Xu, R.** and **Özgüner, U.** Sliding mode control of a class of underactuated systems. *Automatica*, 2008, **44**, 233–241.

25 **Martinez, R., Alvarez, J.,** and **Orlov, Y.** Hybrid sliding-mode-based control of underactuated systems with dry friction. *IEEE Trans. Ind. Electron.*, 2008, **55**(11), 3998–4003.

26 **Wang, W., Liu, X. D.,** and **Yi, J. Q.** Structure design of two types of sliding-mode controllers for a class of under-actuated mechanical systems. *IET Control Theory Appl.*, 2007, **1**(1), 163–172.